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Contents

Preface

The evolution of the state of many systems modeled by linear partial differential equations (PDEs) or linear delay differential equations can be described by operator semigroups. The state of such a system is an element in an infinite-dimensional normed space, whence the name "infinite-dimensional linear system".

The study of operator semigroups is a mature area of functional analysis, which is still very active. The study of observation and control operators for such semigroups is relatively more recent. These operators are needed to model the interaction of a system with the surrounding world via outputs or inputs. The main topics of interest about observation and control operators are admissibility, observability, controllability, stabilizability and detectability. Observation and control operators are an essential ingredient of well-posed linear systems (or more generally system nodes). In this book we deal only with admissibility, observability and controllability. We deal only with operator semigroups acting on Hilbert spaces.

This book is meant to be an elementary introduction into the topics mentioned above. By "elementary" we mean that we assume no prior knowledge of finite-dimensional control theory, and no prior knowledge of operator semigroups or of unbounded operators. We introduce everything needed from these areas. We do assume that the reader has a basic understanding of bounded operators on Hilbert spaces, differential equations, Fourier and Laplace transforms, distributions and Sobolev spaces on n-dimensional domains. Much of the background needed in these areas is summarized in the appendices, often with proofs.

Another meaning of "elementary" is that we only cover results for which we can provide complete proofs. The abstract results are supported by a large number of examples coming from PDEs, worked out in detail. We mention some of the more advanced results, which require advanced tools from functional analysis or PDEs, in our bibliographic comments. One of the glaring omissions of the book is that we do not cover anything based on microlocal analysis.

The concepts of controllability and observability have been set at the center of control theory by the work of R. Kalman in the 1960s and soon they have been generalized to the infinite-dimensional context. Among the early contributors we mention D.L. Russell, H. Fattorini, T. Seidman, A.V. Balakrishnan, R. Triggiani, W. Littman and J.-L. Lions. The latter gave the field an enormous impact with his book [156], which is still a main source of inspiration for many researchers.

Unlike in finite-dimensional control theory, for infinite-dimensional systems there are many different (and not equivalent) concepts of controllability and observability. The strongest concepts are called exact controllability and exact observability, respectively. Exact controllability in time $\tau > 0$ means that any final state can be reached, starting from the initial state zero, by a suitable input signal on the time interval $[0, \tau]$. The dual concept of exact observability in time τ means that if the input is zero, the initial state can be recovered in a continuous way from the output signal on the time interval $[0, \tau]$. We shall establish the exact observability or exact controllability of various (classes of) systems using a variety of techniques. We shall also discuss other concepts of controllability and observability.

Exact controllability is important because it guarantees stabilizability and the existence of a linear quadratic optimal control. Dually, exact observability guarantees the existence of an exponentially converging state estimator and the existence of a linear quadratic optimal filter. Moreover, exact (or final state) observability is useful in identification problems. To include these topics into this book we would have needed at least double the space and ten times the time, and we gave up on them. There are excellent books dealing with these subjects, such as (in alphabetical order) Banks and Kunisch [13], Bensoussan et al. [17], Curtain and Zwart [39], Luo, Guo and Morgul [163] and Staffans [209].

Researchers in the area of observability and controllability tend to belong to either the abstract functional analysis school or to the PDE school. This is true also for the authors, as MT feels more at home with PDEs and GW with functional analysis. By our collaboration we have attempted to combine these two approaches. We believe that such a collaboration is essential for an efficient approach to the subject. More precisely, the functional analytic methods are important to formulate in a precise way the main concepts and to investigate their interconnections. When we try to apply these concepts and results to systems governed by PDEs, we generally have to face new difficulties. To solve these difficulties, quite refined techniques of mathematical analysis might be necessary. In this book the main tools to tackle concrete PDE systems are multipliers, Carleman estimates and non-harmonic Fourier analysis, but results from even more sophisticated fields of mathematics (microlocal analysis, differential geometry, analytic number theory) have been used in the literature.

While we were working on this book, Birgit Jacob from the University of Delft (The Netherlands) with Hans Zwart from the University of Twente (The Netherlands) have achieved an important breakthrough on exact observability for normal semigroups. Birgit has communicated to us their results, so that we could include them (without proof) in the bibliographic notes on Chapter 6.

We are grateful to Emmanuel Humbert from the University of Nancy (France) for accepting to contribute to an appendix on differential calculus. The material in Chapter 14 is to a great extent his work.

Bernhard Haak from the University of Bordeaux has contributed significantly to Section 5.6. Moreover, Proposition 5.4.7 is due to him.

Large parts of the manuscript have been read by our colleagues Karim Ramdani, Takéo Takahashi (both from Nancy) and Xiaowei Zhao (from London) who made many suggestions for improvements. The two figures in Chapter 7, the figure in Chapter 11 and the first figure in Chapter 15 were drawn by Karim Ramdani. Jorge San Martin (from Santiago de Chile) contributed in an important manner to the calculations in Section 15.1. Luc Miller (from Paris) made useful comments on Chapter 6. Sorin Micu (from Craiova) and Jean-Pierre Raymond (from Toulouse) made very useful remarks on Sections 9.2 and 15.2, respectively. Gérald Tenenbaum and François Chargois (both from Nancy) suggested us corrections and simplifications in Sections 8.4 and 14.2. Birgit Jacob, in addition to her help described earlier, has made useful bibliographic comments on Chapters 5 and 6. Other valuable bibliographic comments have been sent to us by Jonathan Partington (from Leeds). Qingchang Zhong (from Liverpool) pointed out some small mistakes and typos. We thank them all for their patience and help.

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Nancy and Tel Aviv

Chapter 1

Observability and Controllability for Finite-dimensional Systems

1.1 Norms and inner products

In this section we recall some basic concepts and results concerning normed vector spaces. Our aim is very modest: to list those facts which are needed in Chapter 1 (the treatment of controllability and observability for finite-dimensional systems). We do not give proofs – our aim is only to clarify our terminology and notation. A proper treatment of this material can be found in many books, of which we mention Brown and Pearcy [23], Halmos [86] and Rudin [194]. Introductions to functional analysis that stress the connections with, and applications in, systems theory are Nikol'skii [178], Partington [181] and Young [240].

Throughout this book, the notation

$$\mathbb{N}, \quad \mathbb{Z}, \quad \mathbb{R}, \quad \mathbb{C}$$

stands for the sets of natural numbers (starting with 1), integer numbers, real numbers and complex numbers, respectively. We denote $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. For the remaining part of this chapter, we assume that the reader is familiar with the basic facts about vector spaces and mathematical analysis.

Let X be a complex vector space. A *norm* on X is a function from X to $[0, \infty)$, denoted $\|x\|$, which satisfies the following assumptions for every $x, z \in X$ and for every $\lambda \in \mathbb{C}$: (1) $\|x+z\| \leqslant \|x\|+\|z\|$, (2) $\|\lambda x\| = |\lambda| \cdot \|x\|$, (3) if $x \neq 0$, then $\|x\| > 0$. A vector space on which a norm has been specified is called a *normed space*. If X is a normed space and $x \in X$, sometimes we write $\|x\|_X$ (or we use other subscripts) instead of $\|x\|$, if we want to avoid a confusion arising from the fact that the same x belongs also to another normed space.

Let X be a complex vector space. An *inner product* on X is a function from $X \times X$ to $\mathbb{C}$, denoted $\langle x, z \rangle$, which satisfies the following assumptions for every $x, y, z \in X$ and every $\lambda \in \mathbb{C}$:

(1) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$,
(2) $\langle \lambda x, z\rangle = \lambda\langle x, z\rangle$,
(3) $\langle x, z\rangle = \overline{\langle z, x\rangle}$,
(4) if $x \neq 0$, then $\langle x, x\rangle > 0$.

A vector space on which an inner product has been specified is called an *inner product space*.

Let X be an inner product space. The *norm induced by the inner product* is the function $\|x\| = \sqrt{\langle x, x\rangle}$. It is easy to see that

$$\|x+y\|^2 = \|x\|^2 + 2\mathrm{Re}\,\langle x, y\rangle + \|y\|^2 \qquad \forall\, x, y \in X\,. \tag{1.1.1}$$

Using here $y = -(\langle x, z\rangle/\|z\|^2)z$, it follows that

$$|\langle x, z\rangle| \leqslant \|x\| \cdot \|z\| \qquad \forall\, x, z \in X\,, \tag{1.1.2}$$

which is called the *Cauchy–Schwarz inequality*. This, together with (1.1.1), implies that $\|x+z\| \leqslant \|x\| + \|z\|$ holds, so that this function is indeed a norm (in the sense defined earlier). Not every norm is induced by an inner product.

The simplest example is to take $X = \mathbb{C}^n$ with the usual inner product given by $\langle x, z\rangle = \sum_{k=1}^{n} x_k \overline{z_k}$. The norm induced by this inner product is called the *Euclidean norm*:

$$\|x\| = \left(\sum_{k=1}^{n} |x_k|^2\right)^{\frac{1}{2}}.$$

If we imagine the above example with $n \to \infty$, we obtain the space called l^2. This consists of all the sequences (x_k) with $x_k \in \mathbb{C}$ such that $\sum_{k\in\mathbb{N}} |x_k|^2 < \infty$. The usual inner product on l^2 is given by

$$\langle x, z\rangle = \sum_{k\in\mathbb{N}} x_k \overline{z_k}\,.$$

Another important example is the space $L^2(J; U)$, where $J \subset \mathbb{R}$ is an interval and U is a finite-dimensional inner product space. This space consists of all the measurable functions $u : J \to U$ for which $\int_J \|u(t)\|^2 \mathrm{d}t < \infty$. In this space we do not distinguish between the two functions u and v if $\int_J \|u(t) - v(t)\| \mathrm{d}t = 0$. Thus, $L^2(J; U)$ is actually a space of equivalence classes of functions. The inner product on $L^2(J; U)$ is

$$\langle u, v\rangle = \int_J \langle u(t), v(t)\rangle \,\mathrm{d}t\,.$$

Now let X be a normed space. A sequence (x_k) with terms in X is called *convergent* if there exists $x^0 \in X$ such that $\lim \|x_k - x^0\| = 0$. In this case we also write $\lim x_k = x^0$ or $x_k \to x^0$ and we call x^0 the *limit* of the sequence (x_k). It is easy to see that if a limit x^0 exists, then it is unique and $\|x^0\| = \lim \|x_k\|$.

Let X be a normed space. The *closure* of a set $L \subset X$, denoted clos L, is the set of the limits of all the convergent sequences with terms in L. We have

$$L \subset \operatorname{clos} L = \operatorname{clos} \operatorname{clos} L\,.$$

L is called *closed* if $\operatorname{clos} L = L$. If V is a subspace of X, then also $\operatorname{clos} V$ is a subspace. Every finite-dimensional subspace of X is closed.

A sequence (x_k) with terms in X is called a *Cauchy sequence* if $\lim \|x_k - x_j\| = 0$. Equivalently, for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for every $k, j \in \mathbb{N}$ with $k, j \geqslant N_\varepsilon$ we have $\|x_k - x_j\| \leqslant \varepsilon$. It is easy to see that every convergent sequence is a Cauchy sequence. However, the converse statement is not true in every normed space X. The normed space X is called *complete* if every Cauchy sequence in X is convergent. In this case, X is also called a *Banach space.* If the norm of a Banach space is induced by an inner product, then the space is also called a *Hilbert space.*

For example, l^2 and $L^2(J;U)$ (with the norms induced by their usual inner products) are Hilbert spaces. Every finite-dimensional normed space is complete.

Assume that X is a Hilbert space and $M \subset X$. The set of all the finite linear combinations of elements of M is denoted by $\operatorname{span} M$ (this is the smallest subspace of X that contains M). The *orthogonal complement of* M is defined by

$$M^\perp = \{x \in X \mid \langle m, x \rangle = 0 \quad \forall\, m \in M\}\,,$$

and this is a closed subspace of X. We have

$$M^{\perp\perp} = \operatorname{clos} \operatorname{span} M\,. \tag{1.1.3}$$

The *Riesz projection theorem* says that if X is a Hilbert space, V is a closed subspace of X and $x \in X$, then there exist unique $v \in V$ and $w \in V^\perp$ such that $x = v + w$. If x, v and w are as above, then clearly $\|x\|^2 = \|v\|^2 + \|w\|^2$ and v is called the *projection* of x onto V.

A set $M \subset X$ is called *orthonormal* if for every $e, f \in M$ we have $\|e\| = 1$ and $f \neq e$ implies $\langle e, f \rangle = 0$. It is easy to see that such a set is linearly independent. An *orthonormal basis* in X is an orthonormal set B with the property $B^\perp = \{0\}$. If an orthonormal basis is finite, then it is also a basis in the usual sense of linear algebra, but this is not true in general (because not every vector can be written as a finite linear combination of the basis vectors).

Let X and Y be normed spaces. A function $T : X \to Y$ is called a *linear operator* if it satisfies the following assumptions for every $x, z \in X$ and for every $\lambda \in \mathbb{C}$: (1) $T(x + z) = T(x) + T(z)$, (2) $T(\lambda x) = \lambda T(x)$. We normally write Tx instead of $T(x)$. A linear operator $T : X \to Y$ is called *bounded* if

$$\sup\{\|Tx\| \mid x \in X,\ \|x\| \leqslant 1\} < \infty\,.$$

This is equivalent to the fact that T is continuous, i.e., $x_n \to x^0$ implies $Tx_n \to Tx^0$. It is easy to verify that if X is finite dimensional, then every linear operator from X to some other normed space Y is continuous.

The set of all the bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. If $Y=X$, then we normally write $\mathcal{L}(X)$ instead of $\mathcal{L}(X,X)$. It is easy to see that $\mathcal{L}(X,Y)$ is a vector space, if we define the addition of operators by $(T+S)x = Tx+Sx$, and the multiplication of an operator with a number by $(\lambda T)x = \lambda(Tx)$. Moreover, for $T\in\mathcal{L}(X,Y)$ and $S\in\mathcal{L}(Y,Z)$, the product ST is an operator in $\mathcal{L}(X,Z)$ defined as the composition of these functions.

The *operator norm* on $\mathcal{L}(X,Y)$ is defined as follows:

$$\|T\| = \sup_{\|x\|\leqslant 1} \|Tx\|.$$

This is indeed a norm, as defined earlier. Moreover, if $T\in\mathcal{L}(X,Y)$ and $S\in\mathcal{L}(Y,Z)$, then $\|ST\|\leqslant\|S\|\cdot\|T\|$. If Y is a Banach space, then so is $\mathcal{L}(X,Y)$.

If $T\in\mathcal{L}(X,Y)$, then the *null-space* (sometimes called the kernel) and the *range* of T are subspaces of X and Y defined, respectively, by

$$\operatorname{Ker} T = \{x\in X \mid Tx=0\}, \qquad \operatorname{Ran} T = \{Tx \mid x\in X\}.$$

$\operatorname{Ker} T$ is always closed. T is called *one-to-one* if $\operatorname{Ker} T=\{0\}$ and it is called *onto* if $\operatorname{Ran} T = Y$. The operator T is *invertible* iff it is one-to-one and onto. In this case, there exists a linear operator $T^{-1}: Y\to X$ such that $T^{-1}T=I$ (the identity operator on X) and $TT^{-1}=I$ (the identity operator on Y). If X and Y are Banach spaces and $T\in\mathcal{L}(X,Y)$ is invertible, then it can be proved (using a result called “the closed-graph theorem”) that the inverse operator is bounded: $T^{-1}\in\mathcal{L}(Y,X)$, see Section 12.1 in Appendix I for more details.

Let X be a Hilbert space and denote $X'=\mathcal{L}(X,\mathbb{C})$. The elements of X' are also called *bounded linear functionals* on X. On X' we define the multiplication with a number in an unusual way, not as we would normally do on a space of operators: if $\xi\in X'$ and $\lambda\in\mathbb{C}$,

$$(\lambda\xi)x = \overline{\lambda}(\xi x) \qquad \forall\, x\in X.$$

We use the operator norm on X'. Then X' is a Hilbert space, called the *dual space* of X. We define the mapping $J_R: X\to X'$ as follows:

$$(J_R z)x = \langle x,z\rangle \qquad \forall\, x\in X. \tag{1.1.4}$$

Due to the special definition of multiplication with a number on X', the mapping J_R is a linear operator. Moreover, it is easy to see from the Cauchy–Schwarz inequality that $\|J_R z\| = \|z\|$ (in particular, $J_R\in\mathcal{L}(X,X')$ and it is one-to-one).

The *Riesz representation theorem* states that J_R is onto. In other words, for every $\xi\in X'$ there exists a unique $z\in X$ such that $J_R z=\xi$. Hence, J_R is invertible. We often identify X' with X, by not distinguishing between z and $J_R z$.

Let X and Y be Hilbert spaces and $T\in\mathcal{L}(X,Y)$. The *adjoint of* T is the operator $T^*\in\mathcal{L}(Y',X')$ defined by

$$(T^*\xi)x = \xi(Tx) \qquad \forall\, x\in X,\ \xi\in Y'. \tag{1.1.5}$$

If we identify X with X' and Y with Y' (this is possible, as we have explained a little earlier), then of course $T^* \in \mathcal{L}(Y, X)$ and (1.1.5) becomes

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \qquad \forall\, x \in X,\ y \in Y\,. \tag{1.1.6}$$

It can be checked that $(ST)^* = T^*S^*$, $T^{**} = T$, $\|T^*\| = \|T\| = \|T^*T\|^{\frac{1}{2}}$ and

$$\text{Ker } T = (\text{Ran } T^*)^\perp, \qquad \text{clos Ran } T = (\text{Ker } T^*)^\perp. \tag{1.1.7}$$

From here it follows easily that

$$\text{Ker } T^*T = \text{Ker } T, \qquad \text{clos Ran } T^*T = \text{clos Ran } T^*. \tag{1.1.8}$$

Moreover, it can be shown that Ran T^*T is closed iff Ran T^* is closed iff Ran T is closed (the last equivalence is known as the closed range theorem).

More background on bounded operators will be given in Appendix I.

1.2 Operators on finite-dimensional spaces

In this section we recall some facts about linear operators acting on finite-dimensional inner product spaces. As in the previous section (and for the same reasons), we do not give proofs. Some good references on linear algebra are Bellman [16], Gantmacher [70], Golub and Van Loan [72], Horn and Johnson [104, 105], Lancaster and Tismenetsky [141] and Marcus and Minc [168].

In this section X, Y and Z denote finite-dimensional inner product spaces. We use the same notation for all the norms.

We denote by I the identity operator on any space. If $T \in \mathcal{L}(X, Y)$ is invertible, then $\dim X = \dim Y$. If $T \in \mathcal{L}(X, Y)$ and $\dim X = \dim Y$, then T is invertible iff it is one-to-one and this happens iff T is onto. If T^{-1} exists, then $\|T^{-1}\| \geqslant \|T\|^{-1}$.

Let $T \in \mathcal{L}(X, Y)$ and let T^* be its adjoint (as defined in (1.1.6)). If we use orthonormal bases in X and Y and represent T, T^* by matrices, then the matrix of T^* is the complex conjugate of the transpose of the matrix of T. The rank of T is defined as rank $T = \dim \text{Ran } T$ and we have rank $T^* =$ rank T.

Let $A \in \mathcal{L}(X)$. A number $\lambda \in \mathbb{C}$ is an *eigenvalue* of A if there exists an $x \in X$, $x \neq 0$, such that $Ax = \lambda x$. In this case, x is called an *eigenvector* of A. The set of all eigenvalues of A is called the *spectrum* of A and it is denoted by $\sigma(A)$. If $\tilde{A}$ is the matrix of A in some basis in X, then $p(s) = \det(sI - \tilde{A})$ is called the *characteristic polynomial* of A (and this is independent of the choice of the basis in X). The set $\sigma(A)$ consists of the zeros of p. The *Cayley–Hamilton theorem* states that $p(A) = 0$. If l eigenvectors of A correspond to l distinct eigenvalues, then the set of these eigenvectors is linearly independent. In particular, if A has $n = \dim X$ distinct eigenvalues, then we can find in X a basis consisting of eigenvectors of A.

We have $|\lambda| \leqslant \|A\|$ for all $\lambda \in \sigma(A)$, and $\lambda \in \sigma(A)$ implies $\overline{\lambda} \in \sigma(A^*)$. We denote by $\rho(A)$ the *resolvent set* of A (the complement of $\sigma(A)$ in $\mathbb{C}$). The function R defined by $R(s) = (sI - A)^{-1}$ is analytic on $\rho(A)$.

An operator $Q \in \mathcal{L}(X, Z)$ is called *isometric* if $Q^*Q = I$ (the identity on X). Equivalently, $\|Qx\| = \|x\|$ holds for every $x \in X$. Q is called *unitary* if it is isometric and onto (i.e., Ran $Q = Z$). If Q is unitary, then $QQ^* = I$ (the identity on Z). If $Q \in \mathcal{L}(X, Z)$ is isometric and $\dim X = \dim Z$, then from Ker $Q = \{0\}$ we see that Q is invertible, and hence unitary. If $Q \in \mathcal{L}(X)$ is unitary, then $|\lambda| = 1$ holds for all $\lambda \in \sigma(Q)$. For example, for every $\varphi \in \mathbb{R}$,

$$Q = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

is unitary in $\mathcal{L}(\mathbb{C}^2)$.

An operator $A \in \mathcal{L}(X)$ is *self-adjoint* if $A^* = A$. This is equivalent to the fact that $\langle Ax, x\rangle \in \mathbb{R}$ for all $x \in X$. We denote by diag $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ a matrix in $\mathbb{C}^{n\times n}$ with the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ on its diagonal and zero everywhere else.

Proposition 1.2.1. *Let $A \in \mathcal{L}(X)$ be self-adjoint and denote $n = \dim X$. Then there exists a unitary $Q \in \mathcal{L}(\mathbb{C}^n, X)$ such that*

$$A = Q\Lambda Q^*, \quad \textit{where} \quad \Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n). \tag{1.2.1}$$

The numbers λ_k appearing above are the eigenvalues of A and they are real.

It follows from this proposition that in (1.2.1) we have $Q = [b_1 \ \cdots \ b_n]$ where $(b_1, \ldots, b_n)$ is an orthonormal basis in X, each b_k is an eigenvector of A (corresponding to the eigenvalue λ_k) and Λ is the matrix of A in this basis.

An operator $P \in \mathcal{L}(X)$ is called *positive* if $\langle Px, x\rangle \geqslant 0$ holds for all $x \in X$. This property is written in the form $P \geqslant 0$. If $P \geqslant 0$, then $P = P^*$, so that the factorization (1.2.1) holds. Moreover, in this case $\lambda_k \geqslant 0$. We can define $P^{\frac{1}{2}}$ by the same formula (1.2.1) in which we replace each λ_k by $\lambda_k^{\frac{1}{2}}$. Then $P^{\frac{1}{2}} \geqslant 0$ and $P^{\frac{1}{2}}P^{\frac{1}{2}} = P$. If $A_0, A_1 \in \mathcal{L}(X)$ are self-adjoint, we write $A_0 \leqslant A_1$ (or $A_1 \geqslant A_0$) if $A_1 - A_0 \geqslant 0$. Note that for any $T \in \mathcal{L}(U, Y)$ we have $T^*T \geqslant 0$. Moreover, it follows from the previous section that Ran $T^*T =$ Ran T^*.

The square roots of the eigenvalues of T^*T are called the *singular values* of T. It follows from the factorization (1.2.1) applied to $A = T^*T$ that

$$\|T\|^2 = \sup_{\|q\|\leqslant 1} \langle \Lambda q, q\rangle,$$

which implies that $\|T\|$ is the largest singular value of T. In particular, if $T^* = T$, then its singular values are $|\lambda_k|$, where $\lambda_k \in \sigma(T)$.

Recall from the previous section that if V is a subspace of X, then every $x \in X$ has a unique decomposition $x = v + w$, where $v \in V$ and $w \in V^\perp$. Therefore, there exists an operator $P_V \in \mathcal{L}(X)$ such that $P_V x = v$. We have

$P_V^2 = P_V$, $P_V = P_V^*$ (these properties imply $P_V \geqslant 0$) and Ran $P_V = V$ (hence Ker $P_V = V^\perp$). This operator is called the *orthogonal projector* onto V.

An operator $P \in \mathcal{L}(X)$ is called *strictly positive* if there exists an $\varepsilon > 0$ such that $P \geqslant \varepsilon I$. This property is written in the form $P > 0$. We have $P > 0$ iff $\langle Px, x\rangle > 0$ holds for every non-zero $x \in X$. If $P = P^*$, then $P > 0$ iff all its eigenvalues are strictly positive. The number ε mentioned earlier can be taken to be the smallest eigenvalue of P. If $P > 0$, then P is invertible and $P^{-1} > 0$.

Suppose that $(b_1, \ldots, b_n)$ is an algebraic basis in X and $A \in \mathcal{L}(X)$. Denote $Q = [b_1 \cdots b_n]$, so that $Q \in \mathcal{L}(\mathbb{C}^n, X)$ is invertible. Then the *matrix of* A *in the basis* $(b_1, \ldots, b_n)$ is

$$\tilde{A} = Q^{-1} A Q .$$

In the following theorem we use the notation diag to construct a block diagonal matrix: if $J_1, J_2, \ldots, J_l$ are square matrices, then diag $(J_1, J_2, \ldots, J_l)$ is the square matrix which has the matrices J_k on its diagonal and zero everywhere else.

Theorem 1.2.2. *If $A \in \mathcal{L}(X)$ then there exists an algebraic basis $(b_1, \ldots, b_n)$ in X such that $\tilde{A}$, the matrix of A in this basis, is*

$$\tilde{A} = \operatorname{diag}\,(J_1, J_2, \ldots, J_l), \qquad J_k \in \mathbb{C}^{d_k \times d_k}, \quad J_k = \lambda_k I + N \tag{1.2.2}$$

(*where* $k = 1, 2, \ldots, l$). *Here N denotes a square matrix* (*of any dimension*) *with* 1 *directly under the diagonal and* 0 *everywhere else.*

Clearly we must have $d_1 + d_2 + \cdots + d_l = n$. It is easy to see that if $N \in \mathbb{C}^{d_k \times d_k}$, then $N^{d_k} = 0$. We have $\sigma(J_k) = \{\lambda_k\}$, whence $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ (there may be repetitions in the finite sequence (λ_k)). The matrices J_k are called *Jordan blocks.* Each Jordan block has only one independent eigenvector. There is an alternative dual statement of the last theorem, in which the matrix N is replaced by N^* (in N^*, the ones are above the diagonal).

Most matrices $A \in \mathbb{C}^{n\times n}$ have n independent eigenvectors (this is the case, for instance, if A has distinct eigenvalues). In this case, choosing the algebraic basis $(b_1, \ldots, b_n)$ to consist of eigenvectors of A, we obtain $l = n$ and $d_k = 1$ in (1.2.2). In this case, $N = 0$ and we obtain $\tilde{A} = \operatorname{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$. The factorization (1.2.1) shows that this is true, in particular, for self-adjoint A. In this very particular case we have the added advantage that the basis can be chosen orthonormal.

1.3 Matrix exponentials

In this section we recall the main facts about e^{tA}, where $t \in \mathbb{R}$, X is a finite-dimensional complex inner product space and $A \in \mathcal{L}(X)$. In this section, we prove our statements. Good references that present (also) matrix exponentials are, for example, Bellman [16], Hirsch and Smale [98], Horn and Johnson [105], Kwakernaak and Sivan [135], Lancaster and Tismenetsky [141] and Perko [183].

For $A \in \mathcal{L}(X)$ and $t \in \mathbb{R}$, the operator e^{tA} is defined by the Taylor series

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

which converges for every $t \in \mathbb{C}$, but we shall only consider real t. The absolute convergence of the above series follows from the fact that its kth term is dominated by the kth term of the scalar Taylor series for $e^{|t|\cdot\|A\|}$:

$$\left\| \frac{t^k}{k!} A^k \right\| \leqslant \frac{|t|^k}{k!} \|A\|^k .$$

This estimate also proves that

$$\|e^{tA}\| \leqslant e^{|t|\cdot\|A\|} \qquad \forall\, t \in \mathbb{R}. \tag{1.3.1}$$

From the definition it follows by a short argument that

$$e^{(t+\tau)A} = e^{tA} e^{\tau A}, \qquad e^{0A} = I,$$

for every $t, \tau \in \mathbb{R}$. Also from the definition of e^{tA} and using also the absolute convergence of the series, it is easy to derive that

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{tA} = A e^{tA} = e^{tA} A \qquad \forall\, t \in \mathbb{R}. \tag{1.3.2}$$

Example 1.3.1. Take $X = \mathbb{C}^2$ and let $A \in \mathcal{L}(X)$ be defined by its matrix

$$A = \begin{bmatrix} \alpha & -\omega \\ \omega & \alpha \end{bmatrix}, \qquad \text{where} \quad \alpha, \omega \in \mathbb{R}.$$

Then from the definition it is not difficult to see that

$$e^{tA} = e^{\alpha t} \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix}.$$

The following simple observation is often useful: if $x \in X$ is an eigenvector of A corresponding to the eigenvalue λ, then $e^{tA}x = e^{t\lambda}x$.

Recall from the previous section that if we represent A by its matrix $\tilde{A}$ in some algebraic basis $(b_1, \ldots, b_n)$, then, denoting $Q = [b_1 \ \cdots \ b_n]$, we have

$$Q \in \mathcal{L}(\mathbb{C}^n, X), \qquad A = Q\tilde{A}Q^{-1}.$$

In this case it follows from the definition of e^{tA} by a simple argument that

$$e^{tA} = Q e^{t\tilde{A}} Q^{-1} \qquad \forall\, t \in \mathbb{R}. \tag{1.3.3}$$

For example, if A is self-adjoint, so that the factorization (1.2.1) holds, then

$$e^{tA} = Q e^{t\Lambda} Q^*, \quad \text{where} \quad e^{t\Lambda} = \operatorname{diag}\left(e^{t\lambda_1}, e^{t\lambda_2}, \ldots, e^{t\lambda_n}\right).$$

According to Theorem 1.2.2 we can always choose the algebraic basis $(b_1, \dots, b_n)$ such that $\tilde{A}$ is as in (1.2.2) (block diagonal with Jordan blocks). Then it is easy to see that e^{tA} is represented (in the same basis) by the block diagonal matrix

$$e^{t\tilde{A}} = \mathrm{diag}\,\left(e^{tJ_1}, e^{tJ_2}, \dots, e^{tJ_l}\right). \tag{1.3.4}$$

From $J_k = \lambda_k I + N \in \mathbb{C}^{d_k \times d_k}$ (see the explanations after (1.2.2)) it follows that

$$e^{tJ_k} = e^{t\lambda_k} e^{tN}, \qquad e^{tN} = I + tN + \frac{t^2}{2!}N^2 + \cdots + \frac{t^m}{m!}N^m,$$

where $m = d_k - 1$. The series defining e^{tN} is finite because $N^{d_k} = 0$. It follows that

$$e^{tJ_k} = e^{t\lambda_k} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ t & 1 & 0 & 0 & \cdots \\ \frac{t^2}{2!} & t & 1 & 0 & \cdots \\ \frac{t^3}{3!} & \frac{t^2}{2!} & t & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{1.3.5}$$

Example 1.3.2. If in a suitable algebraic basis the matrix of A is

$$\tilde{A} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 1 & \mu \end{bmatrix}, \qquad \text{where} \quad \lambda, \mu \in \mathbb{C}, \tag{1.3.6}$$

then the matrix of e^{tA} in the same basis is

$$e^{t\tilde{A}} = \begin{bmatrix} e^{t\lambda} & 0 & 0 \\ 0 & e^{t\mu} & 0 \\ 0 & te^{t\mu} & e^{t\mu} \end{bmatrix}.$$

Proposition 1.3.3. *Denote* $s_0(A) = \sup\{\mathrm{Re}\,\lambda \mid \lambda \in \sigma(A)\}$. *Then for every* $\omega > s_0(A)$ *there exists* $M_\omega \geqslant 1$ *such that*

$$\|e^{tA}\| \leqslant M_\omega e^{\omega t} \qquad \forall\, t \in [0, \infty).$$

Proof. From (1.3.5) we see that there exists $m_k \geqslant 1$ such that

$$\left\|e^{tJ_k}\right\| \leqslant m_k(1 + |t|^m)e^{t\,\mathrm{Re}\,\lambda_k} \qquad \forall\, t \in \mathbb{R},$$

where $m = d_k - 1$. Going back to (1.3.4) we see that there exists $M \geqslant 1$ such that

$$\|e^{t\tilde{A}}\| \leqslant M(1 + |t|^{m_0})e^{ts_0(A)} \qquad \forall\, t \geqslant 0, \tag{1.3.7}$$

where $m_0 = \max\{d_1, d_2, \dots, d_l\} - 1$. Using (1.3.3) together with (1.3.7) implies (after some reasoning) the estimate in the proposition. $\square$

The number $s_0(A)$ introduced above is called the *spectral bound* of A. In the next proposition we compute the Laplace transform of e^{tA}, which is well defined in the right half-plane determined by $s_0(A)$.

Proposition 1.3.4. *For every $s \in \mathbb{C}$ with $\operatorname{Re} s > s_0(A)$ we have*

$$\int_0^\infty e^{-st} e^{tA} \, \mathrm{d}t = (sI - A)^{-1}.$$

Proof. Proposition 1.3.3 implies that the Laplace integral above converges. Let us denote by $R(s)$ the Laplace transform of e^{tA}. We know that $\frac{\mathrm{d}}{\mathrm{d}t} e^{tA} = A e^{tA}$. Applying the Laplace integral to both sides, we obtain that $sR(s) - I = AR(s)$. From here, $(sI - A)R(s) = I$ and the formula in the proposition follows. □

Remark 1.3.5. For any subspace $V \subset X$ the following properties are equivalent: (1) $AV \subset V$, (2) $A^* V^\perp \subset V^\perp$, (3) $e^{tA} V \subset V$ for all t in an interval. Such a subspace V is called *A-invariant.* If A_V denotes the restriction of A to V, then $\sigma(A_V) \subset \sigma(A)$. The proofs of these statements are easy and we omit them.

For any $A \in \mathcal{L}(X)$, we define its *real* and *imaginary parts* by

$$\operatorname{Re} A = \frac{1}{2}(A + A^*), \qquad \operatorname{Im} A = \frac{1}{2i}(A - A^*).$$

These are self-adjoint operators and $A = \operatorname{Re} A + i \operatorname{Im} A$, so that

$$\operatorname{Re} \langle Ax, x \rangle = \langle (\operatorname{Re} A)x, x \rangle.$$

The operator $A \in \mathcal{L}(X)$ is called *dissipative* if $\operatorname{Re} A \leqslant 0$.

Proposition 1.3.6. *If A is dissipative, then $\|e^{tA}\| \leqslant 1$ for all $t \geqslant 0$.*

Proof. For every $x \in X$ and $t \in \mathbb{R}$ we have, using (1.3.2),

$$\frac{\mathrm{d}}{\mathrm{d}t} \|e^{tA}x\|^2 = \langle Ae^{tA}x, e^{tA}x \rangle + \langle e^{tA}x, Ae^{tA}x \rangle = 2\langle (\operatorname{Re} A)e^{tA}x, e^{tA}x \rangle \leqslant 0,$$

so that $\|e^{tA}x\|^2$ is non-increasing. This implies that $\|e^{tA}x\| \leqslant \|x\|$ for all $t \geqslant 0$. □

The operator $A \in \mathcal{L}(X)$ is called *skew-adjoint* if $\operatorname{Re} A = 0$. Equivalently, iA is self-adjoint. For example, the matrix A in Example 1.3.1 is skew-adjoint if $\alpha = 0$.

Proposition 1.3.7. *If A is skew-adjoint, then e^{tA} is unitary for all $t \in \mathbb{R}$.*

Proof. Arguing as in the proof of the previous proposition, we obtain that for every $x \in X$, $\|e^{tA}x\|^2$ is constant (as a function of $t \in \mathbb{R}$). This implies that e^{tA} is isometric, and hence unitary, for all $t \in \mathbb{R}$. □

1.4 Observability and controllability for finite-dimensional linear systems

In the remaining part of this chapter we introduce basic concepts concerning linear time-invariant systems, with emphasis on controllability and observability. We work with systems that have finite-dimensional input, state and output spaces, but the style of our presentation is such as to suit generalizations to infinite-dimensional systems in the later chapters. For good introductory chapters on such systems we refer the reader to D'Azzo and Houpis [46], Friedland [68], Ionescu, Oară and Weiss [109], Kwakernaak and Sivan [135], Maciejowski [164], Rugh [196] and Wonham [237].

Let U, X and Y be finite-dimensional inner product spaces. We denote $n = \dim X$. A finite-dimensional linear time-invariant (LTI) system Σ with input space U, state space X and output space Y is described by the equations

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \\ y(t) = Cz(t) + Du(t), \end{cases} \tag{1.4.1}$$

where $u(t) \in U$, u is the *input function* (or input signal) of Σ, $z(t) \in X$ is its *state* at time t, $y(t) \in Y$ and y is the *output function* (or output signal) of Σ. Usually t is considered to be in the interval $[0, \infty)$ (but occasionally other intervals are considered). In the above equations, A, B, C, D are linear operators such that $A : X \to X$, $B : U \to X$, $C : X \to Y$ and $D : U \to Y$. The differential equation in (1.4.1) has, for any continuous u and any initial state $z(0)$, the unique solution

$$z(t) = e^{tA}z(0) + \int_0^t e^{(t-\sigma)A} Bu(\sigma)\,\mathrm{d}\sigma. \tag{1.4.2}$$

This formula defines the state trajectories $z(\cdot)$ also for input signals that are not continuous, for example for $u \in L^2([0, \infty); U)$. Even for such input functions, $z(t)$ is a continuous function of the time t. Notice that $z(t)$ does not depend on the values $u(\theta)$ for $\theta > t$, a property called *causality*.

Definition 1.4.1. The operator A (or the system Σ) is *stable* if $\lim_{t\to\infty} e^{tA} = 0$.

We see that A is stable iff $s_0(A) < 0$ (this follows from Proposition 1.3.3). Thus, A is stable iff all its eigenvalues are in the open left half-plane of $\mathbb{C}$.

For any $u \in L^2([0, \infty); U)$ and $\tau \geqslant 0$, we denote by $\mathbf{P}_\tau u$ the truncation of u to the interval $[0, \tau]$. For any linear system as above we introduce two families of operators, depending on $\tau \geqslant 0$, $\Phi_\tau \in \mathcal{L}(L^2([0, \infty); U), X)$ and $\Psi_\tau \in \mathcal{L}(X, L^2([0, \infty); Y))$, by

$$\Phi_\tau\, u = \int_0^\tau e^{(\tau-\sigma)A}\, B\, u(\sigma)\,\mathrm{d}\sigma, \quad (\Psi_\tau\, x)(t) = \begin{cases} C\, e^{At} x & \text{for} \ \ t \in [0, \tau], \\ 0 & \text{for} \ \ t > \tau. \end{cases}$$

Note that if in (1.4.1) we have $z(0) = 0$, then $z(\tau) = \Phi_\tau u$. If instead we have $u = 0$ and $z(0) = x$, then $\mathbf{P}_\tau y = \Psi_\tau x$. For this reason, the operators Φ_τ are called the *input maps* of Σ, while Ψ_τ are called the *output maps* of Σ.

We have $\Phi_\tau \mathbf{P}_\tau = \Phi_\tau$ (causality) and $\mathbf{P}_\tau \Psi_\tau = \Psi_\tau$.

For the system Σ described by (1.4.1), the *dual system* Σ^d is described by

$$\begin{cases} \dot{z}^d(t) = A^* z^d(t) + C^* y^d(t), \\ u^d(t) = B^* z^d(t) + D^* y^d(t), \end{cases} \tag{1.4.3}$$

where $y^d(t) \in Y$ is the input function of Σ^d at time t, $z^d(t) \in X$ is its state at time t and $u^d(t) \in U$ is its output function at time t. We denote by Φ_τ^d and Ψ_τ^d the input and the output maps of Σ^d.

In order to express the adjoints of the operators Φ_τ and Ψ_τ, we need the time-reflection operators $\reflectbox{$\mathbf{R}$}_\tau \in \mathcal{L}(L^2([0,\infty); U))$ defined for all $\tau \geqslant 0$ as follows:

$$(\reflectbox{$\mathbf{R}$}_\tau u)(t) = \begin{cases} u(\tau - t) & \text{for } t \in [0, \tau], \\ 0 & \text{for } t > \tau. \end{cases}$$

It will be useful to note that

$$\reflectbox{$\mathbf{R}$}_\tau^* = \reflectbox{$\mathbf{R}$}_\tau \quad \text{and} \quad \reflectbox{$\mathbf{R}$}_\tau^2 = \mathbf{P}_\tau . \tag{1.4.4}$$

The notation introduced so far in this section will be used throughout the section.

Definition 1.4.2. The system Σ (or the pair (A, C)) is *observable* if for some $\tau > 0$ we have $\operatorname{Ker} \Psi_\tau = \{0\}$. The system Σ (or the pair (A, B)) is *controllable* if for some $\tau > 0$ we have $\operatorname{Ran} \Phi_\tau = X$.

Observability and controllability are dual properties, as the following proposition and its corollaries show.

Proposition 1.4.3. *For all $\tau \geqslant 0$ we have $\Phi_\tau^* = \reflectbox{$\mathbf{R}$}_\tau \Psi_\tau^d$.*

Proof. For every $z_0 \in X$ and $u \in L^2([0,\infty); U)$ we have

$$\begin{aligned} \langle \Phi_\tau u, z_0 \rangle &= \int_0^\tau \left\langle e^{(\tau-\sigma)A} B u(\sigma), z_0 \right\rangle \mathrm{d}\sigma \\ &= \int_0^\tau \left\langle u(\sigma), B^* e^{(\tau-\sigma)A^*} z_0 \right\rangle \mathrm{d}\sigma = \langle u, \reflectbox{$\mathbf{R}$}_\tau \Psi_\tau^d z_0 \rangle . \end{aligned}$$

This implies the stated equality. □

We can express Φ_τ^* in terms of A and B as follows:

$$(\Phi_\tau^* x)(t) = B^* e^{(\tau-t)A^*} x \qquad \forall\, t \in [0, \tau].$$

Corollary 1.4.4. *For all $\tau \geqslant 0$ we have* $\operatorname{Ran} \Phi_\tau = \left(\operatorname{Ker} \Psi_\tau^d\right)^\perp$.

Proof. According to (1.1.7) and using the previous proposition, we have

$$(\text{Ran } \Phi_\tau)^\perp = \text{Ker } \Phi_\tau^* = \text{Ker } \boldsymbol{\text{Я}}_\tau \Psi_\tau^d .$$

Since Ker $\boldsymbol{\text{Я}}_\tau \Psi_\tau^d =$ Ker Ψ_τ^d, we obtain that $(\text{Ran } \Phi_\tau)^\perp =$ Ker Ψ_τ^d. Taking orthogonal complements and using (1.1.3), we obtain the desired equality. □

Corollary 1.4.5. *We have* Ran $\Phi_\tau = X$ *if and only if* Ker $\Psi_\tau^d = \{0\}$. *Thus,* (A, B) *is controllable if and only if* (A^*, B^*) *is observable.*

This is an obvious consequence of the previous corollary.

Corollary 1.4.6. *We have* $\Psi_\tau^* = \Phi_\tau^d \boldsymbol{\text{Я}}_\tau$ *and* Ker $\Psi_\tau = \left(\text{Ran } \Phi_\tau^d\right)^\perp$.

Proof. To prove the first statement, we use Proposition 1.4.3 in which we replace Σ by Σ^d; i.e., we replace A by A^*, B by C^* and U by Y. This yields $(\Phi_\tau^d)^* = \boldsymbol{\text{Я}}_\tau \Psi_\tau$. We apply $\boldsymbol{\text{Я}}_\tau$ to both sides and obtain (using (1.4.4) and $\mathbf{P}_\tau \Psi_\tau = \Psi_\tau$) that $\boldsymbol{\text{Я}}_\tau (\Phi_\tau^d)^* = \Psi_\tau$. By taking adjoints (and using again (1.4.4)), we obtain the first statement of the proposition. The second statement follows from the first by using (1.1.7) and the fact that Ran $\Phi_\tau^d \boldsymbol{\text{Я}}_\tau =$ Ran Φ_τ^d. □

Proposition 1.4.7. *We have, for every* $\tau > 0$,

$$\text{Ker } \Psi_\tau = \text{Ker } \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} . \tag{1.4.5}$$

Proof. Let $z_0 \in$ Ker Ψ_τ. Then the analytic function $y(t) = Ce^{tA} z_0$ is zero on the interval $[0, \tau]$, so that its derivatives of any order at $t = 0$ are all zero, so that $CA^k z_0 = 0$ for all integers $k \geqslant 0$. This implies that z_0 is in the null-space of the big matrix appearing in (1.4.5).

Conversely, suppose that $z_0 \in X$ is in the null-space of the big matrix in (1.4.5). This means that $CA^k z_0 = 0$ for $0 \leqslant k \leqslant n - 1$. Since the powers A^k for $k \geqslant n$ are linear combinations of the lower powers of A (this is a consequence of the Cayley–Hamilton theorem mentioned in Section 1.2), it follows that $CA^k z_0 = 0$ for all integers $k \geqslant 0$. Looking at the Taylor series of $y(t) = Ce^{tA} z_0$, it follows that $y(t) = 0$ for all t. Hence, $z_0 \in$ Ker Ψ_τ holds for every $\tau > 0$. □

Note that (1.4.5) implies that Ker Ψ_τ is independent of τ. This space is called the *unobservable space* of the system Σ (or of the pair (A, C)). It can be derived from (1.4.5) that Ker Ψ_τ is the largest subspace of X that is invariant under A and contained in Ker C.

The following corollary is the *Kalman rank condition* for observability.

Corollary 1.4.8. *The pair (A, C) is observable if and only if*

$$\operatorname{rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n. \tag{1.4.6}$$

Indeed, since the big matrix above has n columns, the condition that its null-space is $\{0\}$ is equivalent to its rank being n.

Corollary 1.4.9. *We have, for every $\tau > 0$,*

$$\operatorname{Ran} \Phi_\tau = \operatorname{Ran} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}. \tag{1.4.7}$$

Proof. Combining Proposition 1.4.4 with Proposition 1.4.7, we obtain

$$\operatorname{Ran} \Phi_\tau = \left(\operatorname{Ker} \begin{bmatrix} B^* \\ B^*A^* \\ B^*(A^*)^2 \\ \vdots \\ B^*(A^*)^{n-1} \end{bmatrix} \right)^\perp .$$

Finally, we compute the above orthogonal complement using (1.1.3) and (1.1.7). □

Note that (1.4.7) implies that Ran Φ_τ is independent of τ. This space is called the *controllable space* of the system Σ (or of the pair (A, B)). It can be derived from (1.4.7) that Ran Φ_τ is the smallest subspace of X that is invariant under A and contains Ran B.

The following corollary is the *Kalman rank condition* for controllability.

Corollary 1.4.10. *The pair (A, B) is controllable if and only if*

$$\operatorname{rank} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n. \tag{1.4.8}$$

Indeed, since the big matrix above has n rows, the condition that its range is X is equivalent to its rank being n.

1.5 The Hautus test and Gramians

In this section we present the Hautus test for controllability or observability and we introduce controllability and observability Gramians, both in finite time and on an infinite time interval. While some of the results in the previous section cannot be extended to infinite-dimensional systems, those in this section all can, and this will be a main theme of the later chapters.

We use the same notation as in the previous section: U, X and Y are finite dimensional inner product spaces, $n = \dim X$, and Σ is an LTI system with input space U, state space X and output space Y described by (1.4.1).

The following proposition is known as the *Hautus test* for observability.

Proposition 1.5.1. *The pair (A, C) is observable if and only if*

$$\operatorname{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \qquad \forall\, \lambda \in \sigma(A).$$

Proof. Denote $\mathcal{N} = \operatorname{Ker} \Psi_\tau$ for some $\tau > 0$ (we know from Proposition 1.4.7 that $\mathcal{N}$ is independent of τ). Assume that (A, C) is not observable, so that $\mathcal{N} \neq \{0\}$. It is easy to see that $e^{tA}\mathcal{N} \subset \mathcal{N}$ for all $t \geqslant 0$. According to Remark 1.3.5, this implies $A\mathcal{N} \subset \mathcal{N}$. Let $A_\mathcal{N}$ be the restriction of A to $\mathcal{N}$, so that $A_\mathcal{N} \in \mathcal{L}(\mathcal{N})$. Clearly, $\sigma(A_\mathcal{N}) \subset \sigma(A)$. Since $\mathcal{N} \neq \{0\}$, $\sigma(A_\mathcal{N})$ is not empty. Take $\lambda \in \sigma(A_\mathcal{N})$ and let $x_\lambda \in \mathcal{N}$ be a corresponding eigenvector. Then $Cx_\lambda = (\Psi_\tau x_\lambda)(0) = 0$, so that

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} x_\lambda = 0 \quad \Rightarrow \quad \operatorname{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} < n.$$

Conversely, if $\operatorname{rank} \left[\begin{smallmatrix} A - \lambda I \\ C \end{smallmatrix}\right] < n$ for some $\lambda \in \sigma(A)$, then for some vector $x_\lambda \in X$, $x_\lambda \neq 0$, we have $(A - \lambda I)x_\lambda = 0$ (i.e., x_λ is an eigenvector of A) and $Cx_\lambda = 0$. Then $e^{tA}x_\lambda = e^{\lambda t}x_\lambda$ for all $t \in \mathbb{R}$ and hence $\Psi_\tau x_\lambda = 0$ for all $\tau > 0$. $\square$

Remark 1.5.2. It follows from the last proposition that (A, C) is observable iff $Cz \neq 0$ for every eigenvector z of A.

Remark 1.5.3. We can rewrite the last proposition as follows: (A, C) is observable iff there exists $k > 0$ such that for every $s \in \mathbb{C}$,

$$\|(sI - A)z\|^2 + \|Cz\|^2 \geqslant k^2\|z\|^2 \qquad \forall\, z \in X. \tag{1.5.1}$$

Indeed, it is clear that (1.5.1) implies the property displayed in the proposition. Conversely, if the property in the proposition holds, then clearly

$$\begin{bmatrix} A - sI \\ C \end{bmatrix}^* \begin{bmatrix} A - sI \\ C \end{bmatrix} > 0 \qquad \forall\, s \in \mathbb{C}.$$

The smallest eigenvalue of the above positive matrix, denoted $\lambda(s)$, is a continuous function of s and $\lim_{s\to\infty}\lambda(s) = \infty$. Therefore, there exists $k > 0$ such that $\lambda(s) \geqslant k^2$ for all $s \in \mathbb{C}$. Now it follows that

$$(sI - A)^*(sI - A) + C^*C \geqslant k^2 I \qquad \forall\, s \in \mathbb{C},$$

and from here it is very easy to obtain (1.5.1). We are interested in the formulation (1.5.1) of the Hautus test because it resembles the infinite-dimensional versions of this test, which will be discussed in Sections 6.5 and 6.6.

Remark 1.5.4. We mention that with the same techniques that we used in the proof of the last proposition, with a little extra effort we could have shown that, in fact, for every $A \in \mathcal{L}(X)$, $C \in \mathcal{L}(X, Y)$ and $\tau > 0$,

$$\operatorname{Ker}\,\Psi_\tau = \operatorname{span} \bigcup_{\lambda\in\sigma(A)} \operatorname{Ker} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}.$$

Proposition 1.5.5. *The pair (A, B) is controllable if and only if*

$$\operatorname{rank} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \qquad \forall\, \lambda \in \sigma(A).$$

Proof. According to Corollary 1.4.5, (A, B) is controllable iff (A^*, B^*) is observable. According to Proposition 1.5.1, the latter condition is equivalent to

$$\operatorname{rank} \begin{bmatrix} A^* - \mu I \\ B^* \end{bmatrix} = n \qquad \forall\, \mu \in \sigma(A^*).$$

Since, for every matrix T we have $\operatorname{rank} T = \operatorname{rank} T^*$, and since $\mu \in \sigma(A^*)$ iff $\overline{\mu} \in \sigma(A)$, we obtain the condition stated in the proposition. □

Remark 1.5.6. The dual version of Remark 1.5.4 states that for every $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(U, X)$ and $\tau > 0$,

$$\operatorname{Ran}\,\Phi_\tau = \bigcap_{\lambda\in\sigma(A)} \operatorname{Ran} \begin{bmatrix} A - \lambda I & B \end{bmatrix}.$$

For every $\tau > 0$, we define the *controllability Gramian* R_τ and the *observability Gramian* Q_τ by

$$R_\tau = \Phi_\tau \Phi_\tau^*, \qquad Q_\tau = \Psi_\tau^* \Psi_\tau.$$

Notice that $R_\tau, Q_\tau \in \mathcal{L}(X)$, $R_\tau \geqslant 0$ and $Q_\tau \geqslant 0$. It follows from (1.1.8) that

$$\operatorname{Ran} R_\tau = \operatorname{Ran} \Phi_\tau, \qquad \operatorname{Ker} Q_\tau = \operatorname{Ker} \Psi_\tau.$$

Hence, R_τ is invertible iff (A, B) is controllable and Q_τ is invertible iff (A, C) is observable. Using the definitions of Φ_τ, Ψ_τ, Proposition 1.4.3 and Corollary 1.4.6, we obtain

$$R_\tau = \int_0^\tau e^{tA} B B^* e^{tA^*}\,\mathrm{d}t, \qquad Q_\tau = \int_0^\tau e^{tA^*} C^* C e^{tA}\,\mathrm{d}t. \tag{1.5.2}$$

Proposition 1.5.7. *Suppose that (A, B) is controllable and let $x \in X$, $\tau > 0$. If*

$$u = \Phi_\tau^* R_\tau^{-1} x,$$

then $\Phi_\tau u = x$. Moreover, among all the inputs $v \in L^2([0,\infty); U)$ for which $\Phi_\tau v = x$, u is the unique one that has minimal norm.

Proof. Clearly we have $\Phi_\tau u = \Phi_\tau \Phi_\tau^* R_\tau^{-1} x = x$. If $v \in L^2([0,\infty); U)$ is such that $\Phi_\tau v = x$, then it is clear that $v = u + \varphi$, where $\varphi \in \operatorname{Ker} \Phi_\tau = (\operatorname{Ran} \Phi_\tau^*)^\perp$. Since u and φ are orthogonal to each other, $\|v\|^2 = \|u\|^2 + \|\varphi\|^2$. The minimum of $\|v\|$ is achieved only for $\varphi = 0$, i.e., for $v = u$. $\square$

Remark 1.5.8. The last proposition shows that for a controllable system, the state trajectory can be driven from any initial state to any final state in any positive time. Moreover, the proposition gives a simple (analytic) input function that is needed to achieve this, and which is of minimal norm. Indeed, to drive the system Σ from the initial state $z(0)$ to the final state $z(\tau)$, according to (1.4.2) we must solve $\Phi_\tau u = z(\tau) - e^{\tau A} z(0)$, and this can be solved using the last proposition.

Corollary 1.5.9. *Suppose that (A, B) is controllable. Let $\mathcal{F}$ be the set of all the operators $F \in \mathcal{L}(X, L^2([0,\infty); U))$ for which $\Phi_\tau F = I$. One such operator is $F_0 = \Phi_\tau^* R_\tau^{-1}$. Moreover, F_0 is minimal in the sense that*

$$F_0^* F_0 \leqslant F^* F \qquad \forall\, F \in \mathcal{F}.$$

Indeed, this is an easy consequence of Proposition 1.5.7. Note that $F_0^* F_0 = R_\tau^{-1}$.

The last corollary can be restated in a dual form.

Corollary 1.5.10. *Suppose that (A, C) is observable. Let $\mathcal{H}$ be the set of all the operators $H \in \mathcal{L}(L^2([0,\infty); Y), X)$ for which $H \Psi_\tau = I$. One such operator is $H_0 = Q_\tau^{-1} \Psi_\tau^*$. Moreover, H_0 is minimal in the sense that*

$$H_0 H_0^* \leqslant H H^* \qquad \forall\, H \in \mathcal{H}.$$

Definition 1.5.11. If A is stable, we define the *infinite-time controllability Gramian* $R \in \mathcal{L}(X)$ and the *infinite-time observability Gramian* $Q \in \mathcal{L}(X)$ by

$$R = \lim_{\tau \to \infty} R_\tau, \qquad Q = \lim_{\tau \to \infty} Q_\tau.$$

This definition makes sense, since we can see from (1.5.2) that the above limits exist and

$$R = \int_0^\infty e^{tA} B B^* e^{tA^*} \, \mathrm{d}t, \qquad Q = \int_0^\infty e^{tA^*} C^* C e^{tA} \, \mathrm{d}t.$$

It is clear that $R \geqslant R_\tau \geqslant 0$ and $Q \geqslant Q_\tau \geqslant 0$ (for all $\tau > 0$).

Remark 1.5.12. We shall need the following simple fact: If A is stable and $z \neq 0$, then $\lim_{t \to -\infty} \|e^{tA}z\| = \infty$. Indeed, this follows from

$$\|z\| = \|e^{-tA}e^{tA}z\| \leqslant \|e^{-tA}\| \cdot \|e^{tA}z\|.$$

Proposition 1.5.13. *If A is stable, then the infinite-time Gramians R and Q are the unique solutions in $\mathcal{L}(X)$ of the equations*

$$AR + RA^* = -BB^*, \qquad QA + A^*Q = -C^*C.$$

The equations appearing above are called *Lyapunov equations.* Thanks to these, R and Q are easy to compute numerically (as opposed to R_τ and Q_τ).

Proof. Denote $\Pi(t) = e^{tA}BB^*e^{tA^*}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi(t) = A\Pi(t) + \Pi(t)A^*.$$

Integrating this from 0 to ∞, and taking into account that $\Pi(t) \to 0$, we obtain that $AR + RA^* = -BB^*$. The proof of the formula $QA + A^*Q = -C^*C$ is similar.

To prove the uniqueness of the solution R, suppose that there is another operator $R' \in \mathcal{L}(X)$ satisfying $AR' + R'A^* = -BB^*$. Introducing $\Delta = R - R'$, we obtain $A\Delta + \Delta A^* = 0$, hence by induction $A^n\Delta = \Delta(-A^*)^n$ for all $n \in \mathbb{N}$, whence

$$e^{tA}\Delta = \Delta e^{-tA^*} \qquad \forall\, t \in \mathbb{R}. \tag{1.5.3}$$

If $x \in X$ is such that $\Delta x \neq 0$, then $\lim_{t \to -\infty} \|e^{tA}\Delta x\| = \infty$, according to Remark 1.5.12. This contradicts the fact that the right-hand side of (1.5.3) tends to zero as $t \to -\infty$. Thus, we must have $\Delta x = 0$ for all $x \in X$, i.e., $\Delta = 0$.

The uniqueness of Q is proved similarly. □

Proposition 1.5.14. *With the notation of the last proposition, (A, B) is controllable if and only if $R > 0$. (A, C) is observable if and only if $Q > 0$.*

Proof. If (A, C) is observable, then (as already mentioned) $Q_\tau > 0$ (for every $\tau > 0$). Since $Q \geqslant Q_\tau$, it follows that $Q > 0$. To prove the converse statement, suppose that (A, C) is not observable and take $x \in \operatorname{Ker} \Psi_\tau$, $x \neq 0$. Then $Ce^{tA}x = 0$ for all $t \geqslant 0$, hence $Qx = 0$, which contradicts $Q > 0$.

The proof for $R > 0$ is similar, using the dual system. □

Chapter 2

Operator Semigroups

In this chapter and the following one, we introduce the basics about strongly continuous semigroups of operators on Hilbert spaces, which are also called operator semigroups for short. We concentrate on those aspects which are useful for the later chapters. As a result, there will be many glaring omissions of subjects normally found in the literature about semigroups. For example, we shall ignore analytic semigroups, compact semigroups, spectral mapping theorems and stability theory.

Bibliographic notes. Of the many good books on operator semigroups we mention Butzer and Berens [28], Davies [44], Engel and Nagel [57], Goldstein [71], Hille and Phillips [97] (who started it all), Pazy [182], Tanabe [213]. The books Arendt et al. [8], Bensoussan et al. [17], Curtain and Zwart [39], Ito and Kappel [110], Luo, Guo and Morgul [163], Staffans [209] and Yosida [239] have substantial chapters devoted to this topic.

Prerequisites. In the remainder of this book, we assume that the standard concepts and results of functional analysis are known to the reader. These include the closed-graph theorem, the uniform boundedness theorem, some properties of Hilbert space-valued L^2 functions, Fourier and Laplace transforms. This material can be found in many books, of which we mention Akhiezer and Glazman [2], Bochner and Chandrasekharan [20], Brown and Pearcy [23], Dautray and Lions [42, 43], Dowson [52], Dunford and Schwartz [53], Rudin [194, 195], Yosida [239]. Nevertheless, some sections in our two chapters on operator semigroups are devoted to aspects of functional analysis that are not part of semigroup theory. In particular, we are careful to introduce all the necessary background about unbounded operators. Some results concerning bounded operators on Hilbert spaces are given in Appendix I (Chapter 12). The background on Sobolev spaces is recalled in Appendix II (Chapter 13).

Notation. Throughout this chapter, X is a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. If X and Z are Hilbert spaces, then $\mathcal{L}(X, Z)$ denotes the space of bounded linear operators from X to Z, with the usual (induced) norm. We write $\mathcal{L}(X) = \mathcal{L}(X, X)$. We use an arrow, as in $x_n \to x$, to indicate convergence in norm. Sometimes we put a subscript near a norm or an inner product, such as in $\|z\|_X$, to indicate which norm or inner product we are using. If X and Z are Hilbert spaces, we write the elements of $X \times Z$ either in the form (x, z) (with $x \in X$ and $z \in Z$) or $\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]$. On $X \times Z$ we consider the natural inner product $\langle (x_1, z_1), (x_2, z_2) \rangle = \langle x_1, x_2 \rangle + \langle z_1, z_2 \rangle$. The domain, range and kernel of an operator T will be denoted by $\mathcal{D}(T)$, $\operatorname{Ran} T$ and $\operatorname{Ker} T$, respectively. For any $\alpha \in \mathbb{R}$, we denote $\mathbb{C}_\alpha = \{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha\}$. In particular, the right half-plane $\mathbb{C}_0$ will appear often in our considerations.

For any open interval J and any Hilbert space U, the Sobolev space $\mathcal{H}^1(J; U)$ consists of those locally absolutely continuous functions $z : J \to U$ for which $\frac{\mathrm{d}z}{\mathrm{d}t} \in L^2(J; U)$. The space $\mathcal{H}^2(J; U)$ is defined similarly, but now we require $\frac{\mathrm{d}z}{\mathrm{d}t} \in \mathcal{H}^1(J; U)$. The space $\mathcal{H}^1_0(J; U)$ consists of those functions in $\mathcal{H}^1(J; U)$ which vanish at the endpoints of J (i.e., they have limits equal to zero there). (If J is infinite, then at the infinite endpoints of J, the limit is zero anyway, for any function in $\mathcal{H}^1(J; U)$.) Occasionally we need also the space

$$\mathcal{H}^2_0(J; U) = \left\{ h \in \mathcal{H}^2(J; U) \cap \mathcal{H}^1_0(J; U) \;\middle|\; \frac{\mathrm{d}h}{\mathrm{d}x} \in \mathcal{H}^1_0(J; U) \right\}.$$

For any interval J (not necessarily open), $C(J; X) = C^0(J; X)$ consists of all the continuous functions from J to X, while $C^m(J; X)$ (for $m \in \mathbb{N}$) consists of all the m times differentiable functions from J to X whose derivatives of order $\leqslant m$ are in $C(J; X)$. Functions in $C^m(J; X)$ are also called *functions of class C^m*.

2.1 Strongly continuous semigroups and their generators

We have seen in the previous chapter that the family of operators $(e^{tA})_{t \geqslant 0}$ (where A is a linear operator on a finite-dimensional vector space) is important, as it describes the evolution of the state of a linear system in the absence of an input. If we want to study systems whose state space is a Hilbert space, then we need the natural generalization of such a family to a family of operators acting on a Hilbert space. Different generalizations are possible, but it seems that the right concept is that of a strongly continuous semigroup of operators. The theory of such semigroups is now a standard part of functional analysis, but due to its special importance for us, we devote a chapter to introducing this material from scratch.

Definition 2.1.1. A family $\mathbb{T} = (\mathbb{T}_t)_{t \geqslant 0}$ of operators in $\mathcal{L}(X)$ is a *strongly continuous semigroup* on X if

(1) $\mathbb{T}_0 = I$,
(2) $\mathbb{T}_{t+\tau} = \mathbb{T}_t \mathbb{T}_\tau$ for every $t, \tau \geqslant 0$ (the semigroup property),
(3) $\lim_{t \to 0,\, t > 0} \mathbb{T}_t z = z$ for all $z \in X$ (strong continuity).

The intuitive meaning of such a family of operators is that it describes the evolution of the state of a process, in the following way: If $z_0 \in X$ is the initial state of the process at time $t = 0$, then its state at time $t \geqslant 0$ is $z(t) = \mathbb{T}_t z_0$. Note that $z(t+\tau) = \mathbb{T}_t z(\tau)$, so that the process does not change its nature in time.

A simple but very limited class of strongly continuous semigroups is obtained as follows: Let $A \in \mathcal{L}(X)$ and put (as in Chapter 1)

$$\mathbb{T}_t = e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}. \tag{2.1.1}$$

It is easy to see that this series converges in $\mathcal{L}(X)$ for every $t \geqslant 0$ (in fact, for every $t \in \mathbb{C}$), and the function $\mathbb{T}_t$ is *uniformly continuous*; i.e., we have $\lim_{t \to 0} \|\mathbb{T}_t - I\| = 0$. It is not difficult to prove that the only uniformly continuous semigroups are the ones defined as in (2.1.1), with $A \in \mathcal{L}(X)$ (see, for instance, Pazy [182, p. 2] or Rudin [195, p. 359]). It follows easily from (2.1.1) that

$$\left\|e^{tA}\right\| \leqslant e^{t\|A\|} \qquad \forall\, t \geqslant 0. \tag{2.1.2}$$

The *growth bound* of the strongly continuous semigroup $\mathbb{T}$ is the number $\omega_0(\mathbb{T})$ defined by

$$\omega_0(\mathbb{T}) = \inf_{t \in (0,\infty)} \frac{1}{t} \log \|\mathbb{T}_t\|. \tag{2.1.3}$$

Clearly $\omega_0(\mathbb{T}) \in [-\infty, \infty)$. The name "growth bound" is justified by the following Proposition.

Proposition 2.1.2. *Let $\mathbb{T}$ be a strongly continuous semigroup on X, with growth bound $\omega_0(\mathbb{T})$. Then*

(1) $\omega_0(\mathbb{T}) = \lim_{t \to \infty} \frac{1}{t} \log \|\mathbb{T}_t\|$,
(2) *for any $\omega > \omega_0(\mathbb{T})$ there exists an $M_\omega \in [1, \infty)$ such that*

$$\|\mathbb{T}_t\| \leqslant M_\omega e^{\omega t} \qquad \forall\, t \in [0, \infty), \tag{2.1.4}$$

(3) *the function $\varphi : [0,\infty) \times X \to X$ defined by $\varphi(t,z) = \mathbb{T}_t z$ is continuous (with respect to the product topology).*

Proof. Let $z \in X$. From the right continuity of the function $t \mapsto \mathbb{T}_t z$ at $t = 0$ it follows that there exists a $\tau > 0$ such that this function is bounded on $[0, \tau]$. Because of the semigroup property, the same function is bounded on $[0, T]$, for any $T > 0$. By applying the uniform boundedness theorem, it follows that the function $t \mapsto \|\mathbb{T}_t\|$ is bounded for $t \in [0, T]$, for any $T > 0$.

Now we prove point (1) of the proposition. Let us denote $p(t) = \log \|\mathbb{T}_t\|$. From the semigroup property we have $p(t+\tau) \leqslant p(t) + p(\tau)$. Let us denote by $[t]$ and by $\{t\}$ the integer and the fractionary part of $t \in [0, \infty)$. We have

$$p(t) = p\left([t] + \{t\}\right) \leqslant [t]p(1) + p\left(\{t\}\right).$$

From the first part of this proof we know that $p(\{t\})$ is bounded from above. Dividing by t and taking lim sup, we get

$$\limsup_{t\to\infty} \frac{p(t)}{t} \leqslant p(1).$$

The same formula (with the same proof) holds if we replace p with p_α, where $p_\alpha(t) = p(\alpha t)$, $\alpha \in (0,\infty)$. From this we get

$$\limsup_{t\to\infty} \frac{p(t)}{t} \leqslant \frac{p(\alpha)}{\alpha} \qquad \forall\, \alpha > 0,$$

hence $\limsup_{t\to\infty} \frac{p(t)}{t} \leqslant \inf_{t\in(0,\infty)} \frac{p(t)}{t}$. The opposite inequality obviously holds, so that we get

$$\lim_{t\to\infty} \frac{p(t)}{t} = \inf_{t\in(0,\infty)} \frac{p(t)}{t} = \omega_0(\mathbb{T}).$$

Point (2) follows easily from point (1). Indeed, if $\omega > \omega_0(\mathbb{T})$, then

$$\|\mathbb{T}_t\| \leqslant e^{\omega t} \quad \forall\, t \geqslant t_\omega$$

holds for some $t_\omega \geqslant 0$. Hence, we may put $M_\omega = \sup_{t\in[0,t_\omega]} \|\mathbb{T}_t\|\, e^{-\omega t}$.

We turn to point (3). First we prove that for every fixed $z_0 \in X$, the function $t \to \varphi(t, z_0)$ is continuous. The continuity from the right is clear. To show the continuity from the left, let $t_n \to t_0 > 0$ with $t_n < t_0$. Then $\|\varphi(t_n, z) - \varphi(t_0, z)\| = \|\mathbb{T}_{t_n}(I - \mathbb{T}_{t_0-t_n})z\| \leqslant K\|(I - \mathbb{T}_{t_0-t_n})z\|$, where K is some upper bound for $\|\mathbb{T}_{t_n}\|$. Finally, we prove the continuity of φ. Let $(t_n, z_n) \to (t_0, z_0) \in [0,\infty) \times X$. Then

$$\mathbb{T}_{t_n} z_n - \mathbb{T}_{t_0} z_0 = \mathbb{T}_{t_n}(z_n - z_0) + \mathbb{T}_{t_n} z_0 - \mathbb{T}_{t_0} z_0,$$

which implies that

$$\|\varphi(t_n, z_n) - \varphi(t_0, z_0)\| \leqslant K\|z_n - z_0\| + \|\varphi(t_n, z_0) - \varphi(t_0, z_0)\|,$$

where K is again some upper bound for $\|\mathbb{T}_{t_n}\|$. □

Definition 2.1.3. Let $\mathbb{T}$ be a strongly continuous semigroup on X, with growth bound $\omega_0(\mathbb{T})$. This semigroup is called *exponentially stable* if $\omega_0(\mathbb{T}) < 0$.

Definition 2.1.4. The linear operator $A : \mathcal{D}(A) \to X$, defined by

$$\mathcal{D}(A) = \left\{ z \in X \;\middle|\; \lim_{t\to 0,\ t>0} \frac{\mathbb{T}_t z - z}{t} \text{ exists} \right\},$$

$$Az = \lim_{t\to 0,\ t>0} \frac{\mathbb{T}_t z - z}{t} \qquad \forall\, z \in \mathcal{D}(A),$$

is called the *infinitesimal generator* (or just the *generator*) of the semigroup $\mathbb{T}$.

For example, if $\mathbb{T}_t = e^{tA}$, as discussed around (2.1.1), then its generator is A.

Proposition 2.1.5. *Let $\mathbb{T}$ be a strongly continuous semigroup on X, with generator A. Then for every $z \in \mathcal{D}(A)$ and $t \geqslant 0$ we have that $\mathbb{T}_t z \in \mathcal{D}(A)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{T}_t z = A\mathbb{T}_t z = \mathbb{T}_t Az. \tag{2.1.5}$$

Proof. If $z \in \mathcal{D}(A)$, $t \geqslant 0$ and $\tau > 0$, then

$$\frac{\mathbb{T}_\tau - I}{\tau}\mathbb{T}_t z = \mathbb{T}_t \frac{\mathbb{T}_\tau - I}{\tau} z \to \mathbb{T}_t Az, \quad \text{as} \quad \tau \to 0. \tag{2.1.6}$$

Thus, $\mathbb{T}_t z \in \mathcal{D}(A)$ and $A\mathbb{T}_t z = \mathbb{T}_t Az$. Moreover, (2.1.6) implies that the derivative from the right of $\mathbb{T}_t z$ exists and is equal to $A\mathbb{T}_t z$. We have to show that for $t > 0$, the left derivative of $\mathbb{T}_t z$ also exists and is equal to $\mathbb{T}_t Az$. This will follow from

$$\frac{\mathbb{T}_t z - \mathbb{T}_{t-\tau} z}{\tau} - \mathbb{T}_t Az = \mathbb{T}_{t-\tau}\left[\frac{\mathbb{T}_\tau z - z}{\tau} - Az\right] + (\mathbb{T}_{t-\tau}Az - \mathbb{T}_t Az).$$

Indeed, using Proposition 2.1.2 and the first part of this proof, we see that both terms on the right-hand side above converge to zero as $\tau \to 0$. □

Proposition 2.1.6. *Let $\mathbb{T}$ be a strongly continuous semigroup on X, with generator A. Let $z_0 \in X$ and for every $\tau > 0$ put*

$$z_\tau = \frac{1}{\tau}\int_0^\tau \mathbb{T}_t z_0 \,\mathrm{d}t.$$

Then $z_\tau \in \mathcal{D}(A)$ and $\lim_{\tau \to 0} z_\tau = z_0$.

Proof. The fact that $z_\tau \to z_0$ (as $\tau \to 0$) follows from the continuity of the function $t \mapsto \mathbb{T}_t z_0$ (since z_τ is the average of this function over $[0, \tau]$). For every $\tau, h > 0$,

$$\frac{\mathbb{T}_h - I}{h} z_\tau = \frac{1}{h\tau}\int_\tau^{\tau+h} \mathbb{T}_t z_0 \,\mathrm{d}t - \frac{1}{h\tau}\int_0^h \mathbb{T}_t z_0 \,\mathrm{d}t.$$

Taking limits as $h \to 0$, we get that $z_\tau \in \mathcal{D}(A)$ and $Az_\tau = \frac{1}{\tau}(\mathbb{T}_\tau z_0 - z_0)$. □

Remark 2.1.7. The above proof also shows the following useful fact:

$$\mathbb{T}_\tau z - z = A\int_0^\tau \mathbb{T}_\sigma z \,\mathrm{d}\sigma \qquad \forall\, z \in X.$$

Corollary 2.1.8. *If A is as above, then $\mathcal{D}(A)$ is dense in X.*

Some simple examples of semigroups will be given at the end of Section 2.3.

2.2 The spectrum and the resolvents of an operator

In this section we collect some general facts about the spectrum, the resolvent set and the resolvents of a possibly unbounded operator on a Hilbert space X, without any reference to strongly continuous semigroups of operators. The material is standard in books or chapters on operator theory, such as Akhiezer and Glazman [2], Brezis [22], Davies [45], Dowson [52], Kato [127] and Yosida [239].

Definition 2.2.1. Let X and Z be Hilbert spaces and let $\mathcal{D}(A)$ be a subspace of X. A linear operator $A : \mathcal{D}(A) \to Z$ is called *closed* if its graph, defined by $G(A) = \left\{ \left[\begin{smallmatrix} f \\ Af \end{smallmatrix} \right] \,\middle|\, f \in \mathcal{D}(A) \right\}$, is closed in $X \times Z$.

Clearly, A is closed iff for any sequence (z_n) in $\mathcal{D}(A)$ such that $z_n \to z$ (in X) and $Az_n \to g$ (in Z), we have $z \in \mathcal{D}(A)$ and $Az = g$. It follows that if A is closed, then $\mathcal{D}(A)$ is a Hilbert space with the *graph norm* $\|\cdot\|_{gr}$ defined by

$$\|z\|_{gr}^2 = \|z\|_X^2 + \|Az\|_Z^2 . \tag{2.2.1}$$

An operator $A : \mathcal{D}(A) \to Z$ is called *bounded* if it has a continuous extension to the closure of $\mathcal{D}(A)$. If A is closed, then it follows from the closed-graph theorem that it is bounded iff $\mathcal{D}(A)$ is closed. Clearly, every $T \in \mathcal{L}(X, Z)$ is closed.

Remark 2.2.2. It will be useful to note that if $A : \mathcal{D}(A) \to Z$ is closed, where $\mathcal{D}(A) \subset X$, and if $P \in \mathcal{L}(X, Z)$, then also $A + P$ is closed (the domain of $A + P$ is again $\mathcal{D}(A)$). The proof of this fact is left to the reader.

Definition 2.2.3. If $A : \mathcal{D}(A) \to X$, where $\mathcal{D}(A) \subset X$, then the *resolvent set* of A, denoted $\rho(A)$, is the set of those points $s \in \mathbb{C}$ for which the operator $sI - A : \mathcal{D}(A) \to X$ is invertible and $(sI - A)^{-1} \in \mathcal{L}(X)$. The *spectrum* of A, denoted $\sigma(A)$, is the complement of $\rho(A)$ in $\mathbb{C}$. $(sI - A)^{-1}$ is called a *resolvent* of A.

Remark 2.2.4. If $\rho(A)$ is not empty, then A is closed. Indeed, if $s \in \rho(A)$, then the graph $G(sI - A)$ is the same as $G\left((sI - A)^{-1}\right)$, except the coordinates are in reversed order. Thus, $sI - A$ is closed. By Remark 2.2.2, A is closed.

Remark 2.2.5. If $A : \mathcal{D}(A) \to X$ and $\beta, s \in \rho(A)$, then simple algebraic manipulations show that the following identity holds:

$$(sI - A)^{-1} - (\beta I - A)^{-1} = (\beta - s)(sI - A)^{-1}(\beta I - A)^{-1} .$$

This formula is known as the *resolvent identity.*

Lemma 2.2.6. *If $T \in \mathcal{L}(X)$ is such that $\|T\| < 1$, then $I - T$ is invertible and*

$$(I - T)^{-1} = I + T + T^2 + T^3 + \cdots , \qquad \|(I - T)^{-1}\| \leqslant \frac{1}{1 - \|T\|} .$$

The proof of this lemma is easy and it is left to the reader.

Proposition 2.2.7. *Suppose that $A : \mathcal{D}(A) \to X$, $\mathcal{D}(A) \subset X$ and $\beta \in \rho(A)$. Denote $r_\beta = \|(\beta I - A)^{-1}\|$. If $s \in \mathbb{C}$ is such that $|s - \beta| < \frac{1}{r_\beta}$, then $s \in \rho(A)$ and*

$$\|(sI - A)^{-1}\| \leqslant \frac{r_\beta}{1 - |s - \beta| r_\beta}. \tag{2.2.2}$$

Proof. If we knew that $s \in \rho(A)$, then according to Remark 2.2.5 we would have

$$(sI - A)^{-1} \left[I + (s - \beta)(\beta I - A)^{-1} \right] = (\beta I - A)^{-1}.$$

If $|s - \beta| < \frac{1}{r_\beta}$, then we have $\|(s - \beta)(\beta I - A)^{-1}\| < 1$. According to Lemma 2.2.6, the expression in the square brackets above is invertible. The above formula suggests defining $R_s \in \mathcal{L}(X)$ (our candidate for $(sI - A)^{-1}$) by

$$R_s = (\beta I - A)^{-1} \left[I + (s - \beta)(\beta I - A)^{-1} \right]^{-1}. \tag{2.2.3}$$

Simple algebraic manipulations show that $R_s(sI - A)z = z$ for all $z \in \mathcal{D}(A)$ and $(sI - A)R_s z = z$ for all $z \in X$, hence $s \in \rho(A)$ and $R_s = (sI - A)^{-1}$. Using again Lemma 2.2.6, it is easy to see that $\|R_s\|$ satisfies the estimate (2.2.2). $\square$

Remark 2.2.8. From the last proposition it follows that for any $A : \mathcal{D}(A) \to X$, the set $\rho(A)$ is open, and hence $\sigma(A)$ is closed. It also follows that for every $\beta \in \rho(A)$, $|\lambda - \beta| \geqslant \frac{1}{r_\beta}$ for every $\lambda \in \sigma(A)$, and hence

$$\|(\beta I - A)^{-1}\| \geqslant \frac{1}{\min\limits_{\lambda \in \sigma(A)} |\beta - \lambda|}.$$

Remark 2.2.9. We can use steps from the proof of Proposition 2.2.7 to show that $(sI - A)^{-1}$ is an analytic $\mathcal{L}(X)$-valued function of $s \in \rho(A)$. Indeed, formula (2.2.3), together with Lemma 2.2.6, shows that if $\beta \in \rho(A)$ and $|s - \beta| < \frac{1}{r_\beta}$, then

$$(sI - A)^{-1} = (\beta I - A)^{-1} \sum_{k=0}^{\infty} (\beta - s)^k (\beta I - A)^{-k}.$$

This is a Taylor series around the point β, proving the analyticity at β. In particular,

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^k (sI - A)^{-1} = (-1)^k k! (sI - A)^{-(k+1)} \qquad \forall\, k \in \mathbb{N}. \tag{2.2.4}$$

Proposition 2.2.10. *If $A \in \mathcal{L}(X)$, then $|\lambda| \leqslant \|A\|$ for every $\lambda \in \sigma(A)$.*

Proof. Suppose that $|\lambda| > \|A\|$. Then $\lambda I - A = \lambda \left(I - \frac{A}{\lambda}\right)$. Here, both factors are invertible, the second because of Lemma 2.2.6. Hence, $\lambda \in \rho(A)$. $\square$

For any $A \in \mathcal{L}(X)$, the number

$$r(A) = \max_{\lambda \in \sigma(A)} |\lambda|$$

is called the *spectral radius* of A. It follows from the last proposition that $r(A) \leqslant \|A\|$. A stronger statement will be proved at the end of this section.

Lemma 2.2.11. *If $A \in \mathcal{L}(X)$ and $r > r(A)$, then there exists $m_r \geqslant 0$ such that*

$$\|A^n\| \leqslant m_r r^n \qquad \forall\, n \in \mathbb{N}.$$

Proof. For every $\alpha, \gamma > 0$ we denote

$$\mathbb{D}_\alpha = \{s \in \mathbb{C} \mid |s| < \alpha\}, \qquad \mathcal{C}_\gamma = \{s \in \mathbb{C} \mid |s| = \gamma\}.$$

For $\alpha = \frac{1}{r(A)}$ we define the function $f : \mathbb{D}_\alpha \to \mathcal{L}(X)$ by

$$f(s) = (I - sA)^{-1}.$$

By Remark 2.2.9, f is analytic. According to Lemma 2.2.6, for $|s| \cdot \|A\| < 1$ we have $f(s) = I + sA + s^2A^2 + s^3A^3 + \cdots$. This, together with Cauchy's formula from the theory of analytic functions, implies that for every $\gamma > 0$ such that $\gamma \cdot \|A\| < 1$,

$$A^n = \frac{1}{2\pi i} \int_{\mathcal{C}_\gamma} f(s) \frac{\mathrm{d}s}{s^{n+1}} \qquad \forall\, n \in \mathbb{N}. \tag{2.2.5}$$

By Cauchy's theorem the above formula remains valid for every $\gamma \in (0, \alpha)$. Denoting $c_\gamma = \max_{s \in \mathcal{C}_\gamma} \|f(s)\|$, we obtain that $\|A^n\| \leqslant c_\gamma \frac{1}{\gamma^n}$. Denoting $r = \frac{1}{\gamma}$ and $m_r = c_\gamma$, we obtain the desired estimate. □

Let $A : \mathcal{D}(A) \to X$ with $\mathcal{D}(A) \subset X$. We define the space $\mathcal{D}(A^n)$ recursively:

$$\mathcal{D}(A^n) = \{z \in \mathcal{D}(A) \mid Az \in \mathcal{D}(A^{n-1})\}.$$

The powers of A, $A^n : \mathcal{D}(A^n) \to X$ are defined in the obvious way.

Proposition 2.2.12. *Let $A : \mathcal{D}(A) \to X$ and let p be a polynomial. Then*

$$\sigma(p(A)) = p(\sigma(A)).$$

Moreover, if $0 \in \rho(A)$, then $\sigma(A^{-1}) = \{0\} \cup 1/\sigma(A)$ if $\mathcal{D}(A) \neq X$, and $\sigma(A^{-1}) = 1/\sigma(A)$ if $\mathcal{D}(A) = X$.

Proof. Denote the order of p by n. For any $\lambda \in \mathbb{C}$ we can decompose $\lambda - p(x) = (\gamma_1(\lambda) - x)(\gamma_2(\lambda) - x) \cdots (\gamma_n(\lambda) - x)$, where $p(\gamma_j(\lambda)) = \lambda$. Then we have

$$\lambda I - p(A) = (\gamma_1(\lambda)I - A)(\gamma_2(\lambda)I - A) \cdots (\gamma_n(\lambda)I - A),$$

which shows that $\lambda \in \sigma(p(A))$ iff $\gamma_j(\lambda) \in \sigma(A)$ for at least one $j \in \{1, 2, \ldots, n\}$. The latter condition is equivalent to $\lambda \in p(\sigma(A))$. The statement about A^{-1} is easy (but slightly tedious) to prove and this task is left to the reader. □

Corollary 2.2.13. *Suppose that $A : \mathcal{D}(A) \to X$, where $\mathcal{D}(A) \subset X$ and $\lambda, s \in \mathbb{C}$, $\lambda \neq s$, $s \in \rho(A)$. Then the following statements are equivalent:*

(1) $\lambda \in \sigma(A)$.

(2) $\frac{1}{s-\lambda} \in \sigma((sI - A)^{-1})$.

This follows from the last part of Proposition 2.2.12 by replacing A with $\lambda I - A$.

Remark 2.2.14. It follows from the last proposition and its corollary that $r(A^n) = r(A)^n$ and also that for $s \in \rho(A)$ we have

$$r((sI - A)^{-1}) = \frac{1}{\min\limits_{\lambda \in \sigma(A)} |s - \lambda|}. \tag{2.2.6}$$

This, together with the fact that $\|T\| \geqslant r(T)$ for any $T \in \mathcal{L}(X)$, provides an alternative (but more complicated) proof for the estimate in Remark 2.2.8.

The following proposition is known as the *Gelfand formula*.

Proposition 2.2.15. *If $A \in \mathcal{L}(X)$, then $r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$.*

Proof. According to Remark 2.2.14 we have $r(A^n) = r(A)^n$, so that $r(A)^n \leqslant \|A^n\|$. Using Lemma 2.2.11 we obtain that for every $r > r(A)$ there exists $m_r \geqslant 0$ such that

$$r(A) \leqslant \|A^n\|^{\frac{1}{n}} \leqslant m_r^{\frac{1}{n}} r \qquad \forall\, n \in \mathbb{N}.$$

This shows that

$$r(A) \leqslant \liminf \|A^n\|^{\frac{1}{n}}, \qquad \limsup \|A^n\|^{\frac{1}{n}} \leqslant r \qquad \forall\, r > r(A),$$

and from here it is easy to obtain the formula in the proposition. □

Remark 2.2.16. Suppose that $\mathbb{T}$ is a strongly continuous semigroup on X with growth bound ω_0. Then

$$r(\mathbb{T}_t) = e^{\omega_0 t} \qquad \forall\, t \in [0, \infty).$$

Indeed, according to the Gelfand formula, $\log r(\mathbb{T}_t) = \lim_{n\to\infty} \frac{1}{n} \log \|\mathbb{T}_{nt}\|$. According to part (1) of Proposition 2.1.2, this is equal to $\omega_0 t$.

Definition 2.2.17. If $A : \mathcal{D}(A) \to X$, where $\mathcal{D}(A) \subset X$, then $\lambda \in \mathbb{C}$ is called an *eigenvalue* of A if there exists a $z_\lambda \in \mathcal{D}(A)$, $z_\lambda \neq 0$, such that $Az_\lambda = \lambda z_\lambda$. In this case, z_λ is called an *eigenvector* of A corresponding to λ. The set of all the eigenvalues of A is called the *point spectrum* of A, and it is denoted by $\sigma_p(A)$.

The following proposition is an elementary spectral mapping theorem for the point spectrum of an operator.

Proposition 2.2.18. *Suppose that $A : \mathcal{D}(A) \to X$, where $\mathcal{D}(A) \subset X$ and $\lambda, s \in \mathbb{C}$, $\lambda \neq s$, $s \in \rho(A)$. Then the following statements are equivalent:*

(1) $\lambda \in \sigma_p(A)$.
(2) $\frac{1}{s-\lambda} \in \sigma_p((sI - A)^{-1})$.

If (1), (2) *hold, then the eigenvectors of A corresponding to the eigenvalue λ are the same as the eigenvectors of $(sI - A)^{-1}$ corresponding to the eigenvalue $\frac{1}{s-\lambda}$.*

Proof. Suppose that (1) holds and let $z_\lambda \in X$ be such that $z_\lambda \neq 0$, $Az_\lambda = \lambda z_\lambda$. Then clearly $(sI - A)z_\lambda = (s - \lambda)z_\lambda$. Applying $(sI - A)^{-1}$ to both sides, we obtain that $(sI - A)^{-1}z_\lambda = \frac{1}{s-\lambda}z_\lambda$, so that (2) holds. The converse is proved in the same way, and this argument also shows that the sets of the eigenvectors corresponding to (1) and (2) are the same. $\square$

2.3 The resolvents of a semigroup generator and the space $\mathcal{D}(A^\infty)$

In this section we examine some properties of the resolvents $(sI - A)^{-1}$ of the operator A, which is the generator of a strongly continuous semigroup on X, we introduce the space $\mathcal{D}(A^\infty)$ and we show that it is dense in X.

Proposition 2.3.1. *Let $\mathbb{T}$ be a strongly continuous semigroup on X, with generator A. Then for every $s \in \mathbb{C}$ with $\operatorname{Re} s > \omega_0(\mathbb{T})$ we have $s \in \rho(A)$ (hence, A is closed) and*

$$(sI - A)^{-1}z = \int_0^\infty e^{-st}\,\mathbb{T}_t z\,\mathrm{d}t \qquad \forall\, z \in X\,.$$

Proof. Suppose that $\operatorname{Re} s > \omega_0(\mathbb{T})$. Then it follows from (2.1.4) (with $\omega_0(\mathbb{T}) < \omega < \operatorname{Re} s$) that the integral in the statement of the proposition is absolutely convergent. Define $R_s \in \mathcal{L}(X)$ by $R_s z = \int_0^\infty e^{-st}\,\mathbb{T}_t z\,\mathrm{d}t$. Then for every $h > 0$ and $z \in X$,

$$\begin{aligned}
\frac{\mathbb{T}_h - I}{h} R_s z &= \frac{1}{h}\int_0^\infty e^{-st}\left(\mathbb{T}_{t+h}z - \mathbb{T}_t z\right)\mathrm{d}t \\
&= \frac{1}{h}\int_h^\infty e^{-s(t-h)}\,\mathbb{T}_t z\,\mathrm{d}t - \frac{1}{h}\int_0^\infty e^{-st}\,\mathbb{T}_t z\,\mathrm{d}t \\
&= \frac{e^{sh}-1}{h}\int_0^\infty e^{-st}\,\mathbb{T}_t z\,\mathrm{d}t - \frac{e^{sh}}{h}\int_0^h e^{-st}\,\mathbb{T}_t z\,\mathrm{d}t\,.
\end{aligned}$$

This implies that

$$\lim_{h\to 0} \frac{\mathbb{T}_h - I}{h} R_s z = sR_s z - z; \tag{2.3.1}$$

i.e., $R_s z \in \mathcal{D}(A)$ and $(sI - A)R_s z = z$. Since R_s commutes with $\mathbb{T}$, for $z \in \mathcal{D}(A)$, (2.3.1) can also be written in the form

$$R_s \lim_{h\to 0} \frac{\mathbb{T}_h - I}{h} z = sR_s z - z\,.$$

Thus, $R_s(sI - A)z = z$ for $z \in \mathcal{D}(A)$, so that $s \in \rho(A)$ and $R_s = (sI - A)^{-1}$. $\square$

Remark 2.3.2. From the last proposition we see that $\mathbb{T}$ is uniquely determined by its generator A. Indeed, any continuous function which has a Laplace transform is uniquely determined by this Laplace transform; see Section 12.4.

Corollary 2.3.3. *If $\mathbb{T}$ is a strongly continuous semigroup on X, with generator A, and if M_ω and ω are as in* (2.1.4)*, then*

$$\|(sI - A)^{-1}\| \leqslant \frac{M_\omega}{\operatorname{Re} s - \omega} \qquad \forall\, s \in \mathbb{C}_\omega\,. \tag{2.3.2}$$

Proof. This follows from Proposition 2.3.1, by estimating the integral:

$$\|(sI - A)^{-1}z\| \leqslant \int_0^\infty e^{-(\operatorname{Re} s)t}\,\|\mathbb{T}_t\| \cdot \|z\|\,\mathrm{d}t \qquad \forall\, z \in X\,. \qquad \square$$

It is often needed to approximate elements of X by elements of $\mathcal{D}(A)$ in a natural way. One approach was given in Proposition 2.1.6, another one is given below.

Proposition 2.3.4. *Let $\mathcal{D}(A)$ be a dense subspace of X and let $A : \mathcal{D}(A) \to X$ be such that there exist $\lambda_0 \geqslant 0$ and $m > 0$ such that $(\lambda_0, \infty) \subset \rho(A)$ and*

$$\left\|\lambda(\lambda I - A)^{-1}\right\| \leqslant m \qquad \forall\, \lambda > \lambda_0\,. \tag{2.3.3}$$

Then we have

$$\lim_{\lambda\to\infty} \lambda(\lambda I - A)^{-1}z = z \qquad \forall\, z \in X\,. \tag{2.3.4}$$

Note that if A is the generator of a strongly continuous semigroup on X, then A satisfies the assumption in this proposition, according to Corollary 2.3.3.

Proof. Assume that $\psi \in \mathcal{D}(A)$. Then

$$\lambda(\lambda I - A)^{-1}\psi = (\lambda I - A)^{-1}A\psi + \psi\,.$$

Now (2.3.3) implies that the first term on the right-hand side converges to zero (as $\lambda \to \infty$). Thus, (2.3.4) holds for $\psi \in \mathcal{D}(A)$. If $z \in X$ and $\psi \in \mathcal{D}(A)$, then from

$$\|\lambda(\lambda I - A)^{-1}z - z\| \leqslant \|\lambda(\lambda I - A)^{-1}(z - \psi)\| + \|\lambda(\lambda I - A)^{-1}\psi - \psi\| + \|\psi - z\|$$

we obtain that for all $\lambda \geqslant \lambda_0$,

$$\|\lambda(\lambda I - A)^{-1}z - z\| \leqslant (m + 1)\|(z - \psi)\| + \|\lambda(\lambda I - A)^{-1}\psi - \psi\|\,.$$

The first term on the right-hand side can be made arbitrarily small by a suitable choice of $\psi \in \mathcal{D}(A)$, because $\mathcal{D}(A)$ is dense in X. The second term tends to zero (as $\lambda \to \infty$), as we have proved earlier. Therefore, the left-hand side can be made arbitrarily small by choosing λ large enough. $\square$

Proposition 2.3.5. *Let $\mathbb{T}$ be a strongly continuous semigroup on X with generator A. Let $z_0 \in \mathcal{D}(A)$ and define the function $z : [0,\infty) \to \mathcal{D}(A)$ by $z(t) = \mathbb{T}_t z_0$.*

Then z is continuous, if we consider on $\mathcal{D}(A)$ the graph norm, and we also have $z \in C^1([0,\infty), X)$. Moreover, z is the unique function with the above properties satisfying the initial value problem

$$\dot{z} = Az, \quad z(0) = z_0. \tag{2.3.5}$$

Proof. According to Proposition 2.1.2, z is continuous as an X-valued function. We also have $Az \in C([0,\infty), X)$, because $Az(t) = \mathbb{T}_t A z_0$ and we can invoke again Proposition 2.1.2. Using the definition of the graph norm, it follows that z is continuous as a $\mathcal{D}(A)$-valued function (with the graph norm on $\mathcal{D}(A)$).

According to Proposition 2.1.5, z satisfies (2.3.5). Since $Az \in C([0,\infty), X)$, it follows that $z \in C^1([0,\infty), X)$.

We still have to prove the uniqueness of z with the above properties. Let $v \in C^1([0,\infty), X)$ with values in $\mathcal{D}(A)$ which is continuous from $[0,\infty)$ to $\mathcal{D}(A)$ and such that $\dot{v} = Av$, $v(0) = z_0$. For all $\tau \in [0,t]$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\mathbb{T}_{t-\tau} v(\tau)\right] = \mathbb{T}_{t-\tau} A v(\tau) - \mathbb{T}_{t-\tau} A v(\tau) = 0,$$

whence

$$v(t) = \mathbb{T}_{t-t} v(t) = \mathbb{T}_{t-0} v(0) = \mathbb{T}_t z_0 = z(t). \qquad \square$$

For a stronger version of the above uniqueness property see Proposition 4.1.4.

For every $n \in \mathbb{N}$, the operator A^n (and its domain) have been introduced before Proposition 2.2.12. It is easy to see (using Proposition 2.3.5 and induction) that for every $t \geqslant 0$, $\mathbb{T}_t \mathcal{D}(A^n) \subset \mathcal{D}(A^n)$. We introduce the space

$$\mathcal{D}(A^\infty) = \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n).$$

The following proposition is a strengthening of Corollary 2.1.8.

Proposition 2.3.6. *If A is the generator of a strongly continuous semigroup on X, then $\mathcal{D}(A^\infty)$ is dense in X.*

Proof. We denote by $\mathcal{D}(0,1)$ the space of all infinitely differentiable functions on $(0,1)$ whose support is compact and contained in $(0,1)$. We denote by $\mathbb{T}$ the semigroup generated by A. For every $\varphi \in \mathcal{D}(0,1)$ we define the operator T_φ by

$$T_\varphi z_0 = \int_0^1 \varphi(t) \mathbb{T}_t z_0 \,\mathrm{d}t \qquad \forall\, z_0 \in X. \tag{2.3.6}$$

Take $z_0 \in \mathcal{D}(A)$. It follows from Proposition 2.3.5 that the integral in the definition of $T_\varphi z_0$ may be considered as an integral in $\mathcal{D}(A)$ (with the graph norm) and $T_\varphi z_0 \in \mathcal{D}(A)$. Using integration by parts, it is now easy to see that we have

$$AT_\varphi z_0 \,=\, -\,T_{\varphi'} z_0 \qquad\qquad \forall\, z_0 \in \mathcal{D}(A)\,.$$

This shows that the operator AT_φ has a continuous extension to X. Since A is a closed operator (as we have seen in Proposition 2.3.1), it follows that $T_\varphi z_0 \in \mathcal{D}(A)$ for every $z_0 \in X$ and $AT_\varphi = -T_{\varphi'}$. This identity shows, by induction, that

$$\operatorname{Ran} T_\varphi \subset \mathcal{D}(A^\infty)\,.$$

For every $\tau \in (0,1)$ consider the function $\psi_\tau \in L^2(0,1)$ defined by

$$\psi_\tau(x) \,=\, \begin{cases} \frac{1}{\tau} & \text{if } \ x \in (0,\tau)\,, \\ 0 & \text{else.} \end{cases}$$

We define T_{ψ_τ} by the same formula (2.3.6) (with ψ_τ in place of φ). We know from Proposition 2.1.6 that for every $z_0 \in X$, $\lim_{\tau\to 0} T_{\psi_\tau} z_0 = z_0$. Since $\mathcal{D}(0,1)$ is dense in $L^2[0,1]$, it follows that $\mathcal{D}(A^\infty)$ is dense in X. □

Example 2.3.7. Take $X = L^2[0,\infty)$ and for every $t \in \mathbb{R}$ and $z \in X$, define

$$(\mathbb{T}_t z)(x) \,=\, z(x+t) \qquad\qquad \forall\, x \in [0,\infty)\,.$$

Then $\mathbb{T}$ is a strongly continuous semigroup, called the *unilateral left shift semigroup*. To prove the strong continuity of this semigroup, the easiest approach is to prove it first for functions $z \in X \cap C^1[0,\infty)$ which have compact support (i.e., there exists $\mu \geqslant 0$ such that $z(x) = 0$ for $x \geqslant \mu$). Afterwards, the strong continuity of $\mathbb{T}$ follows from the fact that the set of functions z as above is dense in X and $\|\mathbb{T}_t\| = 1$ for all $t \geqslant 0$. (The argument resembles the last part of the proof of Proposition 2.3.4.)

We claim that the generator of $\mathbb{T}$ is

$$A = \frac{\mathrm{d}}{\mathrm{d}x}\,, \qquad \mathcal{D}(A) \,=\, \mathcal{H}^1(0,\infty)\,.$$

A detailed proof of this claim requires some effort. We know from Proposition 2.3.1 that $1 \in \rho(A)$ and, for every $z \in X$,

$$[(I-A)^{-1}z](x) \,=\, \int_0^\infty e^{-t} z(x+t)\,\mathrm{d}t \,=\, e^x \int_x^\infty e^{-\xi} z(\xi)\,\mathrm{d}\xi$$

holds for almost every $x \in [0,\infty)$. Denoting $\varphi = (I-A)^{-1}z$, it follows that φ is continuous and the above formula holds for all $x \geqslant 0$. We rearrange the formula:

$$\varphi(x) \,=\, e^x \varphi(0) - \int_0^x e^{x-\xi} z(\xi)\,\mathrm{d}\xi \qquad\qquad \forall\, x \geqslant 0\,.$$

This shows that φ is locally absolutely continuous and $\varphi'(x) = \varphi(x) - z(x)$ holds for almost every $x \geqslant 0$. Since both φ and z are in X, it follows that $\varphi' \in X = L^2[0,\infty)$, whence $\varphi \in \mathcal{H}^1(0,\infty)$. Thus, $\mathcal{D}(A) \subset \mathcal{H}^1(0,\infty)$. By the definition of φ, we have $A\varphi = \varphi - z$. Comparing this with the formula $\varphi' = \varphi - z$ derived a little earlier, it follows that

$$A\varphi = \varphi' \qquad \forall\, \varphi \in \mathcal{D}(A).$$

If the inclusion $\mathcal{D}(A) \subset \mathcal{H}^1(0,\infty)$ were strict, then there would exist $\psi \in \mathcal{H}^1(0,\infty)$ such that $\psi \notin \mathcal{D}(A)$. Denote $z = \psi - \psi'$ and put $\varphi = (I-A)^{-1}z$, then $\varphi - \varphi' = z$. Denoting $\eta = \psi - \varphi$ we obtain that $\eta \in \mathcal{H}^1(0,\infty)$ and $\eta' = \eta$, whence $\eta = 0$, so that $\psi \in \mathcal{D}(A)$, which is a contradiction. Thus we have proved our claim.

It is easy to see that every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$ is an eigenvalue of A and a corresponding eigenvector is $z_\lambda(x) = e^{\lambda x}$. Since $\sigma(A)$ is closed, it follows that the closed left half-plane (where $\operatorname{Re} s \leqslant 0$) is contained in $\sigma(A)$. On the other hand, we know from Proposition 2.3.1 that $\mathbb{C}_0 \subset \rho(A)$. Thus, it follows that

$$\sigma(A) = \{s \in \mathbb{C} \mid \operatorname{Re} s \leqslant 0\}.$$

A little exercise in differential equations shows that the points on the imaginary axis are not eigenvalues of A, so that

$$\sigma_p(A) = \{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}.$$

For a detailed discussion of this example and others related to it see also Engel and Nagel [57, Chapter II]. We shall need several times a slight generalization of this example to the case when $X = L^2([0,\infty); Y)$, where Y is a Hilbert space – this will be the case, for example, in the proof of Theorem 4.1.6 and of Lemma 6.1.11.

Example 2.3.8. Let $\tau > 0$, take $X = L^2[0,\tau]$ and for every $t \in \mathbb{R}$ and $z \in X$ define

$$(\mathbb{T}_t z)(x) = \begin{cases} z(x+t) & \text{if } \; x + t \leqslant \tau, \\ 0 & \text{else}. \end{cases}$$

Then $\mathbb{T}$ is a strongly continuous semigroup. Clearly $\mathbb{T}_\tau = 0$ (the semigroup is vanishing in finite time), so that $\omega_0(\mathbb{T}) = -\infty$. It is not difficult to verify that the generator of $\mathbb{T}$ is

$$A = \frac{\mathrm{d}}{\mathrm{d}x}, \qquad \mathcal{D}(A) = \{z \in \mathcal{H}^1(0,\tau) \mid z(\tau) = 0\}$$

and $\sigma(A) = \emptyset$ (this last fact is impossible for bounded operators A).

Example 2.3.9. Take $X = L^2(\mathbb{R})$ and for every $t > 0$ and $z \in X$ define

$$(\mathbb{T}_t z)(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\sigma)^2}{4t}} z(\sigma)\mathrm{d}\sigma \qquad \forall\, x \in \mathbb{R}.$$

We put $\mathbb{T}_0 = I$. Then $\mathbb{T}$ is a strongly continuous semigroup of operators (as we shall see), called the *heat semigroup* on $\mathbb{R}$. It is easier to understand this semigroup if we apply the Fourier transformation $\mathcal{F}$ (with respect to the space variable x) to the definition of $\mathbb{T}$, obtaining that (for almost every $\xi \in \mathbb{R}$)

$$(\mathcal{F}\mathbb{T}_t z)(\xi) = e^{-\xi^2 t}(\mathcal{F}z)(\xi).$$

This formula shows clearly that $\mathbb{T}$ has the semigroup property and $\|\mathbb{T}_t\| \leqslant 1$ for all $t \geqslant 0$. Moreover, the generator of $\mathbb{T}$ can be expressed in terms of Fourier transforms as follows:

$$\mathcal{D}(A) = \left\{ z \in L^2(\mathbb{R}) \;\middle|\; \int_{-\infty}^{\infty} \xi^4 |(\mathcal{F}z)(\xi)|^2 \,\mathrm{d}\xi < \infty \right\},$$
$$(\mathcal{F}Az)(\xi) = -\xi^2 (\mathcal{F}z)(\xi). \tag{2.3.7}$$

From here, applying $\mathcal{F}^{-1}$, we now see that $\mathcal{D}(A)$ is in fact the Sobolev space $\mathcal{H}^2(\mathbb{R})$ and $A = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$. Thus, the functions $\varphi(t,x) = (\mathbb{T}_t z)(x)$ satisfy the one-dimensional heat equation, namely $\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial x^2}$. From (2.3.7) we can derive that $\sigma(A) = (-\infty, 0]$. It is easy to check that this operator has no eigenvalues.

2.4 Invariant subspaces for semigroups

In this section we derive some facts about invariant subspaces for operator semigroups, and the restrictions of operator semigroups to invariant subspaces.

Definition 2.4.1. Let $\mathbb{T}$ be a strongly continuous semigroup on X, with generator A. Let V be a subspace of X (not necessarily closed). The *part of A in V*, denoted by A_V, is the restriction of A to the domain

$$\mathcal{D}(A_V) = \{ z \in \mathcal{D}(A) \cap V \mid Az \in V \}.$$

V is called *invariant under* $\mathbb{T}$ if $\mathbb{T}_t z \in V$ for all $z \in V$ and all $t \geqslant 0$.

The following facts are easy to see: If V is invariant under $\mathbb{T}$, then so is $\operatorname{clos} V$. If V_1 and V_2 are invariant under $\mathbb{T}$, then so are $V_1 \cap V_2$ and $V_1 + V_2$. (The last statement can be generalized to arbitrary infinite intersections and sums.)

The following proposition will be useful here.

Proposition 2.4.2. *If $\mathbb{T}$ is a strongly continuous semigroup on X, with generator A, then*

$$\mathbb{T}_t z = \lim_{n \to \infty} \left(I - \frac{t}{n} A \right)^{-n} z \qquad \forall\, t > 0,\ z \in X. \tag{2.4.1}$$

Proof. For $z \in X$ fixed, we define a continuous function f on $[0, \infty)$ by $f(t) = \mathbb{T}_t z$. We shall denote by $\hat{f}$ the Laplace transform of f (see Section 12.4). Recall the Post–Widder formula (Theorem 12.4.4): For every $t > 0$,

$$f(t) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \hat{f}^{(n)} \left(\frac{n}{t}\right).$$

Since $\hat{f}(s) = (sI - A)^{-1} z$ (see Proposition 2.3.1) and since, by (2.2.4),

$$\hat{f}^{(n)}(s) = (-1)^n n!\, (sI - A)^{-(n+1)} z,$$

we obtain $f(t) = \lim_{n \to \infty} \left(I - \frac{t}{n} A\right)^{-(n+1)} z$. Now using (2.3.4) we get (2.4.1). □

Proposition 2.4.3. *Let $\mathbb{T}$ be a strongly continuous semigroup on X, with generator A. We denote by $\rho_\infty(A)$ the connected component of $\rho(A)$ containing a right half-plane. Let V be a closed subspace of X.*

Then the following conditions are equivalent:

(1) *V is invariant under $\mathbb{T}$.*
(2) *For some $s_0 \in \rho_\infty(A)$ we have $(s_0 I - A)^{-1} V \subset V$.*
(3) *For every $s \in \rho_\infty(A)$ we have $(sI - A)^{-1} V \subset V$.*

Moreover, if one (hence, all) of the above conditions holds, then the restriction of $\mathbb{T}$ to V, denoted by $\mathbb{T}^V$, is a strongly continuous semigroup on V. We have $A(\mathcal{D}(A) \cap V) \subset V$ and the generator of $\mathbb{T}^V$ is the restriction of A to $\mathcal{D}(A) \cap V$.

Note that under the assumptions in the "moreover" part of the above proposition, the restriction of A to $\mathcal{D}(A) \cap V$ is A_V, the part of A in V.

Proof. (1) $\Rightarrow$ (2) follows from Proposition 2.3.1, by taking $\operatorname{Re} s_0 > \omega_0(\mathbb{T})$.

(2) $\Rightarrow$ (3): Take $z \in V$ and $w \in V^\perp$. The function $f(s) = \langle (sI - A)^{-1} z, w \rangle$ is analytic on $\rho_\infty(A)$ according to Remark 2.2.9. It is easy to see, using (2.2.4), that all the derivatives of f at s_0 are zero, so that f is zero on an open disk around s_0. Since $\rho_\infty(A)$ is connected, an analytic function on this domain is uniquely determined by its restriction to an open subset. Hence, $f = 0$, so that $(sI - A)^{-1} z \in V$.

(3) $\Rightarrow$ (1): Take $z \in V$ and $t > 0$. According to (2.4.1),

$$\mathbb{T}_t z = \lim_{n \to \infty} \frac{n}{t} \left(\frac{n}{t} I - A\right)^{-n} z.$$

For n large enough, $\frac{n}{t} \in \rho_\infty(A)$, so that V is invariant for the operator in the limit above. Since V is closed, it follows that it is invariant also for $\mathbb{T}_t$.

If (1) holds, then it is clear that $\mathbb{T}^V$ is a strongly continuous semigroup on V. The remaining statements in the "moreover" part of the proposition follow from the definition of the infinitesimal generator of an operator semigroup. □

Proposition 2.4.4. *Let V be a Hilbert space such that $V \subset X$, with continuous embedding (i.e., the identity operator on V is bounded from V to X). Let $\mathbb{T}$ be a strongly continuous semigroup on X, with generator A.*

If V is invariant under $\mathbb{T}$ and if the restriction of $\mathbb{T}$ to V, denoted by $\mathbb{T}^V$, is strongly continuous on V, then the generator of $\mathbb{T}^V$ is A_V (the part of A in V).

Conversely, if A_V is the generator of a strongly continuous semigroup $\mathbb{T}^V$ on V, then V is invariant under $\mathbb{T}$ and for each $t \geqslant 0$, $\mathbb{T}^V_t$ is the restriction of $\mathbb{T}_t$ to V.

Proof. Suppose that V is invariant under $\mathbb{T}$ and the restriction $\mathbb{T}^V$ is strongly continuous. Denote the generator of $\mathbb{T}^V$ by $\mathcal{A}$. We have to show that $\mathcal{A} = A_V$.

Take $s \in \mathbb{C}$ with $\operatorname{Re} s > \omega_0(\mathbb{T})$. We know from Proposition 2.3.1 that for every $z \in V$, $(sI - \mathcal{A})^{-1}z = \int_0^\infty e^{-st}\mathbb{T}_t z\,\mathrm{d}t$, with integration in V. Because of the continuous embedding $V \subset X$, integration in X yields the same vector. We conclude that

$$(sI - \mathcal{A})^{-1}z = (sI - A)^{-1}z \qquad \forall\, z \in V.$$

From here it is easy to derive that $\mathcal{D}(\mathcal{A}) = \mathcal{D}(A_V)$ and $\mathcal{A} = A_V$.

Conversely, suppose that A_V is the generator of a strongly continuous semigroup $\mathbb{T}^V$ on V (but we do not know that $\mathbb{T}^V$ is a restriction of $\mathbb{T}$). It follows that for $\operatorname{Re} s > \omega_0(\mathbb{T}^V)$ we have $s \in \rho(A_V)$. If s satisfies also $\operatorname{Re} s > \omega_0(\mathbb{T})$, then from the definition of A_V we see that $(sI - A_V)^{-1}z = (sI - A)^{-1}z$, for all $z \in V$. Using Proposition 2.3.1 for $\mathbb{T}^V$, we get that for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \max\{\omega_0(\mathbb{T}^V), \omega_0(\mathbb{T})\}$,

$$(sI - A)^{-1}z = \int_0^\infty e^{-st}\mathbb{T}^V_t z\,\mathrm{d}t \qquad \forall\, z \in V,$$

with integration in V. Because of the continuous embedding $V \subset X$, integration in X would yield the same vector. Using once again Proposition 2.3.1, this time for $\mathbb{T}$, we obtain

$$\int_0^\infty e^{-st}\mathbb{T}_t z\,\mathrm{d}t = \int_0^\infty e^{-st}\mathbb{T}^V_t z\,\mathrm{d}t \qquad \forall\, z \in V,$$

with integration in X on both sides. According to Proposition 12.4.5, we obtain that $\mathbb{T}^V$ is the restriction of $\mathbb{T}$ to V. In particular, V is invariant under $\mathbb{T}$. $\square$

The numbers $\omega_0(\mathbb{T}^V)$ and $\omega_0(\mathbb{T})$ (that have appeared in the last part of the above proof) may be different, as the last part of the following example shows.

Example 2.4.5. We define the *unilateral right shift semigroup* on $X = L^2[0, \infty)$ by

$$(\mathbb{T}_t z)(x) = \begin{cases} z(x - t) & \text{if } \ x - t \geqslant 0, \\ 0 & \text{else} \end{cases} \qquad \forall\, z \in L^2[0, \infty).$$

It is clear that $\mathbb{T}$ satisfies the semigroup property and $\|\mathbb{T}_t\| = 1$ for every $t \geqslant 0$. It is not difficult to verify (using a similar approach as in Example 2.3.7) that indeed $\mathbb{T}$ is strongly continuous. We can check that the generator of this semigroup is

$$A = -\frac{\mathrm{d}}{\mathrm{d}x}, \qquad \mathcal{D}(A) = \left\{z \in \mathcal{H}^1(0,\infty) \mid z(0) = 0\right\} = \mathcal{H}_0^1(0,\infty).$$

It follows easily from Proposition 2.3.1 that $\mathbb{C}_0 \subset \rho(A)$ and

$$[(sI - A)^{-1}z](x) = \int_0^x e^{s(t-x)}\, z(t)\,\mathrm{d}t \qquad \forall\, s \in \mathbb{C}_0,\ x \in [0,\infty).$$

For further comments on this semigroup see Examples 2.8.7 and 2.10.7.

It is clear that for every $\tau > 0$, the closed subspace Ran $\mathbb{T}_\tau$ is invariant under $\mathbb{T}$. Another class of closed invariant subspaces can be constructed as follows: Let F be a finite subset of $\mathbb{C}_0$ and consider the closed subspace V consisting of those $z \in X$ for which $\hat{z}(s) = 0$ for all $s \in F$. (Here, $\hat{z}$ denotes the Laplace transform of z.) It is easy to verify that indeed V is invariant under $\mathbb{T}$. We mention that a complete characterization of the closed invariant subspaces for this semigroup is given by the Beurling–Lax theorem; see, for example, Partington [181, p. 41].

Now we examine a non-closed invariant subspace. Define V as the space of those $z \in X$ for which

$$\int_0^\infty e^{2x}|z(x)|^2\,\mathrm{d}t < \infty,$$

with the norm on V being the square root of the above integral. This is a Hilbert space and the embedding $V \subset X$ is continuous. It is easy to see that V is invariant under $\mathbb{T}$, and the restriction of $\mathbb{T}$ to V, denoted by $\mathbb{T}^V$, is strongly continuous on V. According to Proposition 2.4.4, the generator of $\mathbb{T}^V$ is A_V. The restricted semigroup grows much faster than the original semigroup:

$$\|\mathbb{T}_t^V\|_{\mathcal{L}(V)} = e^t \qquad \forall\, t \geqslant 0.$$

2.5 Riesz bases

In this section we collect some simple facts about Riesz bases, since these will be needed in the next section (and later). Good books treating (among other things) Riesz bases are Akhiezer and Glazman [2], Avdonin and Ivanov [9], Curtain and Zwart [39], Nikol'skii [177], Partington [180] and Young [241].

The Hilbert space l^2 has been introduced in Section 1.1. Let the sequence (e_k) be the standard orthonormal basis in l^2. Thus, e_k has a 1 in the kth position and zero everywhere else. Clearly $\langle e_k, e_j\rangle = 1$ if $k = j$, and it is zero else.

Definition 2.5.1. A sequence (ϕ_k) in a Hilbert space X is called a *Riesz basis* in X if there is an invertible operator $Q \in \mathcal{L}(X, l^2)$ such that $Q\phi_k = e_k$ for all $k \in \mathbb{N}$.

In this case, the sequence $(\tilde{\phi}_k)$, defined by

$$\tilde{\phi}_k = Q^* Q \phi_k,$$

is called the *biorthogonal sequence* to (ϕ_k).

The sequence $\tilde{\phi}_k$ is also a Riesz basis, since $Q(Q^*Q)^{-1}\tilde{\phi}_k = e_k$. Note that

$$\langle \phi_k, \tilde{\phi}_j \rangle = \begin{cases} 1 & \text{if} \quad k = j\,, \\ 0 & \text{else}\,. \end{cases} \tag{2.5.1}$$

Note that (ϕ_k) is an orthonormal basis iff $\tilde{\phi}_k = \phi_k$ for all $k \in \mathbb{N}$.

We remark that the existence of a Riesz basis in X implies that X is separable. Riesz bases can be defined also for non-separable spaces by allowing an arbitrary index set in place of $\mathbb{N}$, but we shall not go into this.

More generally, if (ϕ_k) and $(\tilde{\phi}_k)$ are two sequences in X that satisfy (2.5.1), then we say that $(\tilde{\phi}_k)$ is *biorthogonal* to (ϕ_k).

Proposition 2.5.2. *If (ϕ_k), Q and $(\tilde{\phi}_k)$ are as in Definition* 2.5.1*, then every $z \in X$ can be expressed as*

$$z = \sum_{k \in \mathbb{N}} \langle z, \tilde{\phi}_k \rangle \phi_k\,. \tag{2.5.2}$$

Moreover, denoting $m = 1/\|Q^{-1}\|$ and $M = \|Q\|$, we have

$$m^2 \|z\|^2 \leqslant \sum_{k \in \mathbb{N}} |\langle z, \tilde{\phi}_k \rangle|^2 \leqslant M^2 \|z\|^2 \qquad \forall\, z \in X\,. \tag{2.5.3}$$

Note that if (ϕ_k) is orthonormal, then Q is unitary and hence $m = M = 1$.

Proof. The statement corresponding to (2.5.2) for $Qz \in l^2$ in place of z, l^2 in place of X and $e_k = Q\phi_k$ in place of ϕ_k is easy to verify. Apply Q^{-1} to this equality in l^2 to obtain

$$z = \sum_{k \in \mathbb{N}} \langle Qz, Q\phi_k \rangle \phi_k\,.$$

Using the definition of $\tilde{\phi}_k$, we get formula (2.5.2).

Applying Q to both sides of (2.5.2) and taking norms in l^2, we obtain that

$$\|Qz\|^2 = \sum_{k \in \mathbb{N}} |\langle z, \tilde{\phi}_k \rangle|^2\,.$$

Since

$$\frac{1}{\|Q^{-1}\|} \cdot \|z\| \leqslant \|Qz\| \leqslant \|Q\| \cdot \|z\|,$$

we obtain the estimates (2.5.3). $\square$

Proposition 2.5.3. *If (ϕ_k) is a Riesz basis in X and (a_k) is a sequence in l^2, then the series $\sum_{k\in\mathbb{N}} a_k\phi_k$ is convergent and*

$$\frac{1}{M}\|(a_k)\|_{l^2} \leqslant \left\|\sum_{k\in\mathbb{N}} a_k\phi_k\right\| \leqslant \frac{1}{m}\|(a_k)\|_{l^2}\,, \tag{2.5.4}$$

where m, M are the constants from Proposition 2.5.2.

Conversely, suppose that (ϕ_k) is a sequence in X with the following property: There exist m, M with $0 < m \leqslant M$ such that for every finite sequence $(a_k)_{1\leqslant k\leqslant n}$,

$$\frac{1}{M}\left(\sum_{k=1}^{n}|a_k|^2\right)^{\frac{1}{2}} \leqslant \left\|\sum_{k=1}^{n} a_k\phi_k\right\| \leqslant \frac{1}{m}\left(\sum_{k=1}^{n}|a_k|^2\right)^{\frac{1}{2}}. \tag{2.5.5}$$

Then (ϕ_k) is a Riesz basis in $H_0 = \operatorname{clos\,span}\{\phi_k \mid k\in\mathbb{N}\}$.

Proof. For any $p, n\in\mathbb{N}$ with $p\leqslant n$, it follows from the first half of (2.5.3) that

$$\left\|\sum_{k=p}^{n} a_k\phi_k\right\| \leqslant \frac{1}{m}\cdot\sqrt{\sum_{k=p}^{n}|a_k|^2}\,.$$

From here it is easy to see that the series $\sum_{k\in\mathbb{N}} a_k\phi_k$ is indeed convergent. Denote its sum by z. The estimate (2.5.4) follows now easily from (2.5.3).

Assume now that (ϕ_k) is a sequence in X such that (2.5.5) holds for every finite sequence $(a_k)_{1\leqslant k\leqslant n}$. By the same argument as at the beginning of this proof it follows that for any sequence $(a_k)\in l^2$, the series $\sum_{k\in\mathbb{N}} a_k\phi_k$ is convergent. Taking limits in (2.5.5), we obtain that (2.5.4) holds. It is easy to check that the space of all the vectors in X that can be written in the form $\sum_{k\in\mathbb{N}} a_k\phi_k$, for some $(a_k)\in l^2$, is complete. Hence, this space is H_0. It follows that the operator Q from H_0 to l^2, defined by

$$Q\left(\sum_{k\in\mathbb{N}} a_k\phi_k\right) = \sum_{k\in\mathbb{N}} a_k e_k \qquad \forall\ (a_k)\in l^2\,,$$

where (e_k) is the standard orthonormal basis in l^2, is bounded and invertible. Thus, (ϕ_k) is a Riesz basis in H_0. □

Proposition 2.5.4. *With the notation of Proposition* 2.5.2, *let (λ_k) be a sequence in $\mathbb{C}$. Then the following statements are equivalent:*

(1) *The sequence (λ_k) is bounded.*
(2) *For every $z\in X$, the series*

$$Az = \sum_{k\in\mathbb{N}} \lambda_k\,\langle z, \tilde{\phi}_k\rangle\phi_k$$

is convergent and the operator A thus defined is bounded on X.

Moreover, if the above statements are true, then

$$\sup|\lambda_k| \leqslant \|A\| \leqslant \frac{M}{m} \cdot \sup|\lambda_k|. \tag{2.5.6}$$

Proof. Suppose that (1) holds. It follows from (2.5.3) that for any $z \in X$, the sequence $(\langle z, \tilde{\phi}_k\rangle)$ is in l^2 and its norm is bounded by $M\|z\|$. Now it follows from Proposition 2.5.3 that the series in the definition of Az is convergent and

$$\|Az\| \leqslant \frac{1}{m}\left(\sum_{k\in\mathbb{N}} |\lambda_k\langle z, \tilde{\phi}_k\rangle|^2\right)^{\frac{1}{2}} \leqslant \frac{M}{m}\sup|\lambda_k| \cdot \|z\|.$$

Thus, $A \in \mathcal{L}(X)$ and the second part of (2.5.6) holds.

Conversely, if (2) holds, then $\lambda_k \in \sigma_p(A)$. According to Proposition 2.2.10, the sequence (λ_k) satisfies the first part of (2.5.6) and hence (1) holds. □

We shall also need the following very simple property of Riesz bases.

Proposition 2.5.5. *Let (ϕ_k) be a sequence in a Hilbert space X such that it is a Riesz basis in*

$$H_0 = \text{clos span}\,\{\phi_k \mid k \in \mathbb{N}\}.$$

Let $\phi_0 \in X$ be such that $\phi_0 \notin H_0$. Then the sequence $(\phi_0, \phi_1, \phi_2, \ldots)$ is a Riesz basis in $H_1 = H_0 + \{\lambda\phi_0 \mid \lambda \in \mathbb{C}\}$.

Recall the following simple property of normed spaces: If V, W are subspaces of a normed space, then $V + W = \{v + w \mid v \in V,\ w \in W\}$ is also a subspace. If V is closed and W is finite dimensional, then $V + W$ is closed. Thus, in particular, H_1 in the last proposition is closed, and hence a Hilbert space.

Proof. It is easy to see that every $z \in H_1$ has a unique decomposition $z = \lambda\phi_0 + h$, where $\lambda \in \mathbb{C}$ and $h \in H_0$. Define a linear functional $\xi : H_1 \to \mathbb{C}$ by $\xi z = \lambda$. This functional is bounded, because Ker $\xi = H_0$ is closed.

Let $Q_0 \in \mathcal{L}(H_0, l^2)$ be the invertible operator from Definition 2.5.1 corresponding to the Riesz basis (ϕ_k). We define the operator $Q \in \mathcal{L}(H_1, l^2)$ by

$$Qz = (\xi z,\ (Q_0 z)_1,\ (Q_0 z)_2,\ (Q_0 z)_3,\ \ldots).$$

It is clear that $Q\phi_k = e_{k+1}$ for all $k \in \{0, 1, 2, \ldots\}$. Clearly Q is bounded (because ξ and Q_0 are bounded). Finally, Q is invertible, because

$$Q^{-1}(a_1,\ a_2,\ a_3,\ \ldots) = a_1\phi_0 + Q_0^{-1}(a_2,\ a_3,\ a_4,\ \ldots). \qquad \square$$

2.6 Diagonalizable operators and semigroups

In this section we introduce diagonalizable operators, which can be described entirely in terms of their eigenvalues and eigenvectors, thus having a very simple structure. If a semigroup generator is diagonalizable, then so is the semigroup. Many examples of semigroups discussed in the PDEs literature are diagonalizable.

Definition 2.6.1. Let $A : \mathcal{D}(A) \to X$, where $\mathcal{D}(A) \subset X$. A is called *diagonalizable* if $\rho(A) \neq \emptyset$ and there exists a Riesz basis (ϕ_k) in X consisting of eigenvectors of A.

Note that if A is diagonalizable, then $\mathcal{D}(A)$ is dense in X. Indeed, $\mathcal{D}(A)$ must contain all the finite linear combinations of the eigenvectors of A. Note also that, by Remark 2.2.4, every diagonalizable operator is closed.

The structure of bounded diagonalizable operators has been described in Proposition 2.5.4. For unbounded operators we must be careful with the definition of the domain and the situation is described in the following two propositions.

Proposition 2.6.2. *Let (ϕ_k) be a Riesz basis in X and let $(\tilde{\phi}_k)$ be the biorthogonal sequence to (ϕ_k). Let (λ_k) be a sequence in $\mathbb{C}$ which is not dense in $\mathbb{C}$. Define an operator $\widetilde{A} : \mathcal{D}(\widetilde{A}) \to X$ by*

$$\mathcal{D}(\widetilde{A}) = \left\{ z \in X \;\middle|\; \sum_{k\in\mathbb{N}} \left(1 + |\lambda_k|^2\right) |\langle z, \tilde{\phi}_k\rangle|^2 < \infty \right\}, \tag{2.6.1}$$

$$\widetilde{A}z = \sum_{k\in\mathbb{N}} \lambda_k \langle z, \tilde{\phi}_k\rangle \phi_k \qquad \forall\, z \in \mathcal{D}(\widetilde{A}). \tag{2.6.2}$$

Then $\widetilde{A}$ is diagonalizable, we have $\sigma_p(\widetilde{A}) = \{\lambda_k \mid k \in \mathbb{N}\}$, $\sigma(\widetilde{A})$ is the closure of $\sigma_p(\widetilde{A})$ and for every $s \in \rho(\widetilde{A})$ we have

$$(sI - \widetilde{A})^{-1} z = \sum_{k\in\mathbb{N}} \frac{1}{s - \lambda_k} \langle z, \tilde{\phi}_k\rangle \phi_k \qquad \forall\, z \in X. \tag{2.6.3}$$

Proof. The condition $z \in \mathcal{D}(\widetilde{A})$ implies that the sequence $(a_k) = (\lambda_k \langle z, \tilde{\phi}_k\rangle)$ is in $l^2(\mathbb{N})$. It follows from Proposition 2.5.3 that the definition of $\widetilde{A}$ makes sense, meaning that the series defining $\widetilde{A}z$ is convergent for every $z \in \mathcal{D}(\widetilde{A})$. It is easy to see that $\sigma_p(\widetilde{A}) = \{\lambda_k \mid k \in \mathbb{N}\}$. Take a number s in the complement of the closure of $\sigma_p(\widetilde{A})$. Then the sequence $(|s - \lambda_k|)$ is bounded from below, and it follows from Proposition 2.5.4 that the operator R_s defined below is bounded on X:

$$R_s z = \sum_{k\in\mathbb{N}} \frac{1}{s - \lambda_k} \langle z, \tilde{\phi}_k\rangle \phi_k \qquad \forall\, z \in X.$$

It is easy to see that $R_s(sI - \widetilde{A})z = z$ for all $z \in \mathcal{D}(\widetilde{A})$. On the other hand, it is not difficult to see that for every $z \in X$, $R_s z \in \mathcal{D}(\widetilde{A})$. Then a simple computation shows that

$$(sI - \widetilde{A})R_s z = z \qquad \forall\, z \in X\,.$$

This implies that $s \in \rho(A)$ and $(sI - \widetilde{A})^{-1} = R_s$. □

Proposition 2.6.3. *Let $A : \mathcal{D}(A) \to X$ be diagonalizable. Let (ϕ_k) be a Riesz basis consisting of eigenvectors of A. Let $(\tilde{\phi}_k)$ be the biorthogonal sequence to (ϕ_k) and denote the eigenvalue corresponding to the eigenvector ϕ_k by λ_k. Then*

$$\mathcal{D}(A) = \left\{ z \in X \;\middle|\; \sum_{k\in\mathbb{N}} \left(1 + |\lambda_k|^2\right) |\langle z, \tilde{\phi}_k\rangle|^2 < \infty \right\}, \tag{2.6.4}$$

$$Az = \sum_{k\in\mathbb{N}} \lambda_k \langle z, \tilde{\phi}_k\rangle \phi_k \qquad \forall\, z \in \mathcal{D}(A)\,. \tag{2.6.5}$$

Proof. Let $s \in \rho(A)$. According to Proposition 2.2.18, $(sI - A)^{-1}$ is a diagonalizable (and bounded) operator with the sequence of eigenvalues $\left(\frac{1}{s-\lambda_k}\right)$ and the corresponding sequence of eigenvectors (ϕ_k). Applying $(sI - A)^{-1}$ to both sides of (2.5.2), we get

$$(sI - A)^{-1} z = \sum_{k\in\mathbb{N}} \frac{1}{s - \lambda_k} \langle z, \tilde{\phi}_k\rangle \phi_k \qquad \forall\, z \in X\,. \tag{2.6.6}$$

Define the operator $\widetilde{A}$ by (2.6.1) and (2.6.2). Comparing (2.6.6) with (2.6.3) we see that $(sI - A)^{-1} = (sI - \widetilde{A})^{-1}$, and hence $A = \widetilde{A}$. □

Remark 2.6.4. Combining the last two propositions, we see that if A is diagonalizable, then $\sigma(A)$ is the closure of $\sigma_p(A)$. Applying Proposition 2.5.4 to (2.6.6), we obtain that for every $s \in \rho(A)$,

$$\frac{1}{\inf\limits_{k\in\mathbb{N}} |s - \lambda_k|} \leqslant \|(sI - A)^{-1}\| \leqslant \frac{M}{m} \cdot \frac{1}{\inf\limits_{k\in\mathbb{N}} |s - \lambda_k|}\,.$$

The first inequality above is also an immediate consequence of the more general estimate given in Remark 2.2.8.

Proposition 2.6.5. *With the notation of Proposition 2.6.3, A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X if and only if*

$$\sup_{k\in\mathbb{N}} \operatorname{Re} \lambda_k < \infty\,. \tag{2.6.7}$$

If this is the case, then

$$\sup_{k\in\mathbb{N}} \operatorname{Re} \lambda_k = \omega_0(\mathbb{T}) \tag{2.6.8}$$

and for every $t \geqslant 0$,

$$\mathbb{T}_t z = \sum_{k\in\mathbb{N}} e^{\lambda_k t} \langle z, \tilde{\phi}_k\rangle \phi_k \qquad \forall\, z \in X\,. \tag{2.6.9}$$

A semigroup as in the last proposition is called *diagonalizable*.

Proof. Suppose that (2.6.7) holds. It follows from Proposition 2.5.4 that for each $t \geqslant 0$, (2.6.9) defines a bounded operator $\mathbb{T}_t$ on X. It is easy to see that this family of operators satisfies the semigroup property and it is uniformly bounded for $t \in [0,1]$. It is clear that the function $t \to \mathbb{T}_t z$ is continuous if z is a finite linear combination of the eigenvectors ϕ_k. Since such combinations are dense in X, it follows that $\mathbb{T} = (\mathbb{T}_t)_{t \geqslant 0}$ is a strongly continuous semigroup on X. It is also easy to see that the growth bound of $\mathbb{T}$ is given by formula (2.6.8). Denote the generator of this semigroup by $\widetilde{A}$. It is easy to check that $\widetilde{A}\phi_k = \lambda_k \phi_k$ for all $k \in \mathbb{N}$. Thus $\widetilde{A}$ is a diagonalizable operator, so that its domain is given by (2.6.4). Hence $\widetilde{A} = A$.

Conversely, suppose that A generates a semigroup. According to Proposition 2.3.1, $\rho(A)$ contains a right half-plane and this implies (2.6.7). $\square$

Example 2.6.6. Let (λ_k) be a sequence in $\mathbb{C}$ such that

$$\sup_{k \in \mathbb{N}} \operatorname{Re} \lambda_k = \alpha < \infty .$$

Put $X = l^2$ and let $A : \mathcal{D}(A) \to X$ be defined by

$$(Az)_k = \lambda_k z_k , \qquad \mathcal{D}(A) = \left\{ z \in l^2(\mathbb{N}) \mid \sum_{k \in \mathbb{N}} (1 + |\lambda_k|^2)|z_k|^2 < \infty \right\} .$$

According to Proposition 2.6.2, A is a diagonalizable operator with the sequence of eigenvectors (e_k), which is the standard basis of l^2. By the same proposition, $\sigma(A)$ is the closure in $\mathbb{C}$ of the set $\sigma_p(A) = \{\lambda_k \mid k \in \mathbb{N}\}$ and we have

$$((sI - A)^{-1} z)_k = \frac{z_k}{s - \lambda_k} \qquad \forall\, s \in \rho(A) . \tag{2.6.10}$$

According to Proposition 2.6.5, A is the generator of the semigroup

$$(\mathbb{T}_t\, z)_k = e^{\lambda_k t} z_k \qquad \forall\, k \in \mathbb{N},$$

and the growth bound of this semigroup is $\omega_0(\mathbb{T}) = \alpha$.

Such a semigroup is called a *diagonal* semigroup, and A is also called *diagonal*. We shall use the notation $A = \operatorname{diag}\,(\lambda_k)$ for such an A. Every diagonalizable semigroup $\mathbb{T}$ is similar to a diagonal semigroup, the similarity operator being Q from Definition 2.5.1. This means that $(Q\mathbb{T}_t Q^{-1})_{t \geqslant 0}$ is a diagonal semigroup.

Proposition 2.6.7. *Assume that $A : \mathcal{D}(A) \to X$ is diagonalizable, and its sequence of eigenvalues (λ_k) satisfies, for some $a, b, p \geqslant 0$,*

$$\operatorname{Re} \lambda_k \leqslant 0 , \qquad |\operatorname{Im} \lambda_k| \leqslant a + b|\operatorname{Re} \lambda_k|^p \qquad \forall\, k \in \mathbb{N} .$$

Let $\mathbb{T}$ be the semigroup generated by A. Then

$$\mathbb{T}_t z \in \mathcal{D}(A^\infty) \qquad \forall\, z \in X,\ t > 0 .$$

Proof. Let (ϕ_k) be a Riesz basis in X consisting of eigenvectors of A, let $(\tilde{\phi}_k)$ be the biorthogonal sequence and assume that $A\phi_k = \lambda_k\phi_k$. To prove that $\mathbb{T}_t z \in \mathcal{D}(A)$ for all $z \in X$ and $t > 0$, according to Proposition 2.6.3 we have to show that

$$\sum_{k\in\mathbb{N}}(1 + |\lambda_k|^2) \cdot |\langle \mathbb{T}_t z, \tilde{\phi}_k\rangle|^2 < \infty \qquad \forall\, z \in X,\ t > 0. \tag{2.6.11}$$

According to Proposition 2.6.5, we have

$$\langle \mathbb{T}_t z, \tilde{\phi}_k\rangle = e^{\lambda_k t}\langle z, \tilde{\phi}_k\rangle \qquad \forall\, z \in X\,,\ t \geqslant 0\,.$$

It is easy to see that under the assumptions of the proposition, for every $t > 0$, the sequence $((1 + |\lambda_k|^2)|e^{\lambda_k t}|^2)$ is bounded, because $\operatorname{Re}\lambda_k \leqslant 0$ and

$$(1 + |\lambda_k|^2)|e^{\lambda_k t}|^2 \leqslant (1 + |\operatorname{Re}\lambda_k|^2 + (a + b|\operatorname{Re}\lambda_k|^p)^2)e^{2\operatorname{Re}\lambda_k t}\,.$$

Recall from Proposition 2.5.2 that $\sum_{k\in\mathbb{N}} |\langle z, \tilde{\phi}_k\rangle|^2 < \infty$. Combining these facts, we obtain that (2.6.11) holds, so that $\operatorname{Ran}\mathbb{T}_t \subset \mathcal{D}(A)$ for all $t > 0$.

We prove by induction that for every $n \in \{0, 1, 2, \ldots\}$, the following statement holds: $\operatorname{Ran}\mathbb{T}_t \subset \mathcal{D}(A^{2^n})$ for every $t > 0$. Assume that this statement holds for some $n \in \{0, 1, 2, \ldots\}$ (and every $t > 0$). Choose $\beta \in \rho(A)$, then it follows that

$$(\beta I - A)^{2^n}\mathbb{T}_{\frac{t}{2}} z \in X \qquad \forall\, z \in X\,,\ t > 0\,.$$

Apply $\mathbb{T}_{\frac{t}{2}}$ to both sides, then we obtain that

$$(\beta I - A)^{2^n}\mathbb{T}_t z \in \mathcal{D}(A^{2^n}) \qquad \forall\, z \in X\,,\ t > 0\,.$$

Apply $(\beta I - A)^{-2^n}$ to both sides, which shows that $\operatorname{Ran}\mathbb{T}_t \subset \mathcal{D}(A^{2^{n+1}})$, so that the induction works. Now it is obvious that $\operatorname{Ran}\mathbb{T}_t \subset \mathcal{D}(A^\infty)$. □

Example 2.6.8. Here we construct the semigroup associated with the equations modeling the heat propagation in a rod of length π and with zero temperature at both ends. The connection between this semigroup and the corresponding partial differential one-dimensional heat equation will be explained in Remark 2.6.9.

Let $X = L^2[0, \pi]$ and let A be defined by

$$\mathcal{D}(A) = \mathcal{H}^2(0, \pi) \cap \mathcal{H}^1_0(0, \pi),$$
$$Az = \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} \qquad \forall\, z \in \mathcal{D}(A)\,.$$

For $k \in \mathbb{N}$, let $\phi_k \in \mathcal{D}(A)$ be defined by

$$\phi_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx) \qquad \forall\, x \in (0, \pi)\,.$$

Then (ϕ_k) is an orthonormal basis in X and we have

$$A\phi_k = -k^2\phi_k \qquad \forall\, k \in \mathbb{N}.$$

Simple considerations about the differential equation $Az = f$, with $f \in L^2[0,\pi]$, show that $0 \in \rho(A)$. Thus we have shown that A is diagonalizable.

According to Proposition 2.6.5, A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X given by

$$\mathbb{T}_t z = \sum_{k\in\mathbb{N}} e^{-k^2 t}\langle z, \phi_k\rangle\phi_k \qquad \forall\, t \geqslant 0, \quad z \in X. \tag{2.6.12}$$

It is now clear that this semigroup is exponentially stable. Moreover, according to Proposition 2.6.7, we have $\mathbb{T}_t z \in \mathcal{D}(A^\infty)$ for all $z \in X$ and $t > 0$. For generalizations of this example see Sections 3.5 and 3.6.

Remark 2.6.9. The interpretation in terms of PDEs of the semigroup constructed in Example 2.6.8 is the following: For $w_0 \in \mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi)$ there exists a unique function w continuous from $[0,\infty)$ to $\mathcal{H}^2(0,\pi)\cap\mathcal{H}_0^1(0,\pi)$ (endowed with the $\mathcal{H}^2(0,\pi)$ norm) and continuously differentiable from $[0,\infty)$ to $L^2[0,\pi]$, satisfying

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t), & x \in (0,\pi),\ t \geqslant 0, \\ w(0,t) = 0, \quad w(\pi,t) = 0, & t \in [0,\infty), \\ w(x,0) = w_0(x), & x \in (0,\pi). \end{cases} \tag{2.6.13}$$

Indeed, by setting $z(t) = w(\cdot,t)$, it is easy to check that w satisfies the above conditions iff z is continuous with values in $\mathcal{D}(A)$ (endowed with the graph norm), continuously differentiable with values in X and it satisfies the equations

$$\dot{z}(t) = Az(t) \ \ \forall t \geqslant 0, \quad z(0) = w_0.$$

Since A generates a semigroup on X, we can apply Proposition 2.3.5 to obtain the existence and uniqueness of z (and consequently of w) with the above properties. Moreover, from (2.6.12) it follows that w has the exponential decay property

$$\|w(\cdot,t)\|_{L^2[0,\pi]} \leqslant e^{-t}\|w_0\|_{L^2[0,\pi]} \qquad \forall\, t \geqslant 0.$$

Example 2.6.10. If we model heat propagation in a rod of length π, with zero heat flux at the left end and with the temperature zero imposed at the right end, we obtain equations which differ from (2.6.13) only by a boundary condition:

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t), & x \in (0,\pi),\ t \geqslant 0, \\ \dfrac{\partial w}{\partial x}(0,t) = 0, \quad w(\pi,t) = 0, & t \in [0,\infty), \\ w(x,0) = w_0(x), & x \in (0,\pi). \end{cases} \tag{2.6.14}$$

Let $X = L^2[0, \pi]$ and let A be defined by

$$\mathcal{D}(A) = \left\{ z \in \mathcal{H}^2(0, \pi) \;\middle|\; \frac{\mathrm{d}z}{\mathrm{d}x}(0) = z(\pi) = 0 \right\},$$
$$Az = \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} \qquad \forall\, z \in \mathcal{D}(A).$$

It is easy to check the following properties.

- If $z(t) = w(\cdot, t)$, then w satisfies (2.6.14) iff z is continuous with values in $\mathcal{D}(A)$ (endowed with the graph norm), continuously differentiable with values in X and it satisfies the equations

$$\dot{z}(t) = Az(t) \quad \forall t \geqslant 0, \quad z(0) = w_0 .$$

- The family of functions $(\varphi_k)_{k\in\mathbb{N}}$, defined by

$$\varphi_k(x) = \sqrt{\frac{2}{\pi}} \cos\left[\left(k - \frac{1}{2}\right) x\right] \qquad \forall\, k \in \mathbb{N}, \quad x \in (0, \pi),$$

consists of eigenvectors of A, it is an orthonormal basis in X and the corresponding eigenvalues are

$$\lambda_k = -\left(k - \frac{1}{2}\right)^2 \qquad \forall\, k \in \mathbb{N}.$$

- $0 \in \rho(A)$.

Consequently, A is diagonalizable and, according to Proposition 2.6.5, A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X given by

$$\mathbb{T}_t z = \sum_{k\in\mathbb{N}} e^{-(k-\frac{1}{2})^2 t} \langle z, \varphi_k \rangle \varphi_k \qquad \forall\, t \geqslant 0, \quad z \in X . \tag{2.6.15}$$

Note that this semigroup is also exponentially stable.

Remark 2.6.11. Everything we have said in this section remains valid if we replace $\mathbb{N}$ with another countable index set, such as $\mathbb{Z}$. Sometimes it is more convenient to work with a different index set, as the following example shows.

Example 2.6.12. Let $X = L^2[0, 1]$. For $\alpha \in \mathbb{R}$ we define $A : \mathcal{D}(A) \to X$ by

$$\mathcal{D}(A) = \left\{ z \in \mathcal{H}^1(0, 1) \;\middle|\; z(1) = e^{\alpha} z(0) \right\},$$
$$Az = \frac{\mathrm{d}z}{\mathrm{d}x} \qquad \forall\, z \in \mathcal{D}(A).$$

For $k \in \mathbb{Z}$ we set $\lambda_k = \alpha + 2k\pi i$ and we define $\phi_k \in \mathcal{D}(A)$ by

$$\phi_k(x) = e^{\alpha x} e^{2k\pi i x} \qquad \forall\, x \in (0, 1).$$

Then

$$A\phi_k = \lambda_k \phi_k \qquad \forall\, k \in \mathbb{Z}\,.$$

Define the operator $Q \in \mathcal{L}(X)$ by

$$(Qz)(x) = e^{-\alpha x} z(x) \qquad \forall\, x \in (0,1)\,.$$

Then it is clear that Q is invertible and $(Q\phi_k)$ is an orthonormal basis in X. Hence, (ϕ_k) is a Riesz basis in X.

For $\alpha \neq 0$, elementary considerations show that $0 \in \rho(A)$. For $\alpha = 0$, similar considerations show that $1 \in \rho(A)$. Hence, regardless of α, A is diagonalizable.

According to Proposition 2.6.5, A is the generator of a strongly continuous semigroup on X. Note that for $t \in [0,1]$, $\mathbb{T}_t$ is described by

$$(\mathbb{T}_t z)(x) = \begin{cases} z(x+t) & \text{if} \quad t + x \leqslant 1, \\ e^{\alpha} z(x+t-1) & \text{else}. \end{cases}$$

For other simple examples of diagonalizable semigroups (corresponding to the string equation) see Examples 2.7.13 and 2.7.15.

Example 2.6.13. This example shows the importance of imposing the condition $\rho(A) \neq \emptyset$ in the definition of a diagonalizable operator. We show that whithout this condition, the operator cannot be represented as in Proposition 2.6.3.

Let $X = L^2[0,\pi]$ and let the operator A be defined by

$$\mathcal{D}(A) = \mathcal{H}^2(0,\pi)\,, \qquad Az = \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} \qquad \forall\, z \in \mathcal{D}(A)\,.$$

For $k \in \mathbb{N}$, let $\phi_k \in \mathcal{D}(A)$ be defined as in Example 2.6.8. Then (ϕ_k) is an orthonormal basis in X and (as in Example 2.6.8) we have

$$A\phi_k = -k^2 \phi_k \qquad \forall\, k \in \mathbb{N}\,.$$

Simple considerations about the differential equation $Az = sz$ show that every $s \in \mathbb{C}$ is an eigenvalue of A, so that A is not diagonalizable in the sense of Definition 2.6.1. Formula (2.6.5) does not hold for A. Indeed, consider the constant function $z(x) = 1$ for all $x \in (0,\pi)$. Then $Az = 0$ but formula (2.6.5) would yield a non-zero series (which is not convergent in X).

Let us denote by A_1 the diagonalizable operator introduced in Example 2.6.8 (there, this operator was denoted by A). Then clearly A is an extension of A_1. More precisely, if we denote by V the space of affine functions on $(0,\pi)$, then $\dim V = 2$ and $\mathcal{D}(A) = \mathcal{D}(A_1) + V$. Hence, the graph $G(A)$ is the sum of $G(A_1)$ and a two-dimensional space. Since A_1 is closed, it follows that also A is closed.

2.7 Strongly continuous groups

An operator $T \in \mathcal{L}(X)$ is called *left-invertible* if there exists $T_{\mathrm{left}}^{-1} \in \mathcal{L}(X)$ such that $T_{\mathrm{left}}^{-1}T = I$. It is easy to see that this is equivalent to the existence of $m > 0$ such that

$$\|Tz\| \geqslant m\|z\| \qquad \forall\, z \in X\,.$$

For this reason, left-invertible operators are also called *bounded from below.*

$T \in \mathcal{L}(X)$ is called *right-invertible* if there exists an operator $T_{\mathrm{right}}^{-1} \in \mathcal{L}(X)$ such that $TT_{\mathrm{right}}^{-1} = I$. It is easy to see that this is equivalent to $\mathrm{Ran}\, T = X$ (i.e., T is onto). Indeed, this follows from Proposition 12.1.2 with $F = I$.

Definition 2.7.1. Let $\mathbb{T}$ be a strongly continuous semigroup on X. $\mathbb{T}$ is called *left-invertible* (respectively, *right-invertible*) if for some $\tau > 0$, $\mathbb{T}_\tau$ is left-invertible (respectively, right-invertible). The semigroup is called *invertible* if it is both left-invertible and right-invertible.

Proposition 2.7.2. *Let $\mathbb{T}$ be a strongly continuous semigroup on X.*

If $\mathbb{T}$ is right-invertible, then $\mathbb{T}_t$ is right-invertible for every $t > 0$.

If $\mathbb{T}$ is left-invertible, then $\mathbb{T}_t$ is left-invertible for every $t > 0$.

Proof. In order to prove the first statement, let $\tau > 0$ be such that $\mathbb{T}_\tau$ is onto. Let $t > 0$ and let $n \in \mathbb{N}$ be such that $t \leqslant n\tau$. Clearly, $\mathbb{T}_{n\tau}$ is onto. Put $\varepsilon = n\tau - t$. Then from $\mathbb{T}_{n\tau} = \mathbb{T}_t\mathbb{T}_\varepsilon$ we see that $\mathbb{T}_t$ is onto, so that the first statement holds.

Let $\tau > 0$ be such that $\mathbb{T}_\tau$ is bounded from below. Let $t > 0$ and let $n \in \mathbb{N}$ be such that $t \leqslant n\tau$. Clearly, $\mathbb{T}_{n\tau}$ is bounded from below. Put $\varepsilon = n\tau - t$. Then from $\mathbb{T}_{n\tau} = \mathbb{T}_\varepsilon\mathbb{T}_t$ we see that $\mathbb{T}_t$ is bounded from below. □

Definition 2.7.3. Let X be a Hilbert space. A family $\mathbb{T} = (\mathbb{T}_t)_{t\in\mathbb{R}}$ of operators in $\mathcal{L}(X)$ is a *strongly continuous group* on X if it has properties (1) and (3) from Definition 2.1.1 and (instead of property (2)) it has the group property:

$$\mathbb{T}_{t+\tau} = \mathbb{T}_t\mathbb{T}_\tau \text{ for every } t, \tau \in \mathbb{R}.$$

The generator of such a group is defined in the same way as for semigroups.

Proposition 2.7.4. *Let $\mathbb{T}$ be a strongly continuous semigroup on X and assume that for some $\theta > 0$, $\mathbb{T}_\theta$ is invertible. Then $\mathbb{T}_t$ is invertible for every $t > 0$ and $\mathbb{T}$ can be extended to a strongly continuous group by putting $\mathbb{T}_{-t} = \mathbb{T}_t^{-1}$.*

Proof. The fact that $\mathbb{T}_t$ is invertible for every $t > 0$ follows from Proposition 2.7.2. To verify the group property for the extended family, we multiply the formula expressing the semigroup property with $\mathbb{T}_{-\tau}$ and/or we take the inverse of both sides, in order to cover all the possible cases. □

Remark 2.7.5. Note that in the definition of a strongly continuous group, the only continuity assumption is the right continuity of $\mathbb{T}_t z$ at $t = 0$ (for every $z \in X$). However, using the group property and part (3) of Proposition 2.1.2, it follows that the function $\varphi(t, z) = \mathbb{T}_t z$ is continuous on $\mathbb{R} \times X$ (with the product topology).

Remark 2.7.6. If $\mathbb{T}$ is a strongly continuous group on X, with generator A, then the family $\mathbb{S}$ defined by $\mathbb{S}_t = \mathbb{T}_{-t}$ is another such group, and its generator is $-A$. Indeed, let $\widetilde{A}$ be the generator of $\mathbb{S}$. For $z \in \mathcal{D}(\widetilde{A})$ we have

$$\widetilde{A}z = \lim_{t \to 0,\ t>0} \frac{1}{t}(\mathbb{S}_t z - z) = \lim_{t \to 0,\ t>0} \frac{1}{t}\mathbb{T}_t(\mathbb{S}_t z - z) = \lim_{t \to 0,\ t>0} -\frac{1}{t}(\mathbb{T}_t z - z),$$

which shows that $-A$ is an extension of $\widetilde{A}$. Similarly, we can show (using also the previous remark) that $-\widetilde{A}$ is an extension of A, so that in fact $\widetilde{A} = -A$.

Remark 2.7.7. Let $\mathbb{T}$ be a strongly continuous group on X, with generator A. Then $\sigma(A)$ is contained in a vertical strip in $\mathbb{C}$. Indeed, let us again denote $\mathbb{S}_t = \mathbb{T}_{-t}$ so that, by the previous remark, $\mathbb{S}$ is strongly continuous group with generator $-A$. We know from Proposition 2.3.1 that all $s \in \mathbb{C}$ with $\operatorname{Re} s > \omega_0(\mathbb{T})$ are in $\rho(A)$, and all $s \in \mathbb{C}$ with $\operatorname{Re} s > \omega_0(\mathbb{S})$ are in $\rho(-A)$. Hence,

$$-\omega_0(\mathbb{S}) < \operatorname{Re}\lambda < \omega_0(\mathbb{T}) \qquad \forall\, \lambda \in \sigma(A).$$

Moreover, by Corollary 2.3.3, $(sI - A)^{-1}$ is uniformly bounded for s in any right half-plane to the right of $\omega_0(\mathbb{T})$ and in any left half-plane to the left of $-\omega_0(\mathbb{S})$.

Proposition 2.7.8. *Suppose that $A : \mathcal{D}(A) \to X$ is the generator of a strongly continuous semigroup $\mathbb{T}$ on X, and $-A$ is the generator of a strongly continuous semigroup $\mathbb{S}$ on X. Extend the family $\mathbb{T}$ to all of $\mathbb{R}$ by putting $\mathbb{T}_{-t} = \mathbb{S}_t$, for all $t > 0$. Then $\mathbb{T}$ is a strongly continuous group on X.*

Proof. For $z \in \mathcal{D}(A)$ and $t > 0$ we compute, using Proposition 2.1.5,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{T}_t\mathbb{S}_t z = A\mathbb{T}_t\mathbb{S}_t z + \mathbb{T}_t(-A)\mathbb{S}_t z = 0.$$

This implies that $\mathbb{T}_t\mathbb{S}_t z = z$ for all $t \geqslant 0$. By a similar argument, $\mathbb{S}_t\mathbb{T}_t z = z$ for all $t \geqslant 0$. Since $\mathcal{D}(A)$ is dense in X, we conclude that $\mathbb{T}_t$ is invertible and its inverse is $\mathbb{S}_t$. By Proposition 2.7.4, $\mathbb{T}$ can be extended to a strongly continuous group in the manner described in the proposition. □

Remark 2.7.9. If $\mathbb{T}$ is a diagonalizable semigroup as in Proposition 2.6.5, then it is invertible iff $\inf \operatorname{Re}\lambda_k > -\infty$. In this case, the extension of $\mathbb{T}$ to a strongly continuous group is still given by (2.6.9). All this is easy to verify.

An operator $T \in \mathcal{L}(X)$ is called *isometric* if $T^*T = I$. Equivalently, $\|Tx\| = \|x\|$ holds for all $x \in X$. A strongly continuous semigroup $\mathbb{T}$ on X is called *isometric* if $\mathbb{T}_t$ is isometric for every $t > 0$. (Requiring that $\mathbb{T}_t$ is isometric for one $t > 0$ is not equivalent.) It is clear that an isometric semigroup is left-invertible. A simple example of an isometric semigroup will be given in Section 2.8.

An operator $U \in \mathcal{L}(X)$ is called *unitary* if $UU^* = U^*U = I$. Equivalently, U is isometric and onto (this characterization of unitary operators avoids refering to

U^*). A strongly continuous semigroup $\mathbb{T}$ on X is called *unitary* if $\mathbb{T}_t$ is unitary for every $t > 0$. It is clear that a unitary semigroup can be extended to a group, which is then called a *unitary group*. In Section 3.8 we shall give a simple characterization of the generators of unitary groups (the theorem of Stone). Three simple examples of unitary groups will be given in this section.

Remark 2.7.10. If there is in X an orthonormal basis formed by eigenvectors of A, then $\mathbb{T}$ is unitary iff $\operatorname{Re}\lambda = 0$ for all $\lambda \in \sigma(A)$. This is easy to check, by expressing $\mathbb{T}$ in terms of its eigenvalues and eigenfunctions, as in (2.6.9).

Example 2.7.11. Take $X = L^2(\mathbb{R})$ and for every $t \in \mathbb{R}$ and $z \in X$ define

$$(\mathbb{T}_t z)(x) = z(t+x) \qquad \forall\, x \in \mathbb{R}.$$

Then $\mathbb{T}$ is a unitary group, called the *bilateral left shift group*. (The arguments used for this example resemble those used for Example 2.3.7.) It is not difficult to verify that the generator of $\mathbb{T}$ is $A = \frac{\mathrm{d}}{\mathrm{d}x}$, defined on the Sobolev space $\mathcal{D}(A) = \mathcal{H}^1(\mathbb{R})$, and we have $\sigma(A) = i\mathbb{R}$ (the imaginary axis).

Now let us consider on X the equivalent norm $\|\cdot\|_e$ defined by

$$\|z\|_e^2 = \int_{-\infty}^0 |z(x)|^2\,\mathrm{d}x + 4\int_0^\infty |z(x)|^2\,\mathrm{d}x$$

(with this norm, X is still a Hilbert space). The same group $\mathbb{T}$ introduced earlier will now have (with respect to the new norm) the properties

$$\|\mathbb{T}_t\| = 1 \quad \text{for} \quad t \geqslant 0, \qquad \|\mathbb{T}_t\| = 2 \quad \text{for} \quad t < 0.$$

Example 2.7.12. The semigroups in Example 2.6.12 are invertible. In particular, for $\alpha = 0$ we obtain a unitary group:

$$(\mathbb{T}_t z)(x) = z(t\dot{+}x) \qquad \forall\, x \in [0,1],\ t \in \mathbb{R},$$

where $\dot{+}$ denotes addition modulo 1. This $\mathbb{T}$ is called the *periodic left shift group*. Its generator is $A = \frac{\mathrm{d}}{\mathrm{d}x}$, defined on $\mathcal{D}(A) = \mathcal{H}^1_P(0,1) = \{z \in \mathcal{H}^1(0,1) \mid z(0) = z(1)\}$, and $\sigma(A) = \{2k\pi i,\ k \in \mathbb{Z}\}$. The eigenvectors of A (given in Example 2.6.12) become the standard orthonormal basis in X used for Fourier series.

Example 2.7.13. In this example we construct the semigroup associated with the equations modeling the vibration of an elastic string of length π which is fixed at both ends. The connection between this semigroup and a one-dimensional wave equation (also called the string equation) will be explained in Remark 2.7.14.

Denote $X = \mathcal{H}^1_0(0,\pi) \times L^2[0,\pi]$, which is a Hilbert space with the scalar product

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle = \int_0^\pi \frac{\mathrm{d}f_1}{\mathrm{d}x}(x)\overline{\frac{\mathrm{d}f_2}{\mathrm{d}x}(x)}\,\mathrm{d}x + \int_0^\pi g_1(x)\overline{g_2(x)}\,\mathrm{d}x.$$

We define $A : \mathcal{D}(A) \to X$ by

$$\mathcal{D}(A) = \left[\mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_0(0,\pi)\right] \times \mathcal{H}^1_0(0,\pi),$$
$$A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A).$$

We denote by $\mathbb{Z}^*$ the set of all non-zero integers. For $n \in \mathbb{Z}^*$, denote $\varphi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$. It is known from the theory of Fourier series that the family $(\varphi_n)_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2[0,\pi]$. This implies that the family $(\phi_n)_{n\in\mathbb{Z}^*}$, defined by

$$\phi_n = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{in}\varphi_n \\ \varphi_n \end{bmatrix} \qquad \forall\, n \in \mathbb{Z}^*, \tag{2.7.1}$$

is an orthonormal basis in X. Indeed, it is easy to see that this family is orthonormal in X. If $z = \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in X$ is such that $\langle z, \phi_n\rangle = 0$ for all $n \in \mathbb{Z}^*$, then the same is true for $\overline{z} = \begin{bmatrix} \overline{f} \\ \overline{g} \end{bmatrix}$ (here we have used that $\overline{\phi}_n = -\phi_{-n}$). It follows that $\operatorname{Re} z$ and $\operatorname{Im} z$ are also orthogonal to ϕ_n, for every $n \in \mathbb{Z}^*$. Since

$$\operatorname{Re}\langle \operatorname{Re} z, \phi_n\rangle = \frac{1}{\sqrt{2}} \langle \operatorname{Re} g, \varphi_n\rangle \qquad \forall\, n \in \mathbb{N},$$

we obtain that $\operatorname{Re} g = 0$. By looking at $\operatorname{Re}\langle \operatorname{Im} z, \phi_n\rangle$, we obtain, similarly, that $\operatorname{Im} g = 0$. Thus, $g = 0$. By looking at $\operatorname{Im}\langle \operatorname{Re} z, \phi_n\rangle = \frac{1}{\sqrt{2}}\langle \frac{\mathrm{d}(\operatorname{Re} f)}{\mathrm{d}x}, \frac{\mathrm{d}\varphi}{\mathrm{d}x}\rangle$ for all $n \in \mathbb{N}$, we obtain that $\frac{\mathrm{d}(\operatorname{Re} f)}{\mathrm{d}x}$ is constant. By a similar argument, $\frac{\mathrm{d}(\operatorname{Im} f)}{\mathrm{d}x}$ is constant. Thus, f is an affine function. Since $f(0) = f(\pi) = 0$, we obtain $f = 0$. We have shown that $z = 0$, so that the family $(\phi_n)_{n\in\mathbb{Z}^*}$ is an orthonormal basis in X.

The vectors ϕ_n from (2.7.1) are eigenvectors of A and the corresponding eigenvalues are $\lambda_n = in$, with $n \in \mathbb{Z}^*$. Moreover, it is easy to check that $0 \in \rho(A)$, so that A is diagonalizable. According to Remark 2.7.10 the operator A generates a unitary group $\mathbb{T}$ on X. According to Proposition 2.6.5, $\mathbb{T}$ is given by

$$\mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} = \sum_{n\in\mathbb{Z}^*} e^{int} \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \phi_n \right\rangle \phi_n \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X.$$

From the above relation it follows that

$$\mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}^*} e^{int} \left(\frac{i}{n} \left\langle \frac{\mathrm{d}f}{\mathrm{d}x}, \frac{\mathrm{d}\varphi_n}{\mathrm{d}x} \right\rangle_{L^2[0,\pi]} + \langle g, \varphi_n\rangle_{L^2[0,\pi]} \right) \phi_n. \tag{2.7.2}$$

We shall encounter a generalization of this example in Propositions 3.7.6 and 3.7.7. The existence of an orthonormal basis in X formed of eigenvectors of A follows from the abstract theory, but here we have given an elementary direct proof.

Remark 2.7.14. The interpretation in terms of PDEs of the semigroup constructed in Example 2.7.13 is the following: For $f \in \mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi)$ and $g \in \mathcal{H}_0^1(0,\pi)$, there exists a unique continuous $w : [0,\infty) \to \mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi)$ (endowed with the $\mathcal{H}^2(0,\pi)$ norm), continuously differentiable from $[0,\infty)$ to $\mathcal{H}_0^1(0,\pi)$, satisfying

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2}(x,t) \,=\, \dfrac{\partial^2 w}{\partial x^2}(x,t), & x \in (0,\pi),\ t \geqslant 0, \\ w(0,t) = 0, \quad w(\pi,t) = 0, & t \in [0,\infty), \\ w(x,0) \,=\, f(x), \quad \dfrac{\partial w}{\partial t}(x,0) = g(x), & x \in (0,\pi). \end{cases} \tag{2.7.3}$$

Indeed, by setting

$$z(t) \,=\, \begin{bmatrix} w(\cdot,t) \\ \dfrac{\partial w}{\partial t}(\cdot,t) \end{bmatrix},$$

it is easy to check that w satisfies the above conditions iff z is continuous with values in $\mathcal{D}(A)$ (endowed with the graph norm), continuously differentiable with values in X and it satisfies the equations

$$\dot z(t) \,=\, Az(t) \quad \forall t \geqslant 0, \quad z(0) \,=\, \begin{bmatrix} f \\ g \end{bmatrix}.$$

Since we have shown in Example 2.7.13 that A generates a semigroup on X, we can apply Proposition 2.3.5 to obtain the existence and uniqueness of z (and consequently of w) with the above properties. Moreover, since the semigroup generated by A can be extended to a unitary group, it follows that the solution w of (2.7.3) is defined for $t \in \mathbb{R}$ and it has the "conservation of energy" property*:*

$$\left\| \frac{\partial w}{\partial t}(\cdot,t) \right\|^2_{L^2[0,\pi]} + \left\| \frac{\partial w}{\partial x}(\cdot,t) \right\|^2_{L^2[0,\pi]} \,=\, \|g\|^2_{L^2[0,\pi]} + \left\| \frac{\mathrm{d}f}{\mathrm{d}x} \right\|^2_{L^2[0,\pi]} \qquad \forall\, t \in \mathbb{R}.$$

Example 2.7.15. In this example we construct the semigroup associated with the equations modeling the vibrations of an elastic string which is fixed at the end $x = \pi$ while at the end $x = 0$ it is free to move perpendicularly to the axis of the sting, so that its slope is zero. We indicate how this semigroup is related to the string equation. Since the considerations below are similar to those in Example 2.7.13 and in Remark 2.7.14, we state the results without proof.

Denote

$$\mathcal{H}_R^1(0,\pi) \,=\, \left\{ f \in \mathcal{H}^1(0,\pi) \mid f(\pi) = 0 \right\}.$$

Then $X = \mathcal{H}_R^1(0,\pi) \times L^2[0,\pi]$ is a Hilbert space with the scalar product

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle \,=\, \int_0^\pi \frac{\mathrm{d}f_1}{\mathrm{d}x}(x)\overline{\frac{\mathrm{d}f_2}{\mathrm{d}x}(x)}\,\mathrm{d}x + \int_0^\pi g_1(x)\overline{g_2(x)}\,\mathrm{d}x. \tag{2.7.4}$$

We define $A : \mathcal{D}(A) \to X$ by

$$\mathcal{D}(A) = \left\{ f \in \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_R(0,\pi) \;\middle|\; \frac{\mathrm{d}f}{\mathrm{d}x}(0) = 0 \right\} \times \mathcal{H}^1_R(0,\pi),$$

$$A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A).$$

For $n \in \mathbb{N}$, denote $\varphi_n(x) = \sqrt{\frac{2}{\pi}} \cos\left[\left(n - \frac{1}{2}\right) x\right]$ and $\mu_n = n - \frac{1}{2}$. If $-n \in \mathbb{N}$, we set $\varphi_n = -\varphi_{-n}$ and $\mu_n = -\mu_{-n}$. Then the family

$$\phi_n = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{i\mu_n} \varphi_n \\ \varphi_n \end{bmatrix} \qquad \forall\, n \in \mathbb{Z}^* \tag{2.7.5}$$

is an orthonormal basis in X formed by eigenvectors of A and the corresponding eigenvalues are $\lambda_n = i\mu_n$, with $n \in \mathbb{Z}^*$. Moreover, it is easy to check that $0 \in \rho(A)$, so that A is diagonalizable. By using Remark 2.7.10 and Proposition 2.6.5 we get that A generates a unitary group $\mathbb{T}$ on X, denoted $\mathbb{T}$, which is given by

$$\mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} = \sum_{n \in \mathbb{Z}^*} e^{i\mu_n t} \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \phi_n \right\rangle \phi_n \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X.$$

From the above relation it follows that

$$\mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}^*} e^{i\mu_n t} \left(\frac{i}{\mu_n} \left\langle \frac{\mathrm{d}f}{\mathrm{d}x}, \frac{\mathrm{d}\varphi_n}{\mathrm{d}x} \right\rangle_{L^2[0,\pi]} + \langle g, \varphi_n \rangle_{L^2[0,\pi]} \right) \phi_n. \tag{2.7.6}$$

The interpretation in PDEs terms of the above semigroup is the following: For every $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{D}(A)$, the initial and boundary value problem

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t), & x \in (0,\pi),\ t \geqslant 0, \\ \dfrac{\partial w}{\partial x}(0,t) = 0, \quad w(\pi,t) = 0, & t \in [0,\infty), \\ w(x,0) = f(x), \quad \dfrac{\partial w}{\partial t}(x,0) = g(x), & x \in (0,\pi), \end{cases} \tag{2.7.7}$$

admits a unique solution

$$w \in C([0,\infty); \mathcal{H}^2(0,\pi)) \cap C^1([0,\infty); \mathcal{H}^1(0,\pi))$$

which is given by

$$\begin{bmatrix} w(\cdot,t) \\ \dfrac{\partial w}{\partial t}(\cdot,t) \end{bmatrix} = \mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A).$$

2.8 The adjoint semigroup

Let $A : \mathcal{D}(A) \to X$ be a densely defined operator (by this we mean that $\mathcal{D}(A)$ is dense in X). The *adjoint* of A, denoted A^*, is an operator defined on the domain

$$\mathcal{D}(A^*) = \left\{ y \in X \;\middle|\; \sup_{z \in \mathcal{D}(A),\ z \neq 0} \frac{|\langle Az, y \rangle|}{\|z\|} < \infty \right\}.$$

Equivalently, $y \in \mathcal{D}(A^*)$ iff the functional $z \to \langle Az, y \rangle$ is bounded. Since $\mathcal{D}(A)$ has been assumed to be dense, this functional has a unique bounded extension to all of X. By the Riesz representation theorem, there exists a unique $w \in X$ such that $\langle Az, y \rangle = \langle z, w \rangle$. Then we define $A^* y = w$, so that

$$\langle Az, y \rangle = \langle z, A^* y \rangle \qquad \forall\, z \in \mathcal{D}(A),\ y \in \mathcal{D}(A^*).$$

This is similar to the familiar case when $A \in \mathcal{L}(X)$ and hence $A^* \in \mathcal{L}(X)$.

We denote the orthogonal complement of a subspace $V \subset X$ by $V^\perp$. This is a closed subspace of X (regardless if V is closed or not). It is easy to verify that for every densely defined A, the graph of A^* (as defined in Definition 2.2.1) is

$$G(A^*) = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} G(A)^\perp = \left(\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} G(A) \right)^\perp . \tag{2.8.1}$$

This implies that the operator A^* is closed (see also Rudin [195, pp. 334–335]).

Proposition 2.8.1. *If $A : \mathcal{D}(A) \to X$ is densely defined and closed, then $\mathcal{D}(A^*)$ is dense in X and $A^{**} = A$.*

Proof. If $\mathcal{D}(A^*)$ were not dense, then we could find a $z \in X$ such that $z \neq 0$, $\left[\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right] \in G(A^*)^\perp$. According to (2.8.1) and using the fact that $G(A)$ is closed,

$$G(A^*)^\perp = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} G(A),$$

so that we obtain $\left[\begin{smallmatrix} 0 \\ z \end{smallmatrix}\right] \in G(A)$, which is absurd.

The formula $A^{**} = A$ follows easily by applying (2.8.1) twice. $\square$

Remark 2.8.2. Let $A : \mathcal{D}(A) \to X$ be densely defined. Using the definition of A^*, it is easy to check that

$$(\mathrm{Ran}\ A)^\perp = \mathrm{Ker}\ A^*, \qquad (\mathrm{Ran}\ A^*)^\perp \supset \mathrm{Ker}\ A.$$

If moreover A is closed, then using the fact that $A^{**} = A$, we obtain

$$(\mathrm{Ran}\ A^*)^\perp = \mathrm{Ker}\ A.$$

Remark 2.8.3. If $C : \mathcal{D}(C) \to Y$ is densely defined in X and closed, and if Y is finite dimensional, then $C \in \mathcal{L}(X, Y)$. Indeed, according to Proposition 2.8.1, C^* is densely defined in Y, so that in fact C^* is defined on all of Y, which implies (since $\dim Y < \infty$) that $C^* \in \mathcal{L}(Y, X)$. Since (again by Proposition 2.8.1) $C = C^{**}$, we obtain that $\mathcal{D}(C) = X$ and C is bounded.

Proposition 2.8.4. *Let $A : \mathcal{D}(A) \to X$ be a densely defined operator and let $s \in \rho(A)$. Then $\overline{s} \in \rho(A^*)$ and*

$$\left[(sI - A)^{-1}\right]^* = (\overline{s}I - A^*)^{-1}. \tag{2.8.2}$$

Proof. First we show that Ran $[(sI - A)^{-1}]^* \subset \mathcal{D}(A^*)$ and

$$(\overline{s}I - A^*)[(sI - A)^{-1}]^* = I. \tag{2.8.3}$$

Take $f \in X$ and denote $z = \left[(sI - A)^{-1}\right]^* f$. For every $y \in \mathcal{D}(A)$,

$$\langle Ay, z\rangle = \left\langle (sI - A)^{-1}Ay, f\right\rangle = \left\langle s(sI - A)^{-1}y, f\right\rangle - \langle y, f\rangle.$$

Since $(sI - A)^{-1} \in \mathcal{L}(X)$, the right-hand side above is bounded with respect to y. By the definition of A^*, we obtain that $z \in \mathcal{D}(A^*)$. Moreover,

$$\langle Ay, z\rangle = \left\langle sy, \left[(sI - A)^{-1}\right]^* f\right\rangle - \langle y, f\rangle = \langle sy, z\rangle - \langle y, f\rangle,$$

i.e., $\langle (sI - A)y, z\rangle = \langle y, f\rangle$ for all $y \in \mathcal{D}(A)$. This implies that $(\overline{s}I - A^*)z = f$, so that (2.8.3) holds, in particular $\overline{s}I - A^*$ is onto.

Since $\rho(A)$ is not empty, we know that A is closed, hence $sI - A$ is closed and Remark 2.8.2 applies to it:

$$\text{Ker}\,(\overline{s}I - A^*) = [\text{Ran}\,(sI - A)]^{\perp} = \{0\}.$$

Thus, $(\overline{s}I - A^*)$ is one-to-one, hence it is invertible and we denote its inverse by $(\overline{s}I - A^*)^{-1}$. This operator maps X onto $\mathcal{D}(A^*)$, but we do not know yet that it is bounded. Applying $(\overline{s}I - A^*)^{-1}$ to both sides of (2.8.3), we obtain that (2.8.2) holds. In particular, now we see that $(\overline{s}I - A^*)^{-1}$ is bounded, so that $\overline{s} \in \rho(A^*)$. □

Proposition 2.8.5. *Let $\mathbb{T}$ be a strongly continuous semigroup on X. Then the family of operators $\mathbb{T}^* = (\mathbb{T}_t^*)_{t\geqslant 0}$ is also a strongly continuous semigroup on X, and its generator is A^*.*

Proof. It is clear that $\mathbb{T}^*$ satisfies $\mathbb{T}_0^* = I$ and the semigroup property (the first two properties in Definition 2.1.1). We have to prove the strong continuity of the family $\mathbb{T}^*$. For any $v \in \mathcal{D}(A^*)$, $w \in X$ and $\tau \geqslant 0$ we have, using Remark 2.1.7,

$$\langle (\mathbb{T}_\tau^* - I)v, w\rangle = \langle v, (\mathbb{T}_\tau - I)w\rangle = \left\langle v, A\int_0^\tau \mathbb{T}_\sigma w\,\mathrm{d}\sigma\right\rangle = \left\langle A^*v, \int_0^\tau \mathbb{T}_\sigma w\,\mathrm{d}\sigma\right\rangle.$$

Let $M \geqslant 1$ be such that $\|\mathbb{T}_\sigma\| \leqslant M$ for all $\sigma \in [0,1]$. Then the above formula shows that for $\tau \leqslant 1$ we have $|\langle(\mathbb{T}_\tau^* - I)v, w\rangle| \leqslant M\tau\|A^*v\| \cdot \|w\|$, whence

$$\|\mathbb{T}_\tau^* v - v\| \leqslant M\tau\|A^*v\| \qquad \forall\, v \in \mathcal{D}(A^*),\ \tau \in [0,1]. \tag{2.8.4}$$

Let $\varepsilon > 0$ and $z \in X$. Since $\mathcal{D}(A^*)$ is dense, we can find $v \in \mathcal{D}(A^*)$ such that $\|z - v\| \leqslant \frac{\varepsilon}{2(M+1)}$. According to (2.8.4), we can find $\tau_\varepsilon \in (0,1]$ such that

$$\|\mathbb{T}_\tau^* v - v\| \leqslant \frac{\varepsilon}{2} \qquad \forall\, \tau \leqslant \tau_\varepsilon .$$

Then for $\tau \leqslant \tau_\varepsilon$ we have

$$\begin{aligned}\|\mathbb{T}_\tau^* z - z\| &\leqslant \|\mathbb{T}_\tau^* z - \mathbb{T}_\tau^* v\| + \|\mathbb{T}_\tau^* v - v\| + \|v - z\| \\ &\leqslant (M+1)\|v - z\| + \|\mathbb{T}_\tau^* v - v\| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .\end{aligned}$$

This shows that $\mathbb{T}^*$ is strongly continuous.

It remains to be shown that the generator of $\mathbb{T}^*$ is A^*. Let us denote the generator of $\mathbb{T}^*$ by A^d. According to Proposition 2.3.1 we have, for every $w \in X$,

$$(sI - A^d)^{-1} w = \int_0^\infty e^{-st}\,\mathbb{T}_t^* w \,\mathrm{d}t \quad \text{for } \operatorname{Re} s > \omega_0(\mathbb{T})$$

(we have used the obvious fact that $\omega_0(\mathbb{T}^*) = \omega_0(\mathbb{T})$). On the other hand, taking the inner product of both sides of the formula in Proposition 2.3.1 with $w \in X$ and replacing s by $\bar{s}$, by a simple argument we obtain that

$$\left[(\bar{s}I - A)^{-1}\right]^* w = \int_0^\infty e^{-st}\,\mathbb{T}_t^* w \,\mathrm{d}t \quad \text{for } \operatorname{Re} s > \omega_0(\mathbb{T}).$$

The last two formulas show that $(sI - A^d)^{-1} = \left[(\bar{s}I - A)^{-1}\right]^*$. According to Proposition 2.8.4, for $\operatorname{Re} s > \omega_0(\mathbb{T})$, we have $(sI - A^d)^{-1} = (sI - A^*)^{-1}$. This shows that $A^d = A^*$. $\square$

The semigroup $\mathbb{T}^*$ appearing above is called the *adjoint semigroup* of $\mathbb{T}$.

For the following proposition, the reader should recall the concepts of Riesz basis, biorthogonal sequence and diagonalizable operator, discussed in Section 2.6.

Proposition 2.8.6. *Let $A : \mathcal{D}(A) \to X$ be a diagonalizable operator. Let (ϕ_k) be a Riesz basis consisting of eigenvectors of A, let $(\tilde{\phi}_k)$ be the biorthogonal sequence to (ϕ_k) and denote the eigenvalue corresponding to the eigenvector ϕ_k by λ_k. Then A^* is a diagonalizable operator with the eigenvectors $\tilde{\phi}_k$ and eigenvalues $\overline{\lambda_k}$.*

Proof. Using the representation of A from Proposition 2.6.3, taking the inner product of both sides with $\tilde{\phi}_k$, we obtain

$$\langle Az, \tilde{\phi}_k\rangle = \lambda_k \langle z, \tilde{\phi}_k\rangle \qquad \forall\, z \in \mathcal{D}(A). \tag{2.8.5}$$

This shows that $\tilde{\phi}_k \in \mathcal{D}(A^*)$ and $A^*\tilde{\phi}_k = \overline{\lambda_k}\,\tilde{\phi}_k$. We know from Section 2.6 that $(\tilde{\phi}_k)$ is a Riesz basis in X. Finally, we know from Proposition 2.8.4 that $\rho(A^*)$ is not empty. Thus, A^* is diagonalizable. $\square$

The last proposition, together with Proposition 2.6.3, implies that if A is diagonalizable, then

$$\mathcal{D}(A^*) = \left\{ z \in X \;\middle|\; \sum_{k\in\mathbb{N}} \left(1 + |\lambda_k|^2\right) |\langle z, \phi_k\rangle|^2 < \infty \right\},$$

$$A^* z = \sum_{k\in\mathbb{N}} \overline{\lambda_k}\,\langle z, \phi_k\rangle \tilde{\phi}_k \qquad \forall\, z \in \mathcal{D}(A^*).$$

In particular, if $A = \operatorname{diag}(\lambda_k)$ (see Section 2.6), then $A^* = \operatorname{diag}(\overline{\lambda_k})$.

Example 2.8.7. We list the adjoints of most of the semigroups encountered in earlier examples. We leave it to the reader to verify that these are indeed the corresponding adjoint semigroups and their generators.

For the unilateral left shift semigroup of Example 2.3.7, the adjoint semigroup is the unilateral right shift semigroup from Example 2.4.5. The unilateral right shift semigroup is isometric. Let us denote by A the generator of the unilateral left shift, then the generator of the unilateral right shift (given in Example 2.4.5) is A^*. Thus, in this case, A is an extension of $-A^*$ (but $\sigma(A^*) = \sigma(A)$, a left half-plane).

For the vanishing left shift semigroup on $L^2[0,\tau]$ discussed in Example 2.3.8, the adjoint is the vanishing right shift semigroup:

$$(\mathbb{T}_t^* z)(x) = \begin{cases} z(x-t) & \text{if } \; x - t \geqslant 0, \\ 0 & \text{else}, \end{cases}$$

with generator

$$A^* = -\frac{\mathrm{d}}{\mathrm{d}x}, \qquad \mathcal{D}(A^*) = \{z \in \mathcal{H}^1(0,\tau) \mid z(0) = 0\}.$$

The adjoint of the heat semigroup on $L^2(\mathbb{R})$ introduced in Example 2.3.9 is the same semigroup (and hence $A^* = A$). The same is true for the heat conduction semigroups on $L^2[0,\pi]$ discussed in Examples 2.6.8 and 2.6.10.

The semigroups in the examples of Section 2.7 are unitary, hence their adjoint semigroups are the same as their inverse semigroups and thus the corresponding generators are $-A$. There is no need to write down formulas.

2.9 The embeddings $V \subset H \subset V'$

In this section we explain what it means that two Hilbert spaces are dual with respect to a pivot space. This concept plays an important role in the theory of PDEs as well as in the theory of infinite-dimensional linear systems.

Definition 2.9.1. If V and Z are Hilbert spaces, then an operator $J \in \mathcal{L}(V, Z)$ is called an *isomorphism* from V to Z, also called a *unitary* operator from V to Z, if $J^* J = I$ (the identity on V) and $J J^* = I$ (the identity on Z).

It is easy to verify that $J \in \mathcal{L}(V, Z)$ is unitary iff (a) $\|Jv\| = \|v\|$ for all $v \in V$ (this property means that J is *isometric*) and (b) Ran $J = Z$. Note that the isometric property of J is equivalent to

$$\langle Jv, Jw \rangle_Z = \langle v, w \rangle_V \qquad \forall\, v, w \in V\,.$$

For J as in the last definition, usually we employ the term "isomorphism" when we intend to identify the spaces V and Z, and we use the term "unitary operator" otherwise. Both situations will arise in this section.

For any Hilbert space V, we denote by V' its *dual* (the space of all bounded linear functionals on V). We denote by $\langle \varphi, z \rangle_{V,V'}$ the functional $z \in V'$ applied to $\varphi \in V$, so that $\langle \varphi, z \rangle_{V,V'}$ is linear in φ and antilinear in z (similarly to the inner product on a Hilbert space). We define the pairing also in reversed order:

$$\langle z, \varphi \rangle_{V',V} = \overline{\langle \varphi, z \rangle_{V,V'}}\,,$$

so that again, the pairing is linear in the first component. The norm on V' is

$$\|z\|_{V'} = \sup_{\varphi \in V,\ \|\varphi\|_V \leqslant 1} |\langle z, \varphi \rangle_{V',V}| \qquad \forall\, z \in V'\,.$$

For V and V' as above, there is a natural operator $J_R : V \to V'$ defined by

$$\langle \varphi, J_R v \rangle_{V,V'} = \langle \varphi, v \rangle_V \qquad \forall\, \varphi, v \in V\,.$$

According to the Riesz representation theorem, this J_R is an isomorphism (this has been discussed in Section 1.1). Often, but not always, we identify V with V', by not distinguishing between v and $J_R v$ (for all $v \in V$).

We denote by V'' the *bidual* of V, which is the dual of V'. Clearly, J_R^2 is an isomorphism from V to V''. The isomorphism J_R^2 is "more natural" than J_R, in the sense that it can be generalized to many Banach spaces (called reflexive Banach spaces), while the isomorphism J_R is specific to Hilbert spaces. *For any Hilbert space V, we identify V with V'', by not distinguishing between v and $J_R^2 v$.*

If V and H are Hilbert spaces such that $V \subset H$, we say that *the embedding $V \subset H$ is continuous* if the identity operator on V is in $\mathcal{L}(V, H)$. Equivalently, there exists an $m \geqslant 0$ such that $\|v\|_H \leqslant m \|v\|_V$ holds for all $v \in V$.

Proposition 2.9.2. *Let V and H be Hilbert spaces such that $V \subset H$, densely and with continuous embedding. Define a function $\|\cdot\|_*$ on H by*

$$\|z\|_* = \sup_{\varphi \in V,\ \|\varphi\|_V \leqslant 1} |\langle z, \varphi \rangle_H| \qquad \forall\, z \in H\,.$$

Then $\|\cdot\|_*$ is a norm on H. Let V^* denote the completion of H with respect to this norm. Define the operator $J : V^* \to V'$ as follows: For any $z \in V^*$,

$$\langle Jz, \varphi\rangle_{V',V} = \lim_{n\to\infty} \langle z_n, \varphi\rangle_H \qquad \forall\, \varphi \in V\,,$$

where (z_n) is a sequence in H such that $z_n \to z$ in V^*.

Then J is an isomorphism from V^* to V'.

Proof. It is easy to show that $\|\cdot\|_*$ is a norm on H. It is also easy (but tedious) to prove that the definition of $\langle Jz, \varphi\rangle_{V',V}$ is correct; i.e., the limit exists and it is independent of the choice of the sequence (z_n), as long as $z_n \to z$ in V^*.

Let us show that $Jz \in V'$; i.e., $\langle Jz, \varphi\rangle_{V',V}$ depends continuously on $\varphi \in V$. From the definition of $\|\cdot\|_*$ we see that

$$|\langle z, \varphi\rangle_H| \leqslant \|z\|_* \cdot \|\varphi\|_V \qquad \forall\, z \in H,\ \varphi \in V\,.$$

This implies that for any $z \in V^*$ and any $\varphi \in V$,

$$\begin{aligned} |\langle Jz, \varphi\rangle_{V',V}| &= \lim\nolimits_{n\to\infty} |\langle z_n, \varphi\rangle_H| \\ &\leqslant \lim\nolimits_{n\to\infty} \|z_n\|_* \cdot \|\varphi\|_V = \|z\|_* \cdot \|\varphi\|_V\,. \end{aligned}$$

This shows that $Jz \in V'$ and, moreover, $\|Jz\|_{V'} \leqslant \|z\|_*$, so that $J \in \mathcal{L}(V^*, V')$. It is clear from the definition of J that

$$\langle Jz, \varphi\rangle_{V',V} = \langle z, \varphi\rangle_H \qquad \forall\, z \in H,\ \varphi \in V\,, \tag{2.9.1}$$

hence $\|Jz\|_{V'} = \|z\|_*$ for all $z \in H$. Since H is dense in V^* and J is continuous, we conclude that $\|Jz\|_{V'} = \|z\|_*$ remains valid for all $z \in V^*$.

It remains to show that J is onto. For this, it is enough to show that Ran J is dense in V' (because the previous conclusion implies that Ran J is closed). If Ran J were not dense, then we could find $\varphi \in V'' = V$ such that $\varphi \neq 0$ and $\langle Jz, \varphi\rangle_{V',V} = 0$ for all $z \in V^*$. Choose $z = \varphi$, then according to (2.9.1) we get $\langle J\varphi, \varphi\rangle_{V',V} = \|\varphi\|_H^2 > 0$. This contradiction shows that Ran J is dense, so that J is an isomorphism from V^* to V'. □

In what follows, if V, H and V^* are as in the last proposition, then *we identify V^* with V'*, by not distinguishing between z and Jz (for all $z \in V^*$). Thus, we have

$$V \subset H \subset V'\,,$$

densely and with continuous embeddings. When V' is identified with V^* (as above), then we call V' the *dual of V with respect to the pivot space H*. Also, the norm $\|\cdot\|_*$ on H defined as in the last proposition is called the *dual norm of $\|\cdot\|_V$ with respect to the pivot space H*. We shall often need these concepts.

We mention that V is uniquely determined by V': it consists of those $\varphi \in H$ for which the product $\langle z, \varphi\rangle_H$, regarded as a function of z, has a continuous extension to V'. We also call V the *dual of V' with respect to the pivot space H*.

Proposition 2.9.3. *Let V and H be Hilbert spaces such that $V \subset H$, densely and with continuous embedding, and let $L \in \mathcal{L}(H)$. We denote by V' the dual of V with respect to the pivot space H.*

(1) *If $LV \subset V$, then the restriction of L to V is in $\mathcal{L}(V)$.*
(2) *If $L^*V \subset V$, then L has a unique extension $\tilde{L} \in \mathcal{L}(V')$.*

Proof. To prove (1), we notice that as an operator from V to V, L is closed (we have used the continuous embedding of V into H). Therefore, by the closed-graph theorem, L is bounded as an operator from V to V.

Now we prove (2). To avoid confusion, we use a different notation, namely L^d, for the restriction of L^* to V. We use (1) to conclude that $L^d \in \mathcal{L}(V)$. Hence, $L^{d*} \in \mathcal{L}(V')$ (see (1.1.5)). We claim that L^{d*} is an extension of L; i.e., $L^{d*}z = Lz$ holds for all $z \in H$. For this, it will be enough to show that

$$\langle L^{d*}z, \varphi\rangle_{V',V} = \langle Lz, \varphi\rangle_{V',V} \qquad \forall\, z \in H,\ \varphi \in V.$$

It is clear from (2.9.1) that the right-hand side above can also be written as $\langle Lz, \varphi\rangle_H$. Hence, the formula that we have to prove can be rewritten as

$$\langle z, L^d\varphi\rangle_{V',V} = \langle Lz, \varphi\rangle_H \qquad \forall\, z \in H,\ \varphi \in V.$$

Applying once more (2.9.1), this time to the left-hand side above, we obtain an equivalent identity which is obviously true. Thus, $\tilde{L} = L^{d*}$ is an extension of L.

The uniqueness of $\tilde{L}$ follows from the density of H in V'. □

2.10 The spaces X_1 and X_{-1}

Here we introduce the spaces X_1 and X_{-1}, which are important in the theory of unbounded control and observation operators. X will denote a Hilbert space.

Proposition 2.10.1. *Let $A : \mathcal{D}(A) \to X$ be a densely defined operator with $\rho(A) \neq \emptyset$. Then for every $\beta \in \rho(A)$, the space $\mathcal{D}(A)$ with the norm*

$$\|z\|_1 = \|(\beta I - A)z\| \qquad \forall\, z \in \mathcal{D}(A)$$

is a Hilbert space, denoted X_1. The norms generated as above for different $\beta \in \rho(A)$ are equivalent to the graph norm (defined in (2.2.1)). The embedding $X_1 \subset X$ is continuous. If $L \in \mathcal{L}(X)$ is such that $L\mathcal{D}(A) \subset \mathcal{D}(A)$, then $L \in \mathcal{L}(X_1)$.

Proof. The fact that $\rho(A) \neq \emptyset$ implies that A is closed, hence $\mathcal{D}(A)$ is a Hilbert space with the graph norm $\|\cdot\|_{gr}$ defined in (2.2.1). We show that for every $\beta \in \rho(A)$, $\|\cdot\|_1$ is equivalent to $\|\cdot\|_{gr}$. It is easy to see that for some $c > 0$ we have

$$\|z\|_1 \leqslant c\|z\|_{gr} \qquad \forall\, z \in \mathcal{D}(A).$$

The proof of this estimate uses the fact that $(a+b)^2 \leqslant 2(a^2+b^2)$ holds for all $a, b \in \mathbb{R}$. To prove the estimate in the opposite direction, we use again this simple fact about real numbers, as follows:

$$\begin{aligned} \|z\|_{gr}^2 &= \|z\|^2 + \|(\beta I - A)z - \beta z\|^2 \\ &\leqslant \|z\|^2 + 2\left(\|(\beta I - A)z\|^2 + \beta^2\|z\|^2\right). \end{aligned}$$

From here, using the estimate

$$\|z\| \leqslant \|(\beta I - A)^{-1}\| \cdot \|(\beta I - A)z\|, \tag{2.10.1}$$

we obtain that $\|z\|_{gr} \leqslant k\|z\|_1$ for some $k > 0$ independent of $z \in \mathcal{D}(A)$. Thus we have shown that the various norms $\|\cdot\|_1$ are equivalent to $\|\cdot\|_{gr}$.

The continuity of the embedding $X_1 \subset X$ follows from (2.10.1).

Now consider $L \in \mathcal{L}(X)$ such that L maps $\mathcal{D}(A)$ into itself. Then by part (1) of Proposition 2.9.3 we have that L is continuous on X_1. □

Let A be as in Proposition 2.10.1, then clearly A^* has the same properties. Thus, we can define $X_1^d = \mathcal{D}(A^*)$ with the norm

$$\|z\|_1^d = \left\|(\overline{\beta} I - A^*)z\right\| \qquad \forall\, z \in \mathcal{D}(A^*),$$

where $\overline{\beta} \in \rho(A^*)$, or equivalently, $\beta \in \rho(A)$, and this is a Hilbert space.

Proposition 2.10.2. *Let A be as in Proposition 2.10.1 and take $\beta \in \rho(A)$. We denote by X_{-1} the completion of X with respect to the norm*

$$\|z\|_{-1} = \left\|(\beta I - A)^{-1}z\right\| \qquad \forall\, z \in X. \tag{2.10.2}$$

Then the norms generated as above for different $\beta \in \rho(A)$ are equivalent (in particular, X_{-1} is independent of the choice of β). Moreover, X_{-1} is the dual of X_1^d with respect to the pivot space X (as defined in the previous section).

If $L \in \mathcal{L}(X)$ is such that $L^*\mathcal{D}(A^*) \subset \mathcal{D}(A^*)$, then L has a unique extension to an operator $\tilde{L} \in \mathcal{L}(X_{-1})$.

Proof. Choose the same β to define the norm on X_1^d (we know from the previous proposition, applied to A^*, that the choice of β in the definition of $\|\cdot\|_1^d$ is not important). For every $z \in X$ we have, using Proposition 2.8.4,

$$\begin{aligned} \|z\|_{-1} = \|(\beta I - A)^{-1}z\| &= \sup_{x \in X,\ \|x\| \leqslant 1} |\langle (\beta I - A)^{-1}z, x\rangle| \\ &= \sup_{x \in X,\ \|x\| \leqslant 1} |\langle z, (\overline{\beta} I - A^*)^{-1}x\rangle| \\ &= \sup_{\varphi \in X_1^d,\ \|\varphi\|_1^d \leqslant 1} |\langle z, \varphi\rangle|. \end{aligned}$$

This shows that the norm $\|\cdot\|_{-1}$ is the dual norm of $\|\cdot\|_1^d$ with respect to the pivot space X. Since $\|\cdot\|_1^d$ changes into an equivalent norm if we change β (according to the previous proposition), the same is true for $\|\cdot\|_{-1}$. It follows that X_{-1} is independent of β and it is the dual space of X_1^d with respect to the pivot space X.

The statement concerning L now follows from part (2) of Proposition 2.9.3. $\square$

At the end of this section we shall determine the space X_{-1} for several examples of semigroup generators.

For the following proposition, recall the concept of a unitary operator between two Hilbert spaces, introduced at the beginning of the previous section.

Proposition 2.10.3. *Let $A : \mathcal{D}(A) \to X$ be a densely defined operator with $\rho(A) \neq \emptyset$, let $\beta \in \rho(A)$, let X_1 be as in Proposition* 2.10.1 *and let X_{-1} be as in Proposition* 2.10.2. *Then $A \in \mathcal{L}(X_1, X)$ and A has a unique extension $\tilde{A} \in \mathcal{L}(X, X_{-1})$. Moreover,*

$$(\beta I - A)^{-1} \in \mathcal{L}(X, X_1), \qquad (\beta I - \tilde{A})^{-1} \in \mathcal{L}(X_{-1}, X)$$

(*in particular, $\beta \in \rho(\tilde{A})$*), *and these two operators are unitary.*

Proof. From the definition of $\|z\|_1$ it is clear that $(\beta I - A) \in \mathcal{L}(X_1, X)$ (it is actually norm-preserving). Since X_1 is continuously embedded in X, it follows that also $A \in \mathcal{L}(X_1, X)$, as claimed. By a similar argument, $A^* \in \mathcal{L}(X_1^d, X)$. Let us denote by $\tilde{A}$ the adjoint of $A^* \in \mathcal{L}(X_1^d, X)$, so that (according to the previous proposition) $\tilde{A} \in \mathcal{L}(X, X_{-1})$. (Here, we identify X with its dual.) We claim that $\tilde{A}$ is an extension of A. Indeed, this follows from

$$\langle \tilde{A}z, q\rangle_{X_{-1},X_1^d} = \langle z, A^*q\rangle_X = \langle Az, q\rangle_X \qquad \forall\, z \in \mathcal{D}(A),\ q \in \mathcal{D}(A^*),$$

which shows that $\tilde{A}z = Az$ for all $z \in \mathcal{D}(A)$. The uniqueness of an extension of A to X follows from the fact that $\mathcal{D}(A)$ is dense in X.

Denote $R = (\beta I - A)^{-1} \in \mathcal{L}(X, X_1)$. We have

$$\|Rz\| = \|z\|_{-1} \qquad \forall\, z \in X,$$

which shows that R has a unique extension $\tilde{R} \in \mathcal{L}(X_{-1}, X)$, and $\tilde{R}$ is norm-preserving. From the formulas

$$(\beta I - \tilde{A})\tilde{R}z = z = \tilde{R}(\beta I - \tilde{A})z \qquad \forall\, z \in \mathcal{D}(A)$$

and the fact that $\mathcal{D}(A)$ is dense in X (and hence also in X_{-1}), we conclude that in fact the above formulas hold for every $z \in X$. Thus, $\beta \in \rho(\tilde{A})$ and $(\beta I - \tilde{A})^{-1} = \tilde{R}$.

We have seen earlier that $(\beta I - \tilde{A})^{-1} \in \mathcal{L}(X_{-1}, X)$ is norm-preserving. It is easy to see that also $(\beta I - A)^{-1} \in \mathcal{L}(X, X_1)$ is norm-preserving. Since these operators are obviously invertible, it follows that they are unitary. $\square$

Suppose that A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X. It follows from Propositions 2.10.1 and 2.10.2 that for every $t \geqslant 0$, $\mathbb{T}_t$ has a restriction which is in $\mathcal{L}(X_1)$ and a unique extension $\tilde{\mathbb{T}}_t$ which is in $\mathcal{L}(X_{-1})$. We now show that these new families of operators are similar to the original semigroup.

Proposition 2.10.4. *We use the notation from Proposition* 2.10.3, *and assume that A generates a strongly continuous semigroup $\mathbb{T}$ on X. The restriction of $\mathbb{T}_t$ to X_1 (considered as an operator in $\mathcal{L}(X_1)$) is the image of $\mathbb{T}_t \in \mathcal{L}(X)$ through the unitary operator $(\beta I - A)^{-1} \in \mathcal{L}(X, X_1)$. Therefore, these operators form a strongly continuous semigroup on X_1, whose generator is the restriction of A to $\mathcal{D}(A^2)$.*

The operator $\tilde{\mathbb{T}}_t \in \mathcal{L}(X_{-1})$ is the image of $\mathbb{T}_t \in \mathcal{L}(X)$ through the unitary operator $(\beta I - \tilde{A}) \in \mathcal{L}(X, X_{-1})$. Therefore, these extended operators form a strongly continuous semigroup $\tilde{\mathbb{T}} = (\tilde{\mathbb{T}}_t)_{t \geqslant 0}$ on X_{-1}, whose generator is $\tilde{A}$.

Proof. The fact that $\mathbb{T}_t$ (considered as an operator in $\mathcal{L}(X_1)$) is the image of $\mathbb{T}_t \in \mathcal{L}(X)$ through the unitary operator $(\beta I - A)^{-1} \in \mathcal{L}(X, X_1)$ can be written as follows:

$$\mathbb{T}_t z = (\beta I - A)^{-1} \mathbb{T}_t (\beta I - A) z \qquad \forall\, z \in X_1,$$

which is obviously true. The corresponding statement for $\tilde{\mathbb{T}}_t$ reads

$$\tilde{\mathbb{T}}_t z = (\beta I - \tilde{A}) \mathbb{T}_t (\beta I - \tilde{A})^{-1} z \qquad \forall\, z \in X_{-1},$$

and this is also easy to check by first considering $z \in X$ and then using the density of X in X_{-1}. The generators of the two new semigroups are the images of the old generator A through the same two unitary operators. $\square$

In what follows, we denote the restriction (extension) of $\mathbb{T}_t$ described above by the same symbol $\mathbb{T}_t$, since this is unlikely to lead to confusions. Similarly, the operator $\tilde{A}$ introduced in Proposition 2.10.3 will be denoted by A.

Remark 2.10.5. The construction of X_1 and X_{-1} can be iterated, in both directions, so that we obtain the infinite sequence of spaces

$$\cdots \subset X_2 \subset X_1 \subset X \subset X_{-1} \subset X_{-2} \subset \cdots,$$

each inclusion being dense and with continuous embedding. For each $k \in \mathbb{Z}$, the original semigroup $\mathbb{T}$ has a restriction (or an extension) to X_k which is the image of $\mathbb{T}$ through the unitary operator $(\beta I - A)^{-k} \in \mathcal{L}(X, X_k)$. The space X_{-2} occasionally arises in the proof of theorems in infinite-dimensional systems theory. We are not aware of the occurence of higher order extended spaces.

Remark 2.10.6. As we have explained before Proposition 2.10.2, in the construction of X_1 we may replace A with A^* and β with $\overline{\beta}$, obtaining the space X_1^d. Similarly, in the construction of X_{-1}, we may replace A with A^* and β with $\overline{\beta}$, obtaining a space denoted by X_{-1}^d. For these spaces, we obtain similar results as in the last two propositions (with the adjoint semigroup $\mathbb{T}^*$ in place of $\mathbb{T}$). In particular,

$$X_1^d \subset X \subset X_{-1}^d,$$

densely and with continuous embeddings. As before, we denote the extensions of A^* and of $\mathbb{T}_t^*$ (to X and to X_{-1}^d) by the same symbols, so that $A^* \in \mathcal{L}(X, X_{-1}^d)$. Note that X_{-1}^d is the dual of X_1 with respect to the pivot space X.

Example 2.10.7. We determine here the spaces X_{-1} and X_{-1}^d for the unilateral left shift semigroup from Example 2.3.7, so that $X = L^2[0,\infty)$, $A = \frac{\mathrm{d}}{\mathrm{d}x}$ and $\mathcal{D}(A) = \mathcal{H}^1(0,\infty)$. As mentioned in Example 2.8.7, we have $\mathcal{D}(A^*) = \{z \in \mathcal{H}^1(0,\infty) \mid z(0) = 0\} = \mathcal{H}_0^1(0,\infty)$. According to Proposition 2.10.2, X_{-1} is the dual of $X_1^d = \mathcal{D}(A^*)$ with respect to the pivot space X. According to Definition 13.4.7 in Appendix II, $X_{-1} = \mathcal{H}^{-1}(0,\infty)$. From Proposition 2.3.1 we have

$$\left[(I-A)^{-1}z\right](x) = \int_0^\infty e^{-t} z(x+t)\,\mathrm{d}t \qquad \forall\, z \in X,$$

and the norm $\|z\|_{-1}$ (corresponding to $\beta = 1$ in (2.10.2)) is of course the L^2-norm of the above function. We are not aware of any simpler way to express this norm.

The space X_{-1}^d is the dual of $\mathcal{D}(A)$, so that $X_{-1}^d = \left(\mathcal{H}^1(0,\infty)\right)'$. To express the norm $\|z\|_{-1}^d$ we note that $[(I-A^*)^{-1}z](x) = \int_0^x e^{t-x} z(t)\,\mathrm{d}t$ (see Example 2.4.5). Applying the Fourier transformation $\mathcal{F}$, we obtain

$$(\|z\|_{-1}^d)^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{|(\mathcal{F}z)(\xi)|^2}{1+\xi^2}\,\mathrm{d}\xi \qquad \forall\, z \in X.$$

Example 2.10.8. We consider the heat semigroup from Example 2.3.9, so that $X = L^2(\mathbb{R})$, $A = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $\mathcal{D}(A) = \mathcal{H}^2(\mathbb{R})$ and, as mentioned in Example 2.8.7, $A^* = A$. According to Proposition 2.10.2, X_{-1} is the dual of $X_1^d = \mathcal{H}^2(\mathbb{R})$ with respect to the pivot space $L^2(\mathbb{R})$. According to Theorem 13.5.4 in Appendix II, $\mathcal{H}^2(\mathbb{R}) = \mathcal{H}_0^2(\mathbb{R})$ so that (using Definition 13.4.7 from Appendix II) $X_{-1} = \mathcal{H}^{-2}(\mathbb{R})$. The norm on X_{-1} can be expressed in terms of the Fourier transform as follows:

$$\|z\|_{-1}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{|(\mathcal{F}z)(\xi)|^2}{(1+\xi^2)^2}\,\mathrm{d}\xi.$$

This corresponds to taking $\beta = 1$ in (2.10.2).

Example 2.10.9. If $X = l^2$ and A is diagonal, as in Example 2.6.6, so that $(Az)_k = \lambda_k z_k$, then it is easy to verify (using (2.6.10)) that for any fixed $\beta \in \rho(A)$,

$$\|z\|_{-1}^2 = \sum_{k\in\mathbb{N}} \frac{|z_k|^2}{|\beta - \lambda_k|^2} \qquad \forall\, z \in X.$$

It follows that X_{-1} is the space of all the sequences $z = (z_k)$ for which

$$\sum_{k\in\mathbb{N}} \frac{|z_k|^2}{1+|\lambda_k|^2} < \infty.$$

Moreover, the square root of the above series gives an equivalent norm on X_{-1}.

Proposition 2.10.10. *Let $A : \mathcal{D}(A) \to X$ be the generator of a strongly continuous semigroup $\mathbb{T}$ on X. If $z \in X$ is such that for some $\varepsilon > 0$*

$$\sup_{t\in(0,\varepsilon)} \left\| \frac{\mathbb{T}_t z - z}{t} \right\| < \infty,$$

then $z \in \mathcal{D}(A)$.

Proof. Denote $x_n = n(\mathbb{T}_{\frac{1}{n}} z - z)$, then (x_n) is a bounded sequence in X by assumption. By Alaoglu's theorem (see Lemma 12.2.4 in Appendix I), there exists a subsequence (x_{n_k}) that converges weakly to a vector $x_0 \in X$, as in (12.2.3). On the other hand, it follows from Proposition 2.10.4 that $\lim x_n = Az$ in X_{-1}. Since (by Proposition 2.10.2) X_{-1} is the dual of X_1^d with respect to the pivot space X, it follows that

$$\lim \langle x_n, \varphi \rangle = \langle Az, \varphi \rangle_{X_{-1}, X_1^d} \qquad \forall\, \varphi \in X_1^d.$$

Comparing this with (12.2.3), we see that

$$\langle x_0, \varphi \rangle_{X_{-1}, X_1^d} = \langle Az, \varphi \rangle_{X_{-1}, X_1^d} \qquad \forall\, \varphi \in X_1^d,$$

whence $x_0 = Az$, so that $Az \in X$. Take $\beta \in \rho(A)$, then we obtain $(\beta I - A)z \in X$, which clearly implies that $z \in \mathcal{D}(A)$. □

Remark 2.10.11. In this book, when we work with a semigroup $\mathbb{T}$ acting on a state space X, then by default we identify X with its dual X' (see the text after (1.1.4)). However, sometimes it is more convenient not to do this. For example, if $\mathbb{T}$ is defined on the Sobolev space $X = \mathcal{H}^{-1}(\Omega)$, where Ω is a bounded open set in $\mathbb{R}^n$, then our intuition may be better tuned to regard $X' = \mathcal{H}_0^1(\Omega)$ as the dual space, which corresponds to duality with respect to the pivot space $L^2(\Omega)$. This also corresponds better to arguments involving integration by parts. (We refer the reader to Section 13.4 in Appendix II for the definitions of these Sobolev spaces.)

The material in the Sections 2.8 to 2.10 can be adjusted easily for the situation when X is not identified with X' (however, we always identify X'' with X). Then the adjoint of an operator $A : \mathcal{D}(A) \to X$, where $\mathcal{D}(A)$ is dense in X, is a closed operator $A^* : \mathcal{D}(A^*) \to X'$, where $\mathcal{D}(A^*) \subset X'$. If A generates a semigroup $\mathbb{T}$ on X, then A^* generates the adjoint semigroup $\mathbb{T}^*$ on X'. The spaces X_1^d and X_{-1}^d (see Remark 2.10.6) are such that

$$X_1^d \subset X' \subset X_{-1}^d. \tag{2.10.3}$$

To understand the relationship between the spaces in (2.10.3) and the spaces $X_1 \subset X \subset X_{-1}$, we need to generalize the concept of duality with respect to a pivot space (from Section 2.9) as follows.

Suppose that V and H are Hilbert spaces such that $V \subset H$, densely and with continuous embedding. We do not identify H with its dual H'. Then the dual of

V with respect to the pivot space H is the completion of H' with respect to the norm

$$\|z\|_* = \sup_{\varphi \in V,\ \|\varphi\|_V \leqslant 1} |\langle z, \varphi \rangle_{H',H}| \qquad \forall\, z \in H\,.$$

After this generalization, we may regard X_{-1}^d as the dual of X_1 with respect to the pivot space X, and similarly we may regard X_{-1} as the dual of X_1^d with respect to the pivot space X'.

2.11 Bounded perturbations of a generator

In this section, $A : \mathcal{D}(A) \to X$ is the generator of a strongly continuous semigroup $\mathbb{T}$ on X and $P \in \mathcal{L}(X)$. Our aim is to show that also $A + P$ is the generator of a strongly continuous semigroup on X. We call P a *perturbation* of the generator.

Lemma 2.11.1. *Suppose that $\omega \in \mathbb{R}$ and $M \geqslant 1$ are such that*

$$\|\mathbb{T}_t\| \leqslant M e^{\omega t} \qquad \forall\, t \geqslant 0\,. \tag{2.11.1}$$

Then for $\alpha = \omega + M\|P\|$ we have $\mathbb{C}_\alpha \subset \rho(A+P)$.

Proof. For every $s \in \rho(A)$ we have the factorization

$$sI - A - P = (sI - A)[I - (sI - A)^{-1}P]\,. \tag{2.11.2}$$

According to Corollary 2.3.3 we have $\|(sI - A)^{-1}\| \leqslant \frac{M}{\operatorname{Re} s - \omega}$ for all $s \in \mathbb{C}_\omega$. Thus, for $s \in \mathbb{C}_\alpha$ we have $\|(sI - A)^{-1}P\| < 1$. This implies, according to Lemma 2.2.6, that the second factor on the right-hand side of (2.11.2) has a bounded inverse. Since now $s \in \rho(A)$, it follows that for $s \in \mathbb{C}_\alpha$ we have $s \in \rho(A+P)$ and

$$(sI - A - P)^{-1} = [I - (sI - A)^{-1}P]^{-1}(sI - A)^{-1}\,. \qquad \square$$

Theorem 2.11.2. *Assume again* (2.11.1) *and put $\alpha = \omega + M\|P\|$. Then $A + P : \mathcal{D}(A) \to X$ is the generator of a strongly continuous semigroup $\mathbb{T}^P$ satisfying*

$$\|\mathbb{T}_t^P\| \leqslant M e^{\alpha t} \qquad \forall\, t \geqslant 0\,. \tag{2.11.3}$$

Proof. We define the sequence of families of bounded operators $(\mathbb{S}^n)$ acting on X by induction: $\mathbb{S}_t^0 = \mathbb{T}_t$ (for all $t \geqslant 0$) and for all $n \in \mathbb{N}$,

$$\mathbb{S}_t^n z = \int_0^t \mathbb{T}_{t-\sigma} P \mathbb{S}_\sigma^{n-1} z \,\mathrm{d}\sigma \qquad \forall\, z \in X,\ t \geqslant 0\,.$$

It is easy to check that these families of operators are strongly continuous, meaning that $\lim_{t \to t_0} \mathbb{S}_t^n z = \mathbb{S}_{t_0}^n z$ for all $t_0 \geqslant 0$, $z \in X$ and $n \in \{0, 1, 2, \ldots\}$.

It is easy to show by induction that

$$\|\mathbb{S}_t^n\| \leqslant M^{n+1}\|P\|^n e^{\omega t}\frac{t^n}{n!} \qquad \forall\, n \in \mathbb{N},\ t \geqslant 0. \tag{2.11.4}$$

This implies that the following series is absolutely convergent in $\mathcal{L}(X)$:

$$\mathbb{T}_t^P = \sum_{n=0}^{\infty} \mathbb{S}_t^n \qquad \forall\, t \geqslant 0. \tag{2.11.5}$$

Indeed, we have

$$\|\mathbb{T}_t^P\| \leqslant Me^{\omega t}\sum_{n=0}^{\infty}\frac{(M\|P\|t)^n}{n!} = Me^{\omega t}e^{M\|P\|t},$$

and this also shows that the family $\mathbb{T}^P$ satisfies (2.11.3). Moreover, it follows from the estimates (2.11.4) that the series in (2.11.5) converges uniformly on bounded intervals. This uniform convergence, together with the strong continuity of the terms $\mathbb{S}^n$, implies that the family of operators $\mathbb{T}^P$ is strongly continuous.

Let us show that the family of operators $\mathbb{T}^P$ satisfies the integral equation

$$\mathbb{T}_t^P z = \mathbb{T}_t z + \int_0^t \mathbb{T}_{t-\sigma} P\,\mathbb{T}_\sigma^P z\,\mathrm{d}\sigma \qquad \forall\, z \in X,\ t \geqslant 0. \tag{2.11.6}$$

Indeed, this follows from

$$\mathbb{T}_t^P z = \mathbb{T}_t z + \sum_{n=1}^{\infty}\int_0^t \mathbb{T}_{t-\sigma} P\,\mathbb{S}_\sigma^{n-1} z\,\mathrm{d}\sigma = \mathbb{T}_t z + \int_0^t \mathbb{T}_{t-\sigma} P\sum_{n=1}^{\infty}\mathbb{S}_\sigma^{n-1} z\,\mathrm{d}\sigma,$$

where we have used the local uniform convergence of the series in (2.11.5)

It is easy to see that for every $z \in X$, the function $t \mapsto \mathbb{T}_t^P z$ has a Laplace transform defined (at least) for all $s \in \mathbb{C}_\alpha$, so that we can define $R(s) \in \mathcal{L}(X)$ by

$$R(s)z = \int_0^\infty e^{-st}\,\mathbb{T}_t^P z\,\mathrm{d}t \qquad \forall\, z \in X,\ s \in \mathbb{C}_\alpha.$$

If we apply the Laplace transformation to (2.11.6) and use Proposition 2.3.1, we obtain that for $s \in \mathbb{C}_\alpha$, $R(s)z = (sI-A)^{-1}z + (sI-A)^{-1}PR(s)z$. This shows that $\operatorname{Ran} R(s) \subset \mathcal{D}(A)$. From the last formula we get by elementary algebraic manipulations that for $s \in \mathbb{C}_\alpha$, $(sI-A-P)R(s) = I$. Since according to Lemma 2.11.1 we have $\mathbb{C}_\alpha \subset \rho(A+P)$, it follows that

$$R(s) = (sI-A-P)^{-1} \qquad \forall\, s \in \mathbb{C}_\alpha. \tag{2.11.7}$$

Let us show that the family $\mathbb{T}^P$ satisfies the semigroup property. For $\tau \geqslant 0$ fixed, we have, from (2.11.6), that for every $t \geqslant 0$ and every $z \in X$,

$$\begin{aligned}\mathbb{T}^P_{t+\tau} z &= \mathbb{T}_{t+\tau} z + \mathbb{T}_t \int_0^\tau \mathbb{T}_{\tau-\sigma} P \mathbb{T}^P_\sigma z \,\mathrm{d}\sigma + \int_\tau^{t+\tau} \mathbb{T}_{t+\tau-\sigma} P \mathbb{T}^P_\sigma z \,\mathrm{d}\sigma \\ &= \mathbb{T}_t \left(\mathbb{T}_\tau z + \int_0^\tau \mathbb{T}_{\tau-\sigma} P \mathbb{T}^P_\sigma z \,\mathrm{d}\sigma \right) + \int_\tau^{t+\tau} \mathbb{T}_{t+\tau-\sigma} P \mathbb{T}^P_\sigma z \,\mathrm{d}\sigma ,\end{aligned}$$

whence

$$\mathbb{T}^P_{t+\tau} z = \mathbb{T}_t \mathbb{T}^P_\tau z + \int_0^t \mathbb{T}_{t-\mu} P \mathbb{T}^P_{\mu+\tau} z \,\mathrm{d}\mu . \tag{2.11.8}$$

For $s \in \mathbb{C}_\alpha$ we define $Q(s) \in \mathcal{L}(X)$ by applying the Laplace transformation to the function $t \mapsto \mathbb{T}^P_{t+\tau} z$:

$$Q(s) z = \int_0^\infty e^{-st}\, \mathbb{T}^P_{t+\tau} z \,\mathrm{d}t \qquad \forall\, z \in X,\ s \in \mathbb{C}_\alpha .$$

A computation that is very similar to the one leading to (2.11.7) (and using (2.11.8)) shows that

$$Q(s) = (sI - A - P)^{-1} \mathbb{T}^P_\tau \qquad \forall\, s \in \mathbb{C}_\alpha .$$

Since the continuous function $t \mapsto \mathbb{T}^P_{t+\tau} z$ is uniquely determined by its Laplace transform (see Proposition 12.4.5), the above formula with (2.11.7) yields

$$\mathbb{T}^P_{t+\tau} = \mathbb{T}^P_t \mathbb{T}^P_\tau .$$

Thus we have shown that $\mathbb{T}^P$ is a strongly continuous semigroup on X, and it satisfies the estimate (2.11.3). From Proposition 2.3.1 and from (2.11.7) we see that the generator of this semigroup is $A + P$. □

The above proof could be shortened by using the Banach space version of the Lumer–Phillips theorem; see, for example, Pazy [182, pp. 76–77]. The Hilbert space version of the Lumer–Phillips theorem will be given in Section 3.8. There are many references discussing unbounded perturbations of semigroup generators, and several such perturbation results can be found in the books on operator semigroups that were cited at the beginning of this chapter.

Remark 2.11.3. With A and P as above, it is easy to verify that

$$\begin{aligned}(sI - A - P)^{-1} - (sI - A)^{-1} &= (sI - A)^{-1} P (sI - A - P)^{-1} \\ &= (sI - A - P)^{-1} P (sI - A)^{-1} \qquad \forall\, s \in \rho(A + P) \cap \rho(A) .\end{aligned}$$

These formulas imply that the following norms are equivalent on X:

$$\|z\|_{-1} = \|(\beta I - A)^{-1} z\| , \qquad \|z\|^P_{-1} = \|(\beta I - A - P)^{-1} z\| ,$$

where $\beta \in \rho(A) \cap \rho(A + P)$. Hence, the space X_{-1} with respect to A (see Section 2.10) is the same as with respect to $A + P$. However, $\mathcal{D}(A^2)$ is in general different from $\mathcal{D}((A + P)^2)$, and also the X_{-2} spaces are in general different.

Chapter 3

Semigroups of Contractions

A strongly continuous semigroup $\mathbb{T}$ is called a *contraction semigroup* if $\|\mathbb{T}_t\| \leqslant 1$ for all $t \geqslant 0$. This chapter is a continuation of the previous one: we present basic facts about unbounded operators and strongly continuous semigroups on Hilbert spaces, but now the emphasis is on contraction semigroups and their generators, which are called m-dissipative operators. We also discuss other important classes of operators (self-adjoint, positive and skew-adjoint operators) that arise as generators or as ingredients of generators of contraction semigroups. We also investigate some classes of self-adjoint differential operators: Sturm–Liouville operators and the Dirichlet Laplacian on various domains in $\mathbb{R}^n$.

The notation is the same as in Chapter 2. Our main references for contraction semigroups are Davies [44], Hille and Phillips [97] and Tanabe [213]. For self-adjoint operators and for the Dirichlet Laplacian our sources are also Brezis [22], Courant and Hilbert [37], Rudin [195] and Zuily [246].

3.1 Dissipative and m-dissipative operators

Definition 3.1.1. The operator $A : \mathcal{D}(A) \to X$ is called *dissipative* if

$$\operatorname{Re} \langle Az, z \rangle \leqslant 0 \qquad \forall\, z \in \mathcal{D}(A).$$

We are interested in dissipative operators for the following reason: If $\mathbb{T}$ is a contraction semigroup on X, then its generator A is dissipative and Ran $(I - A) = X$. Conversely, every operator A with these properties generates a contraction semigroup on X. This will follow from the material below and from Section 3.8. The dissipativity of an operator is often easy to check, so that we have an attractive way of establishing that certain PDEs have well-behaved solutions.

Proposition 3.1.2. *The operator $A : \mathcal{D}(A) \to X$ is dissipative if and only if*

$$\|(\lambda I - A)z\| \geqslant \lambda \|z\| \qquad \forall\, z \in \mathcal{D}(A),\ \lambda \in (0, \infty), \tag{3.1.1}$$

which is further equivalent to

$$\|(sI - A)z\| \geqslant (\operatorname{Re} s)\|z\| \qquad \forall\, z \in \mathcal{D}(A),\ s \in \mathbb{C}_0. \tag{3.1.2}$$

Proof. If A is dissipative, then (3.1.1) holds, because for z and λ as in (3.1.1),

$$\|(\lambda I - A)z\|^2 = \lambda^2\|z\|^2 - 2\lambda \operatorname{Re}\langle Az, z\rangle + \|Az\|^2 \geqslant \lambda^2\|z\|^2.$$

Conversely, if (3.1.1) holds, then, for all $\lambda > 0$, we have

$$\operatorname{Re}\langle Az, z\rangle - \frac{1}{2\lambda}\|Az\|^2 = \frac{\lambda^2\|z\|^2 - \|(\lambda I - A)z\|^2}{2\lambda} \leqslant 0.$$

For $\lambda \to \infty$ we obtain that A is dissipative.

Finally, we show that (3.1.1) and (3.1.2) are equivalent. Indeed, it is obvious that the second implies the first. Suppose (3.1.1) holds and let $s \in \mathbb{C}_0$, so that $s = \lambda + i\omega$ for some $\lambda > 0$ and $\omega \in \mathbb{R}$. Since A is dissipative, so is $A - i\omega I$. Writing (3.1.1) with $A - i\omega I$ in place of A, we get (3.1.2). □

Proposition 3.1.3. *Let $A : \mathcal{D}(A) \to X$ be dissipative, with $\mathcal{D}(A)$ dense in X. Then A has a closed extension, which is again dissipative.*

Proof. One such extension (possibly not the only one) is the operator A^{cl} whose graph is the closure of the graph of A. Thus, $z_0 \in \mathcal{D}(A^{cl})$ iff there is a sequence (z_n) in $\mathcal{D}(A)$ such that $z_n \to z_0$ and $Az_n \to y$ for some $y \in X$. In this case, we put $A^{cl}z_0 = y$. To verify that this definition of A^{cl} makes sense, we must check that $A^{cl}z_0$ is independent of the sequence (z_n). Suppose that there is another sequence (z_n') in $\mathcal{D}(A)$ with $z_n' \to z_0$ and $Az_n' \to v$ for some $v \in X$. Put $\delta_n = z_n - z_n'$, then $\delta_n \to 0$ and $A\delta_n \to y - v$. For every $\psi \in \mathcal{D}(A)$ and every $s \in \mathbb{C}$, we have

$$\lim_{n\to\infty} \langle A(\psi + s\delta_n), \psi + s\delta_n\rangle = \langle A\psi, \psi\rangle + s\langle y - v, \psi\rangle.$$

The real part of the left-hand side must be $\leqslant 0$. Since this is true for every $s \in \mathbb{C}$, we obtain that $\langle y - v, \psi\rangle = 0$. Since this is true for all $\psi \in \mathcal{D}(A)$ and $\mathcal{D}(A)$ is dense, we get $y = v$. Thus, the definition of A^{cl} makes sense. Clearly, A^{cl} is closed.

Now we show that A^{cl} is dissipative. If $z_0 \in \mathcal{D}(A^{cl})$, then there exists a sequence (z_n) in $\mathcal{D}(A)$ with $z_n \to z_0$ and $Az_n \to A^{cl}z_0$. We have $\operatorname{Re}\langle A^{cl}z_0, z_0\rangle = \lim_{n\to\infty} \operatorname{Re}\langle Az_n, z_n\rangle \leqslant 0$. Since $\operatorname{Re}\langle Az_n, z_n\rangle \leqslant 0$, A^{cl} is dissipative. □

The operator A^{cl} constructed in the above proof is called the *closure* of A. Obviously, A^{cl} is closed, so that it is equal to its own closure. Not every unbounded operator has a closure, and the first part of the above proof was devoted to showing that under the given assumptions, A has a closure.

Lemma 3.1.4. *Let A be a closed and dissipative operator on the Hilbert space X. Then for every $s \in \mathbb{C}_0$,* Ran $(sI - A)$ *is closed in X.*

Proof. Let (f_n) be a sequence in Ran $(sI - A) = (sI - A)\mathcal{D}(A)$, with $f_n \to f$ in X. Then there exists a sequence (z_n) in $\mathcal{D}(A)$ such that

$$sz_n - Az_n = f_n \qquad \forall\, n \geqslant 1. \tag{3.1.3}$$

The convergence of (f_n) and (3.1.2) imply that $z_n \to z$ in X. Moreover, (3.1.3) implies that $Az_n \to sz - f$ in X. Since A is closed, it follows that $z \in \mathcal{D}(A)$ and $Az = sz - f$, so that $f \in$ Ran $(sI - A)$. □

Lemma 3.1.5. *Let $A : \mathcal{D}(A) \to X$ be dissipative and closed. Then for each $s \in \mathbb{C}_0$, the operator A has a dissipative extension $\tilde{A}$ such that*

$$\mathrm{Ran}\,(sI - \tilde{A}) = X.$$

Proof. Take $s \in \mathbb{C}_0$ and let us denote $N_s = [\mathrm{Ran}\,(sI - A)]^{\perp}$. We claim that $N_s \cap \mathcal{D}(A) = \{0\}$. Indeed, if $v \in N_s \cap \mathcal{D}(A)$, then

$$0 = \langle (sI - A)v, v\rangle = s\|v\|^2 - \langle Av, v\rangle.$$

Taking real parts, we obtain $0 \geqslant (\mathrm{Re}\, s)\|v\|^2$, which implies $v = 0$.

Now define $\mathcal{D}(\tilde{A}) = \mathcal{D}(A) + N_s$ and

$$\tilde{A}(z + v) = Az - \overline{s}v \qquad \forall\, z \in \mathcal{D}(A),\ v \in N_s.$$

We check that $\tilde{A}$ is dissipative. Indeed, for z, v as above,

$$\mathrm{Re}\,\langle \tilde{A}(z + v), z + v\rangle = \mathrm{Re}\,\langle Az, z\rangle - \mathrm{Re}\,\langle (sI - A)z, v\rangle - (\mathrm{Re}\, s)\|v\|^2 \leqslant 0,$$

since $\langle (sI - A)z, v\rangle = 0$. We check that Ran $(sI - \tilde{A}) = X$. Indeed, for $z \in \mathcal{D}(A)$ and $v \in N_s$ we have

$$(sI - \tilde{A})(z + v) = (sI - A)z + (s + \overline{s})v. \tag{3.1.4}$$

By Lemma 3.1.4, Ran $(sI - A)$ is closed, so that every point $x \in X$ can be decomposed as $x = (sI - A)z + u$ for some $z \in \mathcal{D}(A)$ and some $u \in N_s$. This, together with (3.1.4), implies that Ran $(sI - \tilde{A}) = X$. □

Note that this lemma implies the following: If $A \in \mathcal{L}(X)$ is dissipative, then Ran $(sI - A) = X$ for all $s \in \mathbb{C}_0$.

Proposition 3.1.6. *Let $A : \mathcal{D}(A) \to X$ be dissipative and such that* Ran $(sI - A) =$ *X for some $s \in \mathbb{C}_0$. Then $\mathcal{D}(A)$ is dense in X.*

Proof. Let $f \in X$ be such that $\langle f, v\rangle = 0$ for all $v \in \mathcal{D}(A)$. Since $sI - A$ is onto, there exists $v_0 \in \mathcal{D}(A)$ such that $sv_0 - Av_0 = f$. Hence

$$0 = \mathrm{Re}\,\langle f, v_0\rangle = (\mathrm{Re}\, s)\|v_0\|^2 - \mathrm{Re}\,\langle Av_0, v_0\rangle \geqslant (\mathrm{Re}\, s)\|v_0\|^2.$$

Thus $v_0 = 0$, so $f = 0$, so that $\mathcal{D}(A)$ is dense. □

Theorem 3.1.7. *Let $A : \mathcal{D}(A) \to X$ be dissipative. Then the following statements about A are equivalent:*

(1) Ran $(sI - A) = X$ *for some* $s \in \mathbb{C}_0$.
(2) Ran $(sI - A) = X$ *for all* $s \in \mathbb{C}_0$.
(3) $\mathcal{D}(A)$ *is dense and if* $\tilde{A}$ *is a dissipative extension of* A*, then* $\tilde{A} = A$.

Proof. (3) $\Rightarrow$ (2): Suppose that (3) holds, so that in particular $\mathcal{D}(A)$ is dense. By Proposition 3.1.3, A has a closed and dissipative extension A^{cl}. By (3) we have $A^{cl} = A$, so that A is closed. Now take $s \in \mathbb{C}_0$. By Lemma 3.1.5 there exists a dissipative extension of A, denoted $\tilde{A}$, such that Ran $(sI - \tilde{A}) = X$. But according to (3) we must have $\tilde{A} = A$, so that Ran $(sI - A) = X$. Thus, (2) holds.

(2) $\Rightarrow$ (1): This is trivial.

(1) $\Rightarrow$ (3): If (1) holds, then, by Proposition 3.1.6, $\mathcal{D}(A)$ is dense. Let $\tilde{A}$ be a dissipative extension of A and take $z \in \mathcal{D}(\tilde{A})$. By (1) there exists $v \in \mathcal{D}(A)$ such that $(sI - A)v = (sI - \tilde{A})z$, whence $(sI - \tilde{A})(z - v) = 0$. By (3.1.2) we have

$$0 = \|(sI - \tilde{A})(z - v)\| \geqslant (\operatorname{Re} s)\|z - v\|,$$

so that $z = v$. Hence, $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$, so that $\tilde{A} = A$. □

Definition 3.1.8. A dissipative operator is called *maximal dissipative* (for brevity, *m-dissipative*) if it has one (hence, all) the properties listed in Theorem 3.1.7.

Proposition 3.1.9. *For $A : \mathcal{D}(A) \to X$, the following statements are equivalent:*

(a) *A is m-dissipative.*
(b) *We have $(0, \infty) \subset \rho(A)$ (in particular, A is closed) and*

$$\|(\lambda I - A)^{-1}\| \leqslant \frac{1}{\lambda} \qquad \forall\, \lambda \in (0, \infty). \tag{3.1.5}$$

(c) *We have $\mathbb{C}_0 \subset \rho(A)$ (in particular, A is closed) and*

$$\|(sI - A)^{-1}\| \leqslant \frac{1}{\operatorname{Re} s} \qquad \forall\, s \in \mathbb{C}_0. \tag{3.1.6}$$

Proof. (a) $\Rightarrow$ (c): From Theorem 3.1.7 we know that for all $s \in \mathbb{C}_0$ we have Ran $(sI - A) = X$. From (3.1.2) we see that for all $s \in \mathbb{C}_0$, $(sI - A)^{-1}$ is a bounded linear operator on X satisfying (3.1.6).

(c) $\Rightarrow$ (b): This is trivial.

(b) $\Rightarrow$ (a): Clearly (b) implies that (3.1.1) holds, so that, by Proposition 3.1.2, A is dissipative. Clearly, (b) also implies that Ran $(\lambda I - A) = X$ for all $\lambda > 0$. Thus, A satisfies statement (1) from Theorem 3.1.7, so that it is m-dissipative. □

Proposition 3.1.10. *If $A : \mathcal{D}(A) \to X$ is m-dissipative, then A^* is m-dissipative.*

Proof. We know from Proposition 3.1.6 that $\mathcal{D}(A)$ is dense, so that A^* exists. From (3.1.5) and Proposition 2.8.4 we see that $(0, \infty) \subset \rho(A^*)$ and $\|(\lambda I - A^*)^{-1}\| \leqslant \frac{1}{\lambda}$ for all $\lambda > 0$. According to Proposition 3.1.9, A^* is m-dissipative. □

Proposition 3.1.11. *Let $A : \mathcal{D}(A) \to X$ be a densely defined dissipative operator. Then A is m-dissipative if and only if A is closed and A^* is dissipative.*

Proof. Suppose that A is m-dissipative. According to Proposition 3.1.9, A is closed and according to Proposition 3.1.10, A^* is m-dissipative.

Conversely, suppose that A is closed and A^* is dissipative. According to Lemma 3.1.4, Ran $(I - A)$ is closed. By Remark 2.8.2, $[\mathrm{Ran}\ (I - A)]^\perp = \mathrm{Ker}\ (I - A^*)$, and the latter is $\{0\}$ according to Proposition 3.1.2. Thus, Ran $(I - A) = X$, so that A is m-dissipative (by definition). □

Definition 3.1.12. A strongly continuous semigroup $\mathbb{T}$ on X is called a *strongly continuous contraction semigroup* (or just a *contraction semigroup* for the sake of brevity) if $\|\mathbb{T}_t\| \leqslant 1$ holds for all $t \geqslant 0$.

The introduction of m-dissipative operators is motivated by the following result.

Proposition 3.1.13. *If A is the generator of a contraction semigroup on X, then A is m-dissipative.*

Proof. It follows from Proposition 2.3.1 and (2.3.2) that if A is the generator of a contraction semigroup, then $\mathbb{C}_0 \subset \rho(A)$ and

$$\|(sI - A)^{-1}\| \leqslant \frac{1}{\mathrm{Re}\, s} \qquad \forall\, s \in \mathbb{C}_0\,. \tag{3.1.7}$$

Now the proposition follows from this estimate and Proposition 3.1.9. □

Example 3.1.14. Most of the examples in Chapter 2 are contraction semigroups. For example, the unilateral left and right shift semigroups on $L^2[0, \infty)$, the vanishing left shift semigroup on $L^2[0, \tau]$ discussed in Example 2.3.8, the bilateral left shift semigroup on $L^2(\mathbb{R})$, the heat semigroup on $L^2(\mathbb{R})$ discussed in Example 2.3.9, the heat semigroups on $L^2[0, \pi]$ given in Examples 2.6.8 and 2.6.10 and the vibrating string semigroup from Example 2.7.13 are all contraction semigroups.

It is easy to see that a diagonal semigroup on l^2 (see Example 2.6.6) is a contraction semigroup iff $\mathrm{Re}\,\lambda \leqslant 0$ holds for all the eigenvalues λ of its generator.

3.2 Self-adjoint operators

In this section we study self-adjoint operators on a Hilbert space. We denote the Hilbert space by H and the operator by A_0 (instead of using the notation X for the space and A for the operator). The reason for this change of notation is that these operators will often appear in the later chapters not as generators of strongly continuous semigroups but as ingredients of such generators.

Let $A_0 : \mathcal{D}(A_0) \to H$, where $\mathcal{D}(A_0)$ is dense in H. Then A_0 is called *symmetric* if

$$\langle A_0 w, v\rangle \,=\, \langle w, A_0 v\rangle \qquad \forall\, w, v \in \mathcal{D}(A_0)\,.$$

It is easy to see that this is equivalent to $G(A_0) \subset G(A_0^*)$ (recall from Section 2.1 that $G(A_0)$ denotes the graph of A_0). In particular, A_0 is called *self-adjoint* if $A_0 = A_0^*$. (The equality $A_0^* = A_0$ means that $\mathcal{D}(A_0^*) = \mathcal{D}(A_0)$ and $A_0^* x = A_0 x$ for all $x \in \mathcal{D}(A_0)$, or equivalently, $G(A_0^*) = G(A_0)$.)

Note that any self-adjoint operator A_0 is closed. Indeed, since A_0^* is closed (as remarked before Proposition 2.8.1) and $A_0 = A_0^*$, A_0 is closed.

Lemma 3.2.1. *Let $T : \mathcal{D}(T) \to H$ and $x, y \in \mathcal{D}(T)$. Then we have*

$$\begin{aligned} 4\langle Tx, y\rangle = \langle T(x+y), (x+y)\rangle - \langle T(x-y), (x-y)\rangle \\ + i\langle T(x+iy), (x+iy)\rangle - i\langle T(x-iy), (x-iy)\rangle . \end{aligned}$$

The proof is by direct computation and it is left to the reader.

Proposition 3.2.2. *Assume that $A_0 : \mathcal{D}(A_0) \to H$, with $\mathcal{D}(A_0)$ dense in H. Then A_0 is symmetric if and only if for every $z \in \mathcal{D}(A_0)$, we have $\langle A_0 z, z\rangle \in \mathbb{R}$.*

Proof. Suppose that for every $z \in \mathcal{D}(A_0)$, $\langle A_0 z, z\rangle$ is real. It follows from Lemma 3.2.1 that for every $w, v \in \mathcal{D}(A_0)$,

$$\operatorname{Re}\langle A_0 w, v\rangle = \frac{1}{4}\Big[\langle A_0(w+v), (w+v)\rangle - \langle A_0(w-v), (w-v)\rangle\Big] = \operatorname{Re}\langle w, A_0 v\rangle .$$

Replacing in this equality v with iv, we obtain that $\operatorname{Im}\langle A_0 w, v\rangle = \operatorname{Im}\langle w, A_0 v\rangle$, so that A_0 is symmetric. The converse statement (the only if part) is very easy. $\square$

Remark 3.2.3. If $A_0 : \mathcal{D}(A_0) \to H$ is symmetric, then for every $s \in \mathbb{C}$ and every $z \in \mathcal{D}(A_0)$ we have $\|(sI - A_0)z\| \geqslant |\operatorname{Im} s| \cdot \|z\|$. To prove this, we decompose $s = \alpha + i\omega$ with $\alpha, \omega \in \mathbb{R}$ and we notice (using Proposition 3.2.2) that

$$\|(sI - A_0)z\|^2 = \|(\alpha I - A_0)z\|^2 + \omega^2\|z\|^2 . \tag{3.2.1}$$

Proposition 3.2.4. *If $A_0 : \mathcal{D}(A_0) \to H$ is symmetric, $s \in \mathbb{C}$ and both $sI - A_0$ and $\overline{s}I - A_0$ are onto, then A_0 is self-adjoint and $s, \overline{s} \in \rho(A_0)$.*

Proof. By Remark 2.8.2 we have $\operatorname{Ker}(\overline{s}I - A_0^*) = \operatorname{Ker}(sI - A_0^*) = \{0\}$. Since A_0^* is an extension of A_0, by assumption we also have $\operatorname{Ran}(sI - A_0^*) = \operatorname{Ran}(\overline{s}I - A_0^*) = H$. This shows that $sI - A_0^*$ and $\overline{s}I - A_0^*$ are invertible (as functions) and their inverses are everywhere defined on H. Since A_0^* is closed, the operators $(sI - A_0^*)^{-1}$ and $(\overline{s}I - A_0^*)^{-1}$ are also closed. According to the closed-graph theorem, these inverses are bounded. Thus, we have shown that $s, \overline{s} \in \rho(A_0^*)$.

Now we show that in fact $A_0 = A_0^*$. Since A_0^* is an extension of A_0, we only have to show that $\mathcal{D}(A_0^*) \subset \mathcal{D}(A_0)$. Since $sI - A_0$ is onto, for any $z_0 \in \mathcal{D}(A_0^*)$, we can find a $w_0 \in \mathcal{D}(A_0)$ such that $(sI - A_0)w_0 = (sI - A_0^*)z_0$, whence $(sI - A_0^*)(z_0 - w_0) = 0$. But since $\operatorname{Ker}(sI - A_0^*) = \{0\}$ (as we have seen earlier), $z_0 = w_0 \in \mathcal{D}(A_0)$, so that $A_0 = A_0^*$. Combined with our earlier conclusion this means that $s, \overline{s} \in \rho(A_0)$. $\square$

Remark 3.2.5. If we delete from the last proposition the condition that $\overline{s}I - A_0$ is onto, then the conclusion is no longer true. A simple counterexample will be given in Remark 3.7.4 (if we denote $A_0 = iA$).

Proposition 3.2.6. *If $A_0 : \mathcal{D}(A_0) \to H$ is self-adjoint, then $\sigma(A_0) \subset \mathbb{R}$.*

Proof. If $s \in \mathbb{C}$ is not real, then by Remark 3.2.3 we have Ker $(\overline{s}I - A_0) = \{0\}$, so that (by Remark 2.8.2) Ran $(sI - A_0)$ is dense in H. On the other hand, it follows from Remark 3.2.3 that Ran $(sI - A_0)$ is closed, so that Ran $(sI - A_0) = H$. Since, again by Remark 3.2.3, we have Ker $(sI - A_0) = \{0\}$, it follows that $s \in \rho(A_0)$. □

Now recall the notation $r(T)$ introduced in Section 2.2.

Proposition 3.2.7. *If $T \in \mathcal{L}(H)$ is self-adjoint, then $r(T) = \|T\|$.*

Proof. We have $\|T^2\| \geqslant \sup_{\|z\| \leqslant 1} \langle T^2 z, z\rangle = \sup_{\|z\| \leqslant 1} \|Tz\|^2 = \|T\|^2$. On the other hand, it is obvious that $\|T^2\| \leqslant \|T\|^2$, so we conclude that $\|T^2\| = \|T\|^2$. By induction it follows that

$$\|T^{2^m}\| = \|T\|^{2^m} \qquad \forall\, m \in \mathbb{N}.$$

According to the Gelfand formula (Proposition 2.2.15), in which we take $n = 2^m$, we obtain that $r(T) = \|T\|$. □

Proposition 3.2.8. *If $A_0 : \mathcal{D}(A_0) \to H$ is self-adjoint and $s \in \mathbb{C}$, then*

$$\|(sI - A_0)z\| \geqslant \min_{\lambda \in \sigma(A_0)} |s - \lambda| \cdot \|z\| \qquad \forall\, z \in \mathcal{D}(A_0). \tag{3.2.2}$$

If $s \in \rho(A_0)$, then

$$\|(sI - A_0)^{-1}\| = \frac{1}{\min_{\lambda \in \sigma(A_0)} |s - \lambda|}. \tag{3.2.3}$$

Proof. First we show that the proposition holds for real s. If $s \in \sigma(A_0)$, then this is clearly true. If $s \in \rho(A_0) \cap \mathbb{R}$, then according to Proposition 2.8.4, $T = (sI - A_0)^{-1}$ is self-adjoint and hence, by Proposition 3.2.7, we have $\|(sI - A_0)^{-1}\| = r((sI - A_0)^{-1})$. Together with (2.2.6) this shows that (3.2.3) holds for this s. From here it is easy to conclude that also (3.2.2) holds for this s.

Now take $s \in \mathbb{C}$ and decompose it as $s = \alpha + i\omega$, where $\alpha, \omega \in \mathbb{R}$. Using (3.2.1) and (3.2.2) with α in place of s, we obtain

$$\|(sI - A_0)z\|^2 \geqslant \min_{\lambda \in \sigma(A_0)} |\alpha - \lambda|^2 \cdot \|z\|^2 + \omega^2 \|z\|^2 \qquad \forall\, z \in \mathcal{D}(A_0).$$

It is easy to see that this implies (3.2.2). The latter implies that for $s \in \rho(A_0)$,

$$\|(sI - A_0)^{-1}\| \leqslant \frac{1}{\min_{\lambda \in \sigma(A_0)} |s - \lambda|}.$$

The opposite inequality is known to hold from Remark 2.2.8. □

Recall the definition of a diagonalizable operator from Section 2.6. In the definition we have used a Riesz basis indexed by $\mathbb{N}$, but (as we mentioned later in that section) sometimes we prefer to use other countable index sets. In the proposition below, we use an index set $\mathcal{I} \subset \mathbb{Z}$.

Proposition 3.2.9. *If $A_0 : \mathcal{D}(A_0) \to H$ is self-adjoint and diagonalizable, then there exists in H an orthonormal basis $(\varphi_k)_{k\in\mathcal{I}}$ of eigenvectors of A_0 (here, $\mathcal{I} \subset \mathbb{Z}$). Denoting the eigenvalue corresponding to φ_k by λ_k, we have for $\lambda_k \in \mathbb{R}$,*

$$\mathcal{D}(A_0) = \left\{ z \in H \;\middle|\; \sum_{k\in\mathcal{I}} (1+\lambda_k^2)\, |\langle z, \varphi_k\rangle|^2 < \infty \right\} \tag{3.2.4}$$

and

$$A_0 z = \sum_{k\in\mathcal{I}} \lambda_k \langle z, \varphi_k\rangle \varphi_k \qquad \forall\, z \in \mathcal{D}(A_0). \tag{3.2.5}$$

Proof. According to the assumption, there exists in X a Riesz basis (ϕ_k) consisting of eigenvectors of A_0. For each $\lambda \in \sigma_p(A_0)$, we can choose an orthonormal basis in the (closed) subspace $X_\lambda = \operatorname{Ker}(\lambda I - A_0)$. We collect all the vectors in these orthonormal bases into the family $(\varphi_k)_{k\in\mathcal{I}}$ (this is possible, because $\sigma_p(A_0)$ is countable). It is easy to verify that $(\varphi_k)_{k\in\mathcal{I}}$ is an orthonormal set (it is here that we need $A_0^* = A_0$). To show that this set is actually an orthonormal basis, assume that $x \in X$ is orthogonal to all φ_k. Then it is not difficult to show that x is orthogonal to the original sequence (ϕ_k), so that $x = 0$. Now it is clear that the biorthogonal sequence to our orthonormal basis is the same orthonormal basis. Thus, (3.2.4) and (3.2.5) follow from Proposition 2.6.3. $\square$

Remark 3.2.10. It is easy to see that the converse of Proposition 3.2.9 also holds: If $A_0 : \mathcal{D}(A_0) \to H$ is given by (3.2.4) and (3.2.5), where $\lambda_k \in \mathbb{R}$ and $(\varphi_k)_{k\in\mathcal{I}}$ is an orthonormal basis in H, then A_0 is self-adjoint and diagonalizable, with the eigenvectors φ_k. This follows from Propositions 2.6.2 and 2.8.6.

Remark 3.2.11. Here is a statement related to Proposition 3.2.9: If A_0 is diagonalizable with an orthonormal sequence of eigenvectors and with real eigenvalues, then A_0 is self-adjoint. This follows from Proposition 2.6.3 and Remark 3.2.10.

Self-adjoint operators with compact resolvents fit into the framework of the previous proposition. This is a consequence of the spectral representation of self-adjoint and compact operators given in Section 12.2 of Appendix I.

Proposition 3.2.12. *Let H be an infinite-dimensional Hilbert space and let $A_0 : \mathcal{D}(A_0) \to H$ be a self-adjoint operator with compact resolvents. Then A_0 is diagonalizable with an orthonormal basis $(\varphi_k)_{k\in\mathcal{I}}$ of eigenvectors (where $\mathcal{I} \subset \mathbb{Z}$) and the corresponding family of real eigenvalues $(\lambda_k)_{k\in\mathcal{I}}$ satisfies $\lim_{|k|\to\infty} |\lambda_k| = \infty$.*

Proof. According to Corollary 2.2.13 and Propositions 12.2.8 and 12.2.9, $\sigma(A_0)$ consists of at most countably many real eigenvalues. Choose $\alpha \in \mathbb{R} \cap \rho(A_0)$, then

$K = (\alpha I - A_0)^{-1}$ is self-adjoint and compact. According to Theorem 12.2.11, there exists an orthonormal sequence (φ_k) of eigenvectors of K, indexed by $\mathcal{I} \subset \mathbb{Z}$, and a corresponding sequence (μ_k) of real eigenvalues with $\mu_k \neq 0$, $\mu_k \to 0$ and such that the representation (12.2.5) holds. Denote $\mathcal{B} = \{\varphi_k \mid k \in \mathcal{I}\}$. It follows from (12.2.6) that $\mathcal{B}^\perp = \operatorname{Ker} K = \{0\}$, so that $\mathcal{B}$ is an orthonormal basis in H. It follows now from Proposition 2.2.18 that A_0 is diagonal, having the same sequence (φ_k) of eigenvectors and the corresponding eigenvalues are $\lambda_k = \alpha - \frac{1}{\mu_k}$. Since $\mu_k \to 0$, then $|\lambda_k| \to \infty$. □

The eigenvalues of self-adjoint operators with compact resolvents can be characterized by the following min-max principle, called the *Courant–Fischer theorem.*

Proposition 3.2.13. *Let H be an infinite-dimensional Hilbert space and let $A_0 : \mathcal{D}(A_0) \to H$ be a self-adjoint operator with compact resolvents such that the eigenvalues of A_0 are bounded from below. We define the function $R_{A_0} : \mathcal{D}(A_0) \setminus \{0\} \to \mathbb{R}$ by*

$$R_{A_0}(z) = \frac{\langle A_0 z, z\rangle}{\|z\|^2} \qquad \forall\, z \in \mathcal{D}(A_0) \setminus \{0\}. \tag{3.2.6}$$

We order the eigenvalues of A_0 to form an increasing sequence $(\mu_k)_{k\in\mathbb{N}}$ such that each μ_k is repeated as many times as its geometric multiplicity. Then

$$\mu_k = \min_{\substack{V \text{ subspace of } \mathcal{D}(A_0)\\ \dim V = k}} \max_{z\in V\setminus\{0\}} R_{A_0}(z) \qquad \forall\, k \in \mathbb{N}, \tag{3.2.7}$$

$$\mu_k = \max_{\substack{V \text{ subspace of } \mathcal{D}(A_0)\\ \dim V = k-1}} \min_{z\in V^\perp\setminus\{0\}} R_{A_0}(z) \qquad \forall\, k \in \mathbb{N}. \tag{3.2.8}$$

Proof. We know from Proposition 3.2.12 that the eigenvalues $(\lambda_k)_{k\in\mathcal{I}}$ of A_0 are real and they satisfy $\lim |\lambda_k| = \infty$. By combining this fact with the assumption that (λ_k) is bounded from below, it follows that $\lim \lambda_k = \infty$ and that indeed the eigenvalues of A_0 can be ordered to form an increasing sequence $(\mu_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} \mu_k = \infty$. Let $(\varphi_k)_{k\in\mathbb{N}}$ be an orthonormal basis formed of eigenvectors of A_0 such that φ_k is an eigenvector associated with (μ_k) for every $k \in \mathbb{N}$, and denote

$$V_k = \operatorname{span}\{\varphi_1, \dots, \varphi_k\} \qquad \forall\, k \in \mathbb{N}.$$

It is easy to check that $\max_{z\in V_k\setminus\{0\}} R_{A_0}(z) = \mu_k$, so that

$$\mu_k \geqslant \min_{\substack{V \text{ subspace of } \mathcal{D}(A_0)\\ \dim V = k}} \max_{z\in V\setminus\{0\}} R_{A_0}(z) \qquad \forall\, k \in \mathbb{N}. \tag{3.2.9}$$

Let now V be a subspace of $\mathcal{D}(A_0)$ of dimension k and denote $W = V + V_{k-1}$. Then W is a finite-dimensional inner product space if endowed with the inner product inherited from H. Denote $m = \dim W$ and let $V_{k-1}^\perp$ be the orthogonal complement (in W) of V_{k-1}. Then $\dim V_{k-1}^\perp = m - k + 1$ and

$$m \geqslant \dim\left(V_{k-1}^\perp + V\right) = m - k + 1 + k - \dim\left(V_{k-1}^\perp \cap V\right),$$

so that $V \cap V_{k-1}^{\perp}$ contains at least one element different from zero, denoted by w. Since $w \in V_{k-1}^{\perp} \setminus \{0\}$, it follows that there exists an l^2 sequence $(w_p)_{p \geqslant k}$ such that

$$\sum_{p \geqslant k} |w_p|^2 > 0 \quad \text{and} \quad w = \sum_{p \geqslant k} w_p \varphi_p .$$

From the above formulas we get that

$$R_{A_0}(w) = \frac{\sum_{p \geqslant k} \mu_p |w_p|^2}{\sum_{p \geqslant k} |w_p|^2} \geqslant \mu_k ,$$

so that

$$\mu_k \leqslant \min_{\substack{V \text{ subspace of } \mathcal{D}(A_0) \\ \dim V = k}} \max_{z \in V \setminus \{0\}} R_{A_0}(z) \qquad \forall\, k \in \mathbb{N}.$$

The above estimate, together with (3.2.9), gives (3.2.7).

The estimate (3.2.8) can be proved in a similar way. $\square$

3.3 Positive operators

As in the previous section, H will denote a Hilbert space and we usually denote by A_0 an operator defined on a dense subspace $\mathcal{D}(A_0) \subset H$ and with values in H.

Definition 3.3.1. Let $A_0 : \mathcal{D}(A_0) \to H$ be self-adjoint. Then A_0 is *positive* if $\langle A_0 z, z\rangle \geqslant 0$ for all $z \in \mathcal{D}(A_0)$. A_0 is *strictly positive* if for some $m > 0$

$$\langle A_0 z, z\rangle \geqslant m\|z\|^2 \qquad \forall\, z \in \mathcal{D}(A_0). \tag{3.3.1}$$

We write $A_0 \geqslant 0$ (or $A_0 > 0$) to indicate that A_0 is positive (or strictly positive). The notations $A_0 \leqslant 0$, $A_0 < 0$ mean that $-A_0 \geqslant 0$, $-A_0 > 0$, respectively. If A_1 is another self-adjoint operator on H, then the notation $A_1 \geqslant A_0$ means that $\mathcal{D}(A_1) \cap \mathcal{D}(A_0)$ is dense in H and $\langle A_1 z, z\rangle \geqslant \langle A_0 z, z\rangle$ for all $z \in \mathcal{D}(A_1) \cap \mathcal{D}(A_0)$. The meanings of $A_1 > A_0$, $A_0 \leqslant A_1$ and $A_0 < A_1$ are similar, with the obvious modifications. Note that if at least one of the operators A_1, A_0 or $A_1 - A_0$ is bounded on H, then $A_1 \geqslant A_0$ (or $A_1 > A_0$) just means that $A_1 - A_0 \geqslant 0$ (or that $A_1 - A_0 > 0$). Thus, if A_0 satisfies (3.3.1), then we can write $A_0 \geqslant mI$.

Proposition 3.3.2. *Let $A_0 : \mathcal{D}(A_0) \to H$ be such that $A_0 \geqslant mI$, $m > 0$. Then $0 \in \rho(A_0)$, $\|A_0^{-1}\| \leqslant \frac{1}{m}$ and*

$$\langle A_0^{-1} w, w\rangle > 0 \qquad \forall\, w \in H \setminus \{0\}. \tag{3.3.2}$$

Proof. For all $z \in \mathcal{D}(A_0)$ we have $\|A_0 z\| \cdot \|z\| \geqslant \langle A_0 z, z\rangle \geqslant m\|z\|^2$, whence

$$\|A_0 z\| \geqslant m\|z\| \qquad \forall\, z \in \mathcal{D}(A_0). \tag{3.3.3}$$

This inequality and the closedness of A_0 imply that Ran A_0 is closed in H. According to Remark 2.8.2 we have $(\text{Ran } A_0)^\perp = \text{Ker } A_0 = \{0\}$, so that Ran $A_0 = H$. Thus A_0 is invertible and (3.3.3) implies that $\|A_0^{-1}\| \leqslant \frac{1}{m}$. Finally, to prove (3.3.2), take $w \in H$, $w \neq 0$, and denote $z = A_0^{-1}w$. Then $\langle A_0^{-1}w, w\rangle = \langle z, A_0 z\rangle > 0$. $\square$

Proposition 3.3.3. *Let $A_0 : \mathcal{D}(A_0) \to H$ be self-adjoint. Then $A_0 \geqslant 0$ if and only if $\sigma(A_0) \subset [0, \infty)$.*

Proof. Suppose that $A_0 \geqslant 0$. We know from Proposition 3.2.6 that $\sigma(A_0) \subset \mathbb{R}$, so we only have to show that negative numbers are in $\rho(A_0)$. For every $m > 0$ we have $mI + A_0 \geqslant mI$, so that by Proposition 3.3.2, $mI + A_0$ has a bounded inverse, hence $-m \in \rho(A_0)$. Thus we have shown that $\sigma(A_0) \subset [0, \infty)$.

Conversely, suppose that A_0 is self-adjoint and $\sigma(A_0) \subset [0, \infty)$. According to Proposition 3.2.8, for every $m > 0$ we have (using $s = -m$ in (3.2.2))

$$\|(mI + A_0)z\| \geqslant m\|z\| \qquad \forall\, z \in \mathcal{D}(A_0).$$

According to Proposition 3.1.2, $-A_0$ is dissipative, i.e., $\text{Re}\,\langle A_0 z, z\rangle \geqslant 0$ for all $z \in \mathcal{D}(A_0)$. Since A_0 is self-adjoint, this implies that $\langle A_0 z, z\rangle \geqslant 0$, i.e., $A_0 \geqslant 0$. $\square$

Remark 3.3.4. Let $A_0 : \mathcal{D}(A_0) \to H$ be self-adjoint and $\lambda \in \mathbb{R}$. Then $A_0 \geqslant \lambda I$ iff $\sigma(A_0) \subset [\lambda, \infty)$. Indeed, this follows from the last proposition, with $A_0 - \lambda I$ in place of A_0. Hence, $A_0 > 0$ iff $\sigma(A_0) \subset (0, \infty)$.

Proposition 3.3.5. *If $A_0 \geqslant 0$, then $-A_0$ is m-dissipative.*

Proof. Since A_0 is closed (as remarked at the beginning of this section) and clearly $-A_0$ is dissipative, according to Proposition 3.1.11, $-A_0$ is m-dissipative. $\square$

If $A_0 : \mathcal{D}(A_0) \to X$ is self-adjoint, then the spaces X_1 and X_1^d from Section 2.10 coincide. Similarly, their duals with respect to the pivot space X, denoted X_{-1} and X_{-1}^d, coincide. Recall the higher-order space $X_2 = \mathcal{D}(A_0^2)$ introduced in Remark 2.10.5. If the pivot space is denoted by H instead of X, then we write H_2, H_1, H_{-1} instead of X_2, X_1, X_{-1}. Thus, if $A_0 : \mathcal{D}(A_0) \to H$ is self-adjoint, then we have

$$H_2 \subset H_1 \subset H \subset H_{-1},$$

densely and with continuous embeddings. We have $A_0 \in \mathcal{L}(H_2, H_1)$, $A_0 \in \mathcal{L}(H_1, H)$ and A_0 can be extended such that $A_0 \in \mathcal{L}(H, H_{-1})$.

Proposition 3.3.6. *If A_0 is a self-adjoint operator on H, then $A_0^2 \geqslant 0$.*

Proof. It is easy to see that A_0^2 is symmetric. To show that it is self-adjoint, we use Proposition 3.2.4 with $s = -1$. Thus, we have to show that for every $f \in H$ there exists $z \in H_2$ such that $(I + A_0^2)z = f$. The graph norm on H_1 is induced by the inner product

$$\langle z, \varphi\rangle_{gr} = \langle z, \varphi\rangle + \langle A_0 z, A_0 \varphi\rangle.$$

Take $f \in H$. According to the Riesz representation theorem on H_1, with the above inner product, there exists $z \in H_1$ such that

$$\langle z, \varphi \rangle + \langle A_0 z, A_0 \varphi \rangle = \langle f, \varphi \rangle \qquad \forall\, \varphi \in H_1 . \tag{3.3.4}$$

This formula shows that the functional $\varphi \mapsto \langle A_0 z, A_0 \varphi \rangle$ has a continuous extension to H. Therefore, $A_0 z \in \mathcal{D}(A_0^*) = \mathcal{D}(A_0) = H_1$, so that $z \in \mathcal{D}(A_0^2)$. With this information, (3.3.4) can be rewritten as $z + A_0^2 z = f$. We have shown that A_0^2 is self-adjoint. It is obvious that $\langle A_0^2 z, z\rangle \geqslant 0$, so that $A_0^2 \geqslant 0$. □

Remark 3.3.7. If A_0 is a self-adjoint operator on H and $0 \in \rho(A_0)$, then $A_0^2 > 0$. Indeed, from the last proposition we know that $A_0^2 \geqslant 0$. From Proposition 2.2.12 we see that $0 \in \rho(A_0^2)$. Thus, by Remark 3.3.4 we obtain $A_0^2 > 0$.

Example 3.3.8. Let J be an interval in $\mathbb{R}$ and let $f : J \to \mathbb{R}$ be measurable. On $H = L^2(J)$ consider the pointwise multiplication operator A_0 defined by

$$A_0 z(x) = f(x) z(x) \quad \text{for almost every } x \in J,$$
$$\mathcal{D}(A_0) = \left\{ z \in L^2(J) \mid fz \in L^2(J) \right\} .$$

It is not obvious that $\mathcal{D}(A_0)$ is dense in H. To prove this, introduce for each $n \in \mathbb{N}$ the set $J_n = \{x \in J \mid |f(x)| > n\}$. Then (J_n) is a decreasing sequence of measurable sets whose intersection is empty. This implies that, denoting the Lebesgue measure by λ, $\lim_{n\to\infty} \lambda(J_n) = 0$. Denote the characteristic function of $J \setminus J_n$ by χ_n. For every $z \in H$, the sequence of functions (z_n) defined by $z_n = \chi_n z$ has the following properties: $z_n \in \mathcal{D}(A_0)$ and $\lim_{n\to\infty} z_n = z$. This shows that $\mathcal{D}(A_0)$ is indeed dense in H.

It is now easy to see that A_0 is symmetric. Moreover, for any $s \in \mathbb{C} \setminus \mathbb{R}$, the operator $sI - A_0$ is onto. Indeed, for any $g \in H$, the equation $(sI - A_0)z = g$ has a solution given by $z(x) = g(x)/[s - f(x)]$ for almost every $x \in J$, and $\|z\| \leqslant \|g\|/|\mathrm{Im}\, s|$. According to Proposition 3.2.4, A_0 is self-adjoint.

It is interesting to investigate the spectrum of A_0. We define the *essential range* of f, denoted $\mathrm{ess\,Ran}\, f$, as follows: A point $\mu \in \mathbb{R}$ belongs to $\mathrm{ess\,Ran}\, f$ if for any interval D centered at μ, $\lambda(f^{-1}(D)) > 0$. Thus, for a continuous function, its essential range is simply its range. For any measurable function f, changing the values of f on a set of measure zero will not change its essential range. It is now an easy exercise to check that

$$\sigma(A_0) = \mathrm{ess\,Ran}\, f .$$

The following statements are easy to verify: $A_0 \geqslant 0$ iff $f(x) \geqslant 0$ for almost every $x \in J$, $A_0 > 0$ iff there exists $m > 0$ such that $f(x) \geqslant m$ for almost every $x \in J$. A_0 is bounded iff $\mathrm{ess\,Ran}\, f$ is bounded, which is equivalent to $f \in L^\infty(J)$.

Example 3.3.9. Take $H = L^2[0, \infty)$,

$$\mathcal{D}(A_0) = \mathcal{H}^2(0, \infty) \cap \mathcal{H}_0^1(0, \infty), \qquad A_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} .$$

An integration by parts shows that A_0 is symmetric. By elementary techniques from the theory of linear differential equations we can verify that $I + A_0$ is onto. Indeed, for every $f \in L^2[0,\infty)$, the Laplace transform of $z \in \mathcal{D}(A_0)$ satisfying $(I + A_0)z = f$ is given by

$$\hat{z}(s) = \frac{-1}{s+1} \cdot \frac{\hat{f}(s) - \hat{f}(1)}{s-1} \qquad \forall\, s \in \mathbb{C}_0 \setminus \{1\}.$$

For $s = 1$ we take the obvious extension of $\hat{z}$ by continuity. Note that $\frac{\mathrm{d}z}{\mathrm{d}x}(0) = \hat{f}(1)$. We mention that the same conclusion (that $I + A_0$ is onto) could have been obtained also from the Riesz representation theorem on the space $\mathcal{H}_0^1(0,\infty)$. Consequently, by Proposition 3.2.4, the operator A_0 is self-adjoint. Since

$$\langle A_0 z, z\rangle = \int_0^\infty \left|\frac{\mathrm{d}z}{\mathrm{d}x}\right|^2 \mathrm{d}x \qquad \forall\, z \in \mathcal{D}(A_0),$$

we have that A_0 is positive. According to Proposition 3.3.3 we have $\sigma(A_0) \subset [0,\infty)$.

The above properties of A_0 are shared by its counterpart on a bounded interval, introduced in Example 2.6.8. It is easy to check that A_0 has no eigenvalues so that, unlike the operator introduced in Example 2.6.8, the resolvents of A_0 are not compact. Another interesting property is that $\sigma(A_0) = [0,\infty)$. Indeed, for every $\lambda > 0$, the equation $(\lambda^2 I - A_0)v = f$ has no solution for $f(t) = e^{-t}$ (many other functions could be used instead of e^{-t}). To see this, consider that v is a solution of the equation. Then, denoting the Laplace transform of v by $\hat{v}$, we get

$$(\lambda^2 + s^2)\hat{v}(s) - sv'(0) = \frac{1}{s+1}.$$

This shows that $\hat{v}$ is rational and has poles at $\pm i\lambda$, so that v cannot be in $L^2[0,\infty)$. Thus, $\lambda^2 \in \sigma(A)$. This being true for every $\lambda > 0$, we obtain that $\sigma(A_0) = [0,\infty)$.

3.4 The spaces $H_{\frac{1}{2}}$ and $H_{-\frac{1}{2}}$

In this section, H is a Hilbert space and $A_0 : \mathcal{D}(A_0) \to H$ is strictly positive ($A_0 > 0$). We shall introduce the square root of A_0, based on the concept of the square root of a bounded positive operator (see Section 12.3 in Appendix I). Then we define the space $H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}})$ with a suitable norm, and $H_{-\frac{1}{2}}$ will be the dual of $H_{\frac{1}{2}}$ with respect to the pivot space H. These spaces are useful in the analysis of certain systems described by PDEs which are of second order in time.

To introduce $A_0^{\frac{1}{2}}$, we use the facts that $A_0^{-1} \in \mathcal{L}(H)$, $A_0^{-1} \geqslant 0$ and Ker $A_0^{-1} = \{0\}$, which follow from Proposition 3.3.2. Denote

$$A_0^{-\frac{1}{2}} = (A_0^{-1})^{\frac{1}{2}}, \qquad \mathcal{D}(A_0^{\frac{1}{2}}) = \operatorname{Ran} A_0^{-\frac{1}{2}}.$$

Then $A_0^{-\frac{1}{2}} : H \to \mathcal{D}(A_0^{\frac{1}{2}})$ is invertible on its range and its (possibly unbounded) inverse is denoted by $A_0^{\frac{1}{2}}$. Thus, by definition, $A_0^{\frac{1}{2}} = ((A_0^{-1})^{\frac{1}{2}})^{-1}$.

Proposition 3.4.1. *For A_0 as above, we have $A_0^{\frac{1}{2}} > 0$.*

Proof. Since $A_0^{-\frac{1}{2}}$ is self-adjoint, it follows from Proposition 2.8.4 that $A_0^{\frac{1}{2}}$ is self-adjoint. According to Proposition 2.2.12, $\sigma(A_0^{\frac{1}{2}}) = (\sigma(A_0))^{\frac{1}{2}}$. Since $A_0 > 0$, by Remark 3.3.4, we have $\sigma(A_0) \subset [\lambda, \infty)$ for some $\lambda > 0$, hence $\sigma(A_0^{\frac{1}{2}}) \subset [\lambda^{\frac{1}{2}}, \infty)$, hence (using again Remark 3.3.4) $A_0^{\frac{1}{2}} \geqslant \lambda^{\frac{1}{2}} I$. □

Remark 3.4.2. For A_0 as above, there is a unique operator $S : \mathcal{D}(S) \to H$ with the properties that $S \geqslant 0$, $S^2 = A_0$, and this is $A_0^{\frac{1}{2}}$. Indeed, clearly $S > 0$, hence we have $S^{-1} \in \mathcal{L}(H)$ and $S^{-1} \geqslant 0$ (see Proposition 3.3.2). We have $S^{-2} = A_0^{-1}$, so that according to the uniqueness part of Theorem 12.3.4 we have $S^{-1} = A_0^{-\frac{1}{2}}$.

We define H_1 as the space $\mathcal{D}(A_0)$ with the norm $\|f\|_1 = \|A_0 f\|$, which is equivalent to the graph norm of A_0 and it is induced by the inner product

$$\langle f, g\rangle_1 = \langle A_0 f, A_0 g\rangle \qquad \forall\, f, g \in H_1 .$$

Similarly, we define the Hilbert space $H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}})$ with the norm $\|f\|_{\frac{1}{2}} = \|A_0^{\frac{1}{2}} f\|$, which is equivalent to the graph norm of $A_0^{\frac{1}{2}}$ and it is induced by

$$\langle f, g\rangle_{\frac{1}{2}} = \langle A_0^{\frac{1}{2}} f, A_0^{\frac{1}{2}} g\rangle \qquad \forall\, f, g \in H_{\frac{1}{2}} .$$

Clearly, if $f \in \mathcal{D}(A_0)$, then the above formula simplifies to $\langle f, g\rangle_{\frac{1}{2}} = \langle A_0 f, g\rangle$.

Proposition 3.4.3. *We have $H_1 \subset H_{\frac{1}{2}} \subset H$, densely and with continuous embeddings. Moreover, $A_0^{\frac{1}{2}} \in \mathcal{L}(H_1, H_{\frac{1}{2}})$ and $A_0^{\frac{1}{2}} \in \mathcal{L}(H_{\frac{1}{2}}, H)$ are unitary.*

Proof. From the definitions it is clear that $H_1 \subset H_{\frac{1}{2}} \subset H$. Since $A_0^{\frac{1}{2}}$ is self-adjoint, its domain $H_{\frac{1}{2}}$ is dense in H. To prove that H_1 is dense in $H_{\frac{1}{2}}$, take $z \in H_{\frac{1}{2}}$, so that $z = A_0^{-\frac{1}{2}} x$ for some $x \in H$. Let (x_n) be a sequence in $H_{\frac{1}{2}}$ such that $x_n \to x$. It is easy to see that $A_0^{-\frac{1}{2}} x_n \in H_1$ and $A_0^{-\frac{1}{2}} x_n \to A_0^{-\frac{1}{2}} x = z$ in $H_{\frac{1}{2}}$.

The continuity of the embeddings follows immediately from the definition of the norm on these spaces and the fact that $A_0^{\frac{1}{2}} > 0$ (see Proposition 3.4.1). The fact that $A_0^{\frac{1}{2}}$ is unitary between the spaces indicated in the proposition is an immediate consequence of the definition of the norm on these spaces. □

Remark 3.4.4. It follows from the last proposition that $H_{\frac{1}{2}}$ may also be regarded as the completion of $\mathcal{D}(A_0)$ with respect to the norm

$$\|f\|_{\frac{1}{2}} = \sqrt{\langle A_0 f, f\rangle} \qquad \forall\, f \in \mathcal{D}(A_0).$$

We define the spaces $H_{-\frac{1}{2}}$ and H_{-1} as the duals of $H_{\frac{1}{2}}$ and H_1, respectively, with respect to the pivot space H (see Section 2.9). Then we have the dense and continuous embeddings

$$H_1 \subset H_{\frac{1}{2}} \subset H \subset H_{-\frac{1}{2}} \subset H_{-1} .$$

Proposition 3.4.5. *$A_0^{\frac{1}{2}}$ and A_0 have unique extensions such that*

$$A_0^{\frac{1}{2}} \in \mathcal{L}(H, H_{-\frac{1}{2}}), \qquad A_0 \in \mathcal{L}(H, H_{-1}). \tag{3.4.1}$$

Using the inverses of these extensions, the norms on $H_{-\frac{1}{2}}$ and on H_{-1} can also be expressed as

$$\|z\|_{-\frac{1}{2}} = \|A_0^{-\frac{1}{2}} z\|, \qquad \|z\|_{-1} = \|A_0^{-1} z\|,$$

so that the operators in (3.4.1) *are unitary. These operators can also be regarded as strictly positive* (*densely defined*) *operators on $H_{-\frac{1}{2}}$ and on H_{-1}, respectively.*

Proof. We can apply Propositions 2.10.2 and 2.10.3 (with H in place of X, A_0 in place of A and 0 in place of β) to conclude that A_0 has unique extension in $\mathcal{L}(H, H_{-1})$, A_0^{-1} has a unique extension in $\mathcal{L}(H_{-1}, H)$ and these operators are unitary. This implies, in particular, that the norm on H_{-1} can indeed be expressed as stated. The strict positivity of the extended A_0 follows from the fact that it is the image of the original $A_0 : \mathcal{D}(A_0) \to H$ through the unitary operator $A_0 \in \mathcal{L}(H, H_{-1})$.

Repeating the above argument with $A_0^{\frac{1}{2}}$ in place of A_0 and $H_{-\frac{1}{2}}$ in place of H_{-1}, we obtain the remaining statements in the proposition. □

Corollary 3.4.6. *A_0 has a unique extension such that*

$$A_0 \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}}),$$

and this is unitary. Moreover, this extension of A_0 can be regarded as a strictly positive (*densely defined*) *operator on $H_{-\frac{1}{2}}$.*

Proof. We know from the previous proposition that $A_0^{\frac{1}{2}} \in \mathcal{L}(H_{\frac{1}{2}}, H)$ and $A_0^{\frac{1}{2}} \in \mathcal{L}(H, H_{-\frac{1}{2}})$. The combination of these unitary operators is a unitary operator $\tilde{A}_0 \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$ and $\tilde{A}_0$ is clearly an extension of the original A_0. If we regard A_0 as a densely defined strictly positive operator on H, then $\tilde{A}_0$ is the image of A_0 through the unitary operator $A_0^{\frac{1}{2}} \in \mathcal{L}(H, H_{-\frac{1}{2}})$, so that $\tilde{A}_0$ is a strictly positive densely defined operator on $H_{-\frac{1}{2}}$. As in the previous proposition, we use the notation A_0 for extensions of the original A_0 by continuity, in particular for $\tilde{A}_0$. □

Remark 3.4.7. In this remark, we use the notation $\tilde{A}_0$ for the extension of A_0 to a strictly positive (densely defined) operator on $\tilde{H} = H_{-\frac{1}{2}}$, introduced in the last corollary. Then $\tilde{H}_1 = \mathcal{D}(\tilde{A}_0) = H_{\frac{1}{2}}$ and $\tilde{H}_{\frac{1}{2}} = \mathcal{D}(\tilde{A}_0^{\frac{1}{2}}) = H$, with equal norms. The proof is straightforward and we leave it to the reader.

In the particular case when $A_0 > 0$ is diagonalizable (see Section 2.6), it follows from Proposition 3.2.9 that there exists an orthonormal basis (φ_k) in H consisting of eigenvectors of A_0 (here we take $k \in \mathbb{N}$). If we denote the corresponding sequence of eigenvalues of A_0 by (λ_k), then A_0 can be written as in (3.2.4) and (3.2.5). In this case, there are simple explicit formulas for $A_0^{\frac{1}{2}}$ and for its domain, as follows.

Proposition 3.4.8. *Suppose that A_0 is diagonalizable, with the orthonormal basis of eigenvectors (φ_k) and the corresponding sequence of eigenvalues (λ_k). Then*

$$\mathcal{D}(A_0^{\frac{1}{2}}) = \left\{ z \in H \;\middle|\; \sum_{k=1}^{\infty} \lambda_k \left|\langle z, \varphi_k \rangle\right|^2 < \infty \right\}, \tag{3.4.2}$$

$$A_0^{\frac{1}{2}} z = \sum_{k=1}^{\infty} \lambda_k^{\frac{1}{2}} \langle z, \varphi_k \rangle \varphi_k \qquad \forall\, z \in \mathcal{D}(A_0^{\frac{1}{2}}). \tag{3.4.3}$$

Moreover, the dual space $H_{-\frac{1}{2}} = H'_{\frac{1}{2}}$ can also be described as

$$H_{-\frac{1}{2}} = \left\{ z \in H_{-1} \;\middle|\; \sum_{k=1}^{\infty} \lambda_k^{-1} \left|\langle z, \varphi_k \rangle\right|^2 < \infty \right\}, \tag{3.4.4}$$

and its norm is

$$\|z\|_{-\frac{1}{2}} = \left(\sum_{k=1}^{\infty} \lambda_k^{-1} \left|\langle z, \varphi_k \rangle\right|^2 \right)^{\frac{1}{2}} \qquad \forall\, z \in H_{-\frac{1}{2}}. \tag{3.4.5}$$

Proof. We shall need several times the approximation formula

$$z = \lim_{N \to \infty} \sum_{k=1}^{N} \langle z, \varphi_k \rangle \varphi_k \ \text{ in } \ H_\alpha \qquad \forall\, z \in H_\alpha, \tag{3.4.6}$$

which is true in any of the spaces H_α under consideration ($\alpha = 1, \frac{1}{2}, 0, -\frac{1}{2}, -1$) and in which the coefficients $\langle z, \varphi_k \rangle$ (understood as a duality pairing between $z \in H_\alpha$ and $\varphi_k \in H_{-\alpha}$) depend on z but they are independent of α.

In order to prove (3.4.2) we recall that $\mathcal{D}(A_0^{\frac{1}{2}})$ is the completion of $\mathcal{D}(A_0)$ with respect to the norm $\|z\|_{\frac{1}{2}} = \sqrt{\langle A_0 z, z \rangle}$. Using (3.2.5) we obtain that

$$\|z\|_{\frac{1}{2}} = \left(\sum_{k=1}^{\infty} \lambda_k \left|\langle z, \phi_k \rangle\right|^2 \right)^{\frac{1}{2}} \qquad \forall z \in H_1.$$

Using (3.4.6) we obtain that the above formula for $\|z\|_{\frac{1}{2}}$ remains valid for all $z \in H_{\frac{1}{2}}$ and (3.4.2) holds. To prove (3.4.3) we notice that the operator defined by the right-hand side of (3.4.3) is positive and its square is A_0. Because of the uniqueness of the square root (see Remark 3.4.2) this operator is in fact $A_0^{\frac{1}{2}}$.

The formula for $A_0^{-\frac{1}{2}}$ is easy to obtain from (3.4.3): replace $\lambda_k^{\frac{1}{2}}$ with $\lambda_k^{-\frac{1}{2}}$. (Indeed, this is true for $z \in H$ and by continuous extension it must be true for $z \in H_{-\frac{1}{2}}$.) From here and from Proposition 3.4.5, formula (3.4.5) follows.

To prove (3.4.4), note that $H_{-\frac{1}{2}}$ is the completion of H with respect to the norm in (3.4.5). Now use again the approximation (3.4.6). $\square$

Proposition 3.4.9. *Let $A_0 > 0$ and $Q = Q^* \in \mathcal{L}(H)$ be such that $A_1 = A_0 + Q > 0$. We define the norm $\|\cdot\|_1'$ induced by A_1 on $\mathcal{D}(A_0)$ by $\|z\|_1' = \|A_1 z\|$. Then the norms $\|\cdot\|_1'$ and $\|\cdot\|_1$ are equivalent. Moreover, $\mathcal{D}(A_1^{\frac{1}{2}}) = \mathcal{D}(A_0^{\frac{1}{2}})$ and the norm $\|\cdot\|_{\frac{1}{2}}'$ defined on $\mathcal{D}(A_0^{\frac{1}{2}})$ by $\|z\|_{\frac{1}{2}}' = \|A_1^{\frac{1}{2}} z\|$ is equivalent to $\|\cdot\|_{\frac{1}{2}}$.*

Proof. Let $m > 0$ be such that $A_0 \geqslant mI$. Then for all $z \in \mathcal{D}(A_0)$,

$$\|z\|_1' = \|(A_0 + Q)z\| \leqslant \|A_0 z\| + \|Q\| \cdot \|z\| \leqslant \left(1 + \frac{\|Q\|}{m}\right) \|A_0 z\|,$$

so that the norm $\|\cdot\|_1$ is stronger than $\|\cdot\|_1'$. By a very similar argument, the norm $\|\cdot\|_1'$ is stronger than $\|\cdot\|_1$. Thus, these two norms on $\mathcal{D}(A_0)$ are equivalent.

Note that there exists a number $k \geqslant 0$ such that $Q \leqslant kA_0$. Indeed, denoting $k = \frac{\|Q\|}{m}$ we have

$$Q \leqslant \|Q\| \cdot I = \frac{\|Q\|}{m} \cdot mI \leqslant kA_0 .$$

We know from Remark 3.4.4 that $\mathcal{D}(A_1^{\frac{1}{2}})$ is the completion of $\mathcal{D}(A_0)$ with respect to the norm

$$\|z\|_{\frac{1}{2}}' = \sqrt{\langle A_1 z, z\rangle}.$$

Since

$$\langle A_1 z, z\rangle = \langle A_0 z, z\rangle + \langle Qz, z\rangle \leqslant (1 + k)\langle A_0 z, z\rangle,$$

the norm $\|\cdot\|_{\frac{1}{2}}'$ on $\mathcal{D}(A_0)$ is stronger than $\|\cdot\|_{\frac{1}{2}}$. By a similar argument, the norm $\|\cdot\|_{\frac{1}{2}}$ is stronger than $\|\cdot\|_{\frac{1}{2}}'$. Thus, these two norms are equivalent, whence $\mathcal{D}(A_0^{\frac{1}{2}}) = \mathcal{D}(A_1^{\frac{1}{2}})$ and the extensions of the two norms to $\mathcal{D}(A_0^{\frac{1}{2}})$ are also equivalent. $\square$

Remark 3.4.10. We state without proof two results (probably the simplest ones) from an area of functional analysis called interpolation theory. We use the notation of Proposition 3.4.3. The first statement is as follows: If $L \in \mathcal{L}(H)$ is such that $LH_1 \subset H_1$ (hence $L \in \mathcal{L}(H_1)$), then also $LH_{\frac{1}{2}} \subset H_{\frac{1}{2}}$ (hence $L \in \mathcal{L}(H_{\frac{1}{2}})$). This is a particular case of Lions and Magenes [157, Theorem 5.1, Chapter 1].

The second statement is as follows: Suppose that $\mathbb{T} = (\mathbb{T}_t)_{t \geqslant 0}$ is a family of operators in $\mathcal{L}(H)$ such that $\mathbb{T}_t H_1 \subset H_1$ for all $t \geqslant 0$,

$$\lim_{t \to 0} \mathbb{T}_t z = z \quad (\text{in } H) \qquad \forall\, z \in H, \tag{3.4.7}$$

and a property similar to (3.4.7) holds with H_1 in place of H everywhere. Then a similar property holds also with $H_{\frac{1}{2}}$ in place of H. This is a particular case of [157, Theorem 5.2, Chapter 1]. Thus, it follows that if $\mathbb{T}$ is a strongly continuous semigroup on both H and H_1, then it is also on $H_{\frac{1}{2}}$.

For positive operators, the Courant–Fischer theorem (Proposition 3.2.13) can be reformulated as follows.

Proposition 3.4.11. *Let H be an infinite-dimensional Hilbert space and let $A_0 : \mathcal{D}(A_0) \to H$ be a positive operator with compact resolvents. We order the eigenvalues of A_0 to form an increasing sequence $(\mu_k)_{k \in \mathbb{N}}$ such that each μ_k is repeated as many times as its geometric multiplicity. Then*

$$\mu_k = \min_{\substack{V \text{ subspace of } H_{\frac{1}{2}} \\ \dim V = k}} \ \max_{z \in V \setminus \{0\}} \frac{\left\| A_0^{\frac{1}{2}} z \right\|^2}{\|z\|^2} \qquad \forall\, k \in \mathbb{N}, \tag{3.4.8}$$

$$\mu_k = \max_{\substack{V \text{ subspace of } H_{\frac{1}{2}} \\ \dim V = k-1}} \ \min_{z \in V^\perp \setminus \{0\}} \frac{\left\| A_0^{\frac{1}{2}} z \right\|^2}{\|z\|^2} \qquad \forall\, k \in \mathbb{N}.$$

Note that we have replaced the space $\mathcal{D}(A_0)$ in Proposition 3.2.13 with the larger space $H_{\frac{1}{2}}$, but this does not change the result. This can be seen either by a density argument, or by redoing the proof of Proposition 3.2.13 in the new context.

Example 3.4.12. Let $H = L^2[0, \pi]$ and let $A_0 : \mathcal{D}(A_0) \to H$ be the operator defined by

$$\mathcal{D}(A_0) = \left\{ z \in \mathcal{H}^2(0, \pi) \;\middle|\; \frac{\mathrm{d}z}{\mathrm{d}x}(0) = z(\pi) = 0 \right\},$$

$$A_0 z = -\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} \qquad \forall\, z \in \mathcal{D}(A_0).$$

Note that $A_0 = -A$, where A is the operator introduced in Example 2.6.10, so that A_0 is diagonalizable, with the eigenvalues

$$\lambda_k = \left(k - \frac{1}{2} \right)^2 \qquad \forall\, k \in \mathbb{N},$$

and with an orthonormal basis of eigenvectors, given in Example 2.6.10. By Remark 3.2.11, A_0 is self-adjoint. Since $\lambda_k \geqslant 1/4$, it follows that $A_0 > 0$. Moreover, a simple integration by parts shows that

$$\langle A_0 z, z\rangle = \left\|\frac{\mathrm{d}z}{\mathrm{d}x}\right\|^2 \qquad \forall\, z \in \mathcal{D}(A_0), \tag{3.4.9}$$

so that the space $H_{\frac{1}{2}}$ (which is $\mathcal{D}(A_0^{\frac{1}{2}})$ with the graph norm) is the completion of $\mathcal{D}(A_0)$ with respect to the norm

$$\|z\|_{\frac{1}{2}} = \left\|\frac{\mathrm{d}z}{\mathrm{d}x}\right\|.$$

On $\mathcal{D}(A_0)$, this norm is obviously equivalent to the norm $\|z\|'_{\frac{1}{2}} = \sqrt{\|z\|^2 + \|z\|^2_{\frac{1}{2}}}$, which is the standard norm on $\mathcal{H}^1(0,\pi)$ (see (13.4.1)). It is easy to check (using the density of $\mathcal{D}(0,\pi)$ in $\mathcal{H}^1_0(0,\pi)$) that the closure of $\mathcal{D}(A_0)$ in $\mathcal{H}^1(0,\pi)$ is

$$\mathcal{H}^1_R(0,\pi) = \left\{ f \in \mathcal{H}^1(0,\pi) \ \mid \ f(\pi) = 0 \right\}.$$

Therefore we conclude that $H_{\frac{1}{2}} = \mathcal{H}^1_R(0,\pi)$.

Example 3.4.13. Let $H = L^2[0,1]$ and let $A_0 : \mathcal{D}(A_0) \to H$ be the operator defined by

$$\mathcal{D}(A_0) = \mathcal{H}^4(0,1) \cap \mathcal{H}^2_0(0,1),$$
$$A_0 f = \frac{\mathrm{d}^4 f}{\mathrm{d}x^4} \qquad \forall\, f \in \mathcal{D}(A_0)$$

(for the notation $\mathcal{H}^2_0(0,1)$ see the beginning of Chapter 2). A simple integration by parts shows that

$$\langle A_0 f, g\rangle = \left\langle \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}, \frac{\mathrm{d}^2 g}{\mathrm{d}x^2}\right\rangle = \langle f, A_0 g\rangle \qquad \forall\, f, g \in \mathcal{D}(A_0), \tag{3.4.10}$$

so that A_0 is symmetric. Simple considerations about the differential equation $A_0 f = g$, with $g \in L^2[0,1]$, show that A_0 is onto. Thus according to Proposition 3.2.4, A_0 is self-adjoint and $0 \in \rho(A_0)$. Since we can see from (3.4.10) that $A_0 \geqslant 0$, it follows that $\sigma(A_0) \subset (0,\infty)$, so that by Remark 3.3.4, $A_0 > 0$.

In order to compute $H_{\frac{1}{2}}$ we note that, according to Remark 3.4.4 and formula (3.4.10), the space $H_{\frac{1}{2}}$ (which is $\mathcal{D}(A_0^{\frac{1}{2}})$ with the graph norm) is the completion of $\mathcal{D}(A_0)$ with respect to the norm

$$\|f\|_{\frac{1}{2}} = \sqrt{\langle A_0 f, f\rangle} = \left\|\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right\|.$$

It is not difficult to check that the above norm is equivalent on $\mathcal{D}(A_0)$ to the standard norm of $\mathcal{H}^2(0,1)$. Since $\mathcal{D}(A_0)$ is dense in $\mathcal{H}^2_0(0,1)$ with the $\mathcal{H}^2$ norm, we obtain that $H_{\frac{1}{2}} = \mathcal{H}^2_0(0,1)$.

3.5 Sturm–Liouville operators

In this section we investigate an important class of self-adjoint operators. More precisely, we consider Sturm–Liouville operators, which are linear second order differential operators acting on a dense domain in $L^2(J)$, where J is an interval. These operators occur in the study of linear PDEs in one space dimension, with possibly variable coefficients.

Throughout this section $a \in \mathcal{H}^1(0,\pi)$ and $b \in L^\infty[0,\pi]$ are real-valued, there exists $m > 0$ with $a(x) \geqslant m$ for all $x \in [0,\pi]$ and we denote $H = L^2[0,\pi]$.

Proposition 3.5.1. *Let $A_0 : \mathcal{D}(A_0) \to H$ be the operator defined by*

$$\begin{aligned} \mathcal{D}(A_0) &= \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_0(0,\pi), \\ A_0 z &= -\frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}z}{\mathrm{d}x}\right) \qquad \forall\, z \in \mathcal{D}(A_0). \end{aligned}$$

Then $A_0 > 0$ and

$$H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}}) = \mathcal{H}^1_0(0,\pi), \qquad H_{-\frac{1}{2}} = \mathcal{H}^{-1}(0,\pi). \tag{3.5.1}$$

Proof. The operator A_0 is symmetric. Indeed, from a simple integration by parts,

$$\langle A_0 z, w\rangle = \int_0^\pi a(x)\frac{\mathrm{d}z}{\mathrm{d}x}\overline{\frac{\mathrm{d}w}{\mathrm{d}x}}\,\mathrm{d}x = \langle z, A_0 w\rangle \qquad \forall\, z,w \in \mathcal{D}(A_0). \tag{3.5.2}$$

Simple considerations about the differential equation $A_0 z = f$, with $f \in L^2[0,\pi]$, using the fact that $\frac{1}{a} \in \mathcal{H}^1(0,\pi)$, show that A_0 is onto. Thus according to Proposition 3.2.4, A_0 is self-adjoint and $0 \in \rho(A_0)$. Since we can see from (3.5.2) that $A_0 \geqslant 0$, it follows that $\sigma(A_0) \subset (0,\infty)$, so that by Remark 3.3.4, $A_0 > 0$.

In order to prove (3.5.1) we note that, according to Remark 3.4.4 and formula (3.5.2), the space $H_{\frac{1}{2}}$ (which is $\mathcal{D}(A_0^{\frac{1}{2}})$ with the graph norm) is the completion of $\mathcal{D}(A_0)$ with respect to the norm

$$\|z\|_{\frac{1}{2}} = \sqrt{\langle A_0 z, z\rangle} = \left(\int_0^\pi a(x)\left|\frac{\mathrm{d}z}{\mathrm{d}x}\right|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

For $a = 1$ this would be the standard norm on $\mathcal{H}^1_0(0,\pi)$. Our assumptions on a imply that $\|\cdot\|_{\frac{1}{2}}$ is equivalent to the standard norm on $\mathcal{H}^1_0(0,\pi)$. Since $\mathcal{D}(A_0)$ is dense in $\mathcal{H}^1_0(0,\pi)$ with the standard norm, we obtain (3.5.1). □

Proposition 3.5.2. *Let $A_1 : \mathcal{D}(A_1) \to H$ be the operator defined by*

$$\begin{aligned} \mathcal{D}(A_1) &= \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_0(0,\pi), \\ A_1 z &= -\frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}z}{\mathrm{d}x}\right) + bz \qquad \forall\, z \in \mathcal{D}(A_1), \end{aligned}$$

with a and b as at the beginning of the section.

Then A_1 is self-adjoint, it has compact resolvents and there is an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ in H consisting of eigenvectors of A_1. If $\lambda \in \mathbb{R}$ is such that $\lambda + b(x) \geqslant 0$ for almost every $x \in [0,\pi]$, then $\sigma(A_1) \subset (-\lambda, \infty)$. The sequence (λ_k) of the eigenvalues of A_1 is such that $\lim \lambda_k = \infty$. Each eigenvalue of A_1 is simple (i.e., its geometric multiplicity is one). If b is such that $A_1 > 0$ (for example, this is the case if $b(x) \geqslant 0$ for almost every $x \in [0,\pi]$), then $\mathcal{D}(A_1^{\frac{1}{2}}) = \mathcal{H}_0^1(0,\pi)$.

Proof. We introduce the operator $M \in \mathcal{L}(X)$ by

$$(Mz)(x) = b(x)z(x) \qquad \forall\, x \in [0,\pi]\,.$$

It is easy to check that M is self-adjoint (in fact it belongs to the class described in Example 3.3.8). The boundedness of M follows from $b \in L^\infty[0,\pi]$. We have $A_1 = A_0 + M$, where $A_0 > 0$ is the operator introduced in Proposition 3.5.1, so that A_1 is self-adjoint. If $\lambda \geqslant 0$ is such that $\lambda + b(x) \geqslant 0$ for almost every $x \in [0,\pi]$, then clearly $\lambda I + M \geqslant 0$ and hence $\lambda I + A_1 > 0$. This implies that $\sigma(A_1) \subset (-\lambda, \infty)$.

The operator A_1 is a generalization of $-A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ from Example 2.6.8 (indeed, $-A$ corresponds to taking $a = 1$ and $b = 0$). We want to show that A_1 is diagonalizable, and we do this by using the fact (already shown in Example 2.6.8) that A is diagonalizable, with the eigenvalues $-k^2$ (where $k \in \mathbb{N}$). It follows from the results in Section 2.6 that $(-A)^{-1}$ is diagonalizable with the eigenvalues $1/k^2$. According to Corollary 12.2.10 from Appendix I, $(-A)^{-1}$ is compact. Since $\mathcal{D}(A_1) = \mathcal{D}(A)$ and A is closed, it follows from the closed-graph theorem that $L = -A(\lambda I + A_1)^{-1}$ is in $\mathcal{L}(H)$. Therefore, $(\lambda I + A_1)^{-1} = (-A)^{-1}L$ is compact. According to Proposition 3.2.12, A_1 is diagonalizable, there is an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ in H consisting of eigenvectors of A_1 and the sequence (λ_k) of the eigenvalues of A_1 satisfies $\lim |\lambda_k| = \infty$. Since $\lambda_k > -\lambda$, it follows that $\lim \lambda_k = \infty$.

To show that each eigenvalue of A_1 is simple, we notice that an eigenvector z corresponding to the eigenvalue λ must satisfy $az'' + a'z' + (\lambda - b)z = 0$, and such a z is completely determined by its initial values $z(0) = 0$ and $z'(0)$. Thus, any solution z is a multiple of the solution obtained for $z'(0) = 1$.

If $b(x) \geqslant 0$ for almost every x, then $M \geqslant 0$ and hence $A_1 = A_0 + M > 0$ (but we may have $A_1 > 0$ also for other b). If $A_1 > 0$, then the property $\mathcal{D}(A_1^{\frac{1}{2}}) = \mathcal{D}(A_0^{\frac{1}{2}})$ follows from Proposition 3.4.9 (with M in place of Q). $\square$

Remark 3.5.3. According to Proposition 2.6.5, $-A_1$ is the generator of a strongly continuous semigroup $\mathbb{T}$ on H:

$$\mathbb{T}_t z = \sum_{k\in\mathbb{N}} e^{-\lambda_k t}\langle z, \varphi_k\rangle \varphi_k\,.$$

It is easy to see that for every $t > 0$, $\mathbb{T}_t$ is self-adjoint (see Remark 3.2.10) and compact. This semigroup corresponds to a non-homogeneous heat equation that is a slight generalization of the one described in Remark 2.6.9:

$$\frac{\partial w}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(a(x)\frac{\partial w}{\partial x}(x,t)\right) - b(x)w(x,t), \qquad x \in (0,\pi),\ t \geqslant 0,$$

with Dirichlet boundary conditions $w(0,t) = w(\pi,t) = 0$.

In order to have more information on the eigenvalues of A_1 we first do a change of variables, by using the function $g : [0,\pi] \to \mathbb{R}$ defined by

$$g(x) = \int_0^x \frac{\mathrm{d}\xi}{\sqrt{a(\xi)}} \qquad \forall\, x \in [0,\pi]. \tag{3.5.3}$$

Since a is bounded from below, we clearly have that g is one-to-one and onto from $[0,\pi]$ to $[0,l]$, where $l = \int_0^\pi \frac{\mathrm{d}x}{\sqrt{a(x)}}$. We can thus introduce the function $h : [0,l] \to [0,\pi]$ defined by $h = g^{-1}$.

Lemma 3.5.4. *With the above notation, assume that $a \in C^2[0,\pi]$ and $b \in C[0,\pi]$, let $\varphi \in \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_0(0,\pi)$ and let $\psi : [0,l] \to \mathbb{C}$ be defined by*

$$\psi(s) = [a(h(s))]^{\frac{1}{4}}\, \varphi(h(s)) \qquad \forall\, s \in [0,l].$$

Then φ is an eigenvector of A_1 corresponding to the eigenvalue λ if and only if

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}s^2} + r\psi = \lambda\psi,$$

where $r \in C[0,l]$ is defined, for every $s \in [0,l]$, by

$$r(s) = \frac{a((h(s))}{16}\left\{4a(h(s))\frac{\mathrm{d}^2 a}{\mathrm{d}x^2}(h(s)) - \left[\frac{\mathrm{d}a}{\mathrm{d}x}(h(s))\right]^2\right\} + b(h(s)). \tag{3.5.4}$$

Proof. It is not difficult to check that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(a(x)\frac{\mathrm{d}\varphi}{\mathrm{d}x}(x)\right) &= a^{-\frac{1}{4}}(x)\frac{\mathrm{d}^2\psi}{\mathrm{d}s^2}(g(x)) \\ &\quad + \frac{a^{-\frac{5}{4}}(x)}{16}\left\{4a(x)\frac{\mathrm{d}^2 a}{\mathrm{d}x^2}(x) - \left[\frac{\mathrm{d}a}{\mathrm{d}x}(x)\right]^2\right\}\psi(g(x)).\end{aligned}$$

The above relation implies, after some simple calculations, our claim. □

Proposition 3.5.5. *Assume that $a \in C^2[0,\pi]$ and $b \in L^\infty(0,\pi)$. Then the eigenvalues of A_1 can be ordered to form a strictly increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ satisfying*

$$\left|\lambda_k - \frac{k^2\pi^2}{l^2}\right| \leqslant C \qquad \forall\, k \in \mathbb{N}, \tag{3.5.5}$$

where $l = \int_0^\pi \frac{\mathrm{d}x}{\sqrt{a(x)}}$ and $C > 0$ is a constant depending only on a and b.

Proof. We know from Proposition 3.5.2 that the eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ of A_1 are simple and that $\lim \lambda_k = \infty$. Thus, without loss of generality, we may assume that (λ_k) is strictly increasing.

Now we introduce the operator $A_2 : \mathcal{D}(A_2) \to L^2[0,l]$ defined by

$$\mathcal{D}(A_2) = \mathcal{H}^2(0,l) \cap \mathcal{H}^1_0(0,l), \quad A_2 z = -\frac{\mathrm{d}^2\psi}{\mathrm{d}s^2} + r\psi \qquad \forall\, \psi \in \mathcal{D}(A_2),$$

where $r \in C[0,l]$ is defined, for every $s \in [0,l]$, by

$$r(s) = \frac{a((h(s))}{16}\left\{4a(h(s))\frac{\mathrm{d}^2 a}{\mathrm{d}x^2}(h(s)) - \left[\frac{\mathrm{d}a}{\mathrm{d}x}(h(s))\right]^2\right\} + b(h(s)).$$

The above definition of A_2 and Lemma 3.5.4 imply that φ is an eigenfunction of A_1 corresponding to the eigenvalue λ iff ψ is an eigenfunction of A_2 corresponding to the same eigenvalue λ. It is clear that the eigenvalues of A_2 are bounded from below so that, according to Proposition 3.2.13, they can be ordered to form an increasing sequence $(\mu_k)_{k\in\mathbb{N}}$ and we have

$$\mu_k = \min_{\substack{V \text{ subspace of } \mathcal{D}(A_2) \\ \dim V = k}} \; \max_{z \in V\setminus\{0\}} R_{A_2}(z) \qquad \forall\, k \in \mathbb{N}, \tag{3.5.6}$$

where R_{A_2} is defined as in (3.2.6). We set $\mathcal{D}(A_3) = \mathcal{D}(A_2)$ and we define $A_3 : \mathcal{D}(A_3) \to L^2[0,l]$ by

$$A_3 z = -\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} \qquad \forall\, \psi \in \mathcal{D}(A_3).$$

Clearly $A_3 > 0$ is diagonalizable and the kth eigenvalue of A_3, with $k \in \mathbb{N}$, is $\frac{k^2\pi^2}{l^2}$. By applying Proposition 3.2.13 it follows that

$$\frac{k^2\pi^2}{l^2} = \min_{\substack{V \text{ subspace of } \mathcal{D}(A_3) \\ \dim V = k}} \; \max_{z \in V\setminus\{0\}} R_{A_3}(z) \qquad \forall\, k \in \mathbb{N}. \tag{3.5.7}$$

On the other hand, it is easy to see that

$$|R_{A_2}(z) - R_{A_3}(z)| \leqslant \|r\|_{L^\infty(0,l)} \qquad \forall\, z \in \mathcal{D}(A_2) \setminus \{0\}.$$

This estimate, together with (3.5.6), (3.5.7) and the fact that A_1 and A_2 have the same eigenvalues, yields the estimate (3.5.5) with $C = \|r\|_{L^\infty[0,l]}$. $\square$

3.6 The Dirichlet Laplacian

In this section we investigate an important example of an unbounded positive operator derived from the Laplacian on a domain in $\mathbb{R}^n$. This operator appears in the study of heat, wave, Schrödinger and plate equations. We shall frequently use concepts and results from Appendix II (Sobolev spaces).

Suppose that $\Omega \subset \mathbb{R}^n$ is an open bounded set. We denote by $\mathcal{D}(\Omega)$ the space of $\mathbb{C}$-valued C^∞ functions with compact support in Ω, and by $\mathcal{D}'(\Omega)$ the space of distributions on Ω. The operators $\frac{\partial}{\partial x_k}$ are continuous on $\mathcal{D}'(\Omega)$ with a certain concept of convergence (see Section 13.2 in Appendix II for details). We introduce the *Laplacian* Δ, a partial differential operator defined by

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2},$$

which acts on distributions in $\mathcal{D}'(\Omega)$. We shall define a self-adjoint operator A_0 by restricting $-\Delta$ to a space of functions which, in a certain sense, are zero on the boundary of Ω. To make the definition of A_0 precise, we need some preliminaries.

We denote by $\mathcal{H}^1(\Omega)$ the space of those $\varphi \in L^2(\Omega)$ for which the gradient $\nabla\varphi = \left(\frac{\partial\varphi}{\partial x_1}, \dots, \frac{\partial\varphi}{\partial x_n}\right)$ (in the sense of distributions in $\mathcal{D}'(\Omega)$) is in $L^2(\Omega;\mathbb{C}^n)$.

According to Proposition 13.4.2, $\mathcal{H}^1(\Omega)$ is a Hilbert space with the norm $\|\cdot\|_{\mathcal{H}^1}$ defined by

$$\|\varphi\|^2_{\mathcal{H}^1} = \|\varphi\|^2_{L^2} + \|\nabla\varphi\|^2_{L^2}.$$

It will be useful to note that for every $z \in \mathcal{H}^1(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$,

$$\langle \Delta z, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = -\int_\Omega \nabla z \cdot \nabla\varphi \,\mathrm{d}x, \tag{3.6.1}$$

where $\cdot$ denotes the usual inner product in $\mathbb{C}^n$. We denote by $\mathcal{H}^1_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $\mathcal{H}^1(\Omega)$. Clearly, the space $\mathcal{H}^1_0(\Omega)$ is a Hilbert space.

To understand this space better, assume for a moment that the boundary of Ω, denoted $\partial\Omega$, is Lipschitz. (We refer the reader to Section 13.5 in Appendix II for the definition of a Lipschitz boundary.) This implies that the boundary trace (restriction to the boundary) of any $\varphi \in \mathcal{H}^1(\Omega)$ is well defined as an element of $L^2(\partial\Omega)$; see Section 13.6. Then $\mathcal{H}^1_0(\Omega)$ is precisely the space of those $\varphi \in \mathcal{H}^1(\Omega)$ for which the trace (the restriction) of φ on $\partial\Omega$ is zero; see Proposition 13.6.2. Thus, any $\varphi \in \mathcal{H}^1_0(\Omega)$ satisfies

$$\varphi(x) = 0 \quad \text{for} \quad x \in \partial\Omega.$$

This boundary condition imposed on φ is called a *homogeneous Dirichlet boundary condition.* In what follows we do not assume that Ω has a Lipschitz boundary.

According to Proposition 13.4.10 in Appendix II, the Poincaré inequality holds for Ω: there exists $m > 0$ such that

$$\int_\Omega |\nabla\varphi(x)|^2 \mathrm{d}x \geqslant m \int_\Omega |\varphi(x)|^2 \mathrm{d}x \qquad \forall\, \varphi \in \mathcal{H}_0^1(\Omega).$$

Here, $|a|$ denotes the Euclidean norm of the vector $a \in \mathbb{C}^n$. This implies that on $\mathcal{H}_0^1(\Omega)$ the norm inherited from $\mathcal{H}^1(\Omega)$ is equivalent to the following norm:

$$\|\varphi\|_{\mathcal{H}_0^1} = \|\nabla\varphi\|_{L^2}. \tag{3.6.2}$$

In this section, we use the above norm on $\mathcal{H}_0^1(\Omega)$ and the corresponding inner product. We define the operator $A_0 : \mathcal{D}(A_0) \to L^2(\Omega)$ by

$$\mathcal{D}(A_0) = \left\{\phi \in \mathcal{H}_0^1(\Omega) \;\middle|\; \Delta\phi \in L^2(\Omega)\right\}, \qquad A_0\phi = -\Delta\phi. \tag{3.6.3}$$

The space $\mathcal{H}^{-1}(\Omega)$ is defined as the dual of $\mathcal{H}_0^1(\Omega)$ with respect to the pivot space $L^2(\Omega)$; see Section 13.4 in Appendix II.

Proposition 3.6.1. *The operator A_0 defined above is strictly positive and*

$$\mathcal{D}\left(A_0^{\frac{1}{2}}\right) = \mathcal{H}_0^1(\Omega). \tag{3.6.4}$$

If $H = L^2(\Omega)$ and the spaces $H_{\frac{1}{2}}$ and $H_{-\frac{1}{2}}$ are defined as in Section 3.4, then

$$H_{\frac{1}{2}} = \mathcal{H}_0^1(\Omega), \qquad H_{-\frac{1}{2}} = \mathcal{H}^{-1}(\Omega).$$

The norm on $H_{\frac{1}{2}}$ as introduced in Section 3.4 is the same as in (3.6.2).

The operator $-A_0$ is called the *Dirichlet Laplacian* on Ω. (We note that the Dirichlet Laplacian can be defined also for domains Ω that are not bounded, and if the Poincaré inequality holds for Ω, then the above proposition is true.)

Proof. Suppose that $\varphi,\, \psi \in \mathcal{D}(A_0)$. Then, according to (3.6.1),

$$\langle A_0\varphi, \psi\rangle_{L^2} = -\int_\Omega \Delta\varphi\,\overline{\psi}\,\mathrm{d}x = \int_\Omega \nabla\varphi\cdot\nabla\overline{\psi}\,\mathrm{d}x = \langle\varphi, A_0\psi\rangle_{L^2}, \tag{3.6.5}$$

so that A_0 is symmetric. According to Proposition 3.2.4, in order to show that A_0 is self-adjoint it suffices to show that A_0 is onto. For this, we take $f \in L^2(\Omega)$ and we prove the existence of $z \in \mathcal{D}(A_0)$ such that $A_0 z = f$. First note that the mapping $\varphi \to \int_\Omega \varphi\overline{f}\,\mathrm{d}x$ is a bounded linear functional on $\mathcal{H}_0^1(\Omega)$. By the Riesz representation theorem, there exists $z \in \mathcal{H}_0^1(\Omega)$ such that

$$\langle\varphi, z\rangle_{\mathcal{H}_0^1} = \langle\varphi, f\rangle_{L^2} \qquad \forall\, \varphi \in \mathcal{H}_0^1(\Omega). \tag{3.6.6}$$

This implies, by using (3.6.1), that

$$\langle -\Delta z, \overline{\varphi}\rangle_{\mathcal{D}',\mathcal{D}} = \langle z, \varphi\rangle_{\mathcal{H}_0^1} = \langle f, \varphi\rangle_{L^2} \qquad \forall\, \varphi \in \mathcal{D}(\Omega).$$

This shows that $-\Delta z = f$ in $\mathcal{D}'(\Omega)$. Since $f \in L^2(\Omega)$, we get that

$$\Delta z \in L^2(\Omega) \quad \text{and} \quad -\Delta z = f \quad \text{in} \quad L^2(\Omega)\,.$$

Thus $z \in \mathcal{D}(A_0)$ and $A_0 z = f$, hence A_0 is onto. Thus, A_0 is self-adjoint. It is clear from (3.6.5) that

$$\langle A_0 z, z\rangle = \|\nabla z\|_{L^2}^2 \qquad \forall\, z \in \mathcal{D}(A_0)\,. \tag{3.6.7}$$

Using this and the Poincaré inequality, we see that $A_0 > 0$.

According to Remark 3.4.4, $H_{\frac{1}{2}}$ may be regarded as the completion of $H_1 = \mathcal{D}(A_0)$ with respect to the norm $\|z\|_{\frac{1}{2}} = \langle A_0 z, z\rangle^{\frac{1}{2}}$. Thus, according to (3.6.7), $H_{\frac{1}{2}}$ is the completion of H_1 with respect to the norm defined in (3.6.2). By using the fact that $\mathcal{H}_0^1(\Omega)$ with the norm in (3.6.2) is complete, it follows that $H_{\frac{1}{2}} \subset \mathcal{H}_0^1(\Omega)$. On the other hand, $\mathcal{D}(A_0) \supset \mathcal{D}(\Omega)$. Since the completion of $\mathcal{D}(\Omega)$ with respect to the norm in (3.6.2) is $\mathcal{H}_0^1(\Omega)$, it follows that $H_{\frac{1}{2}} \supset \mathcal{H}_0^1(\Omega)$. Thus we have $H_{\frac{1}{2}} = \mathcal{H}_0^1(\Omega)$. By definition $H_{-\frac{1}{2}}$ is the dual space of $H_{\frac{1}{2}}$ with respect to the pivot space $H = L^2(\Omega)$. By using the definition of $\mathcal{H}^{-1}(\Omega)$ we conclude that $H_{-\frac{1}{2}} = \mathcal{H}^{-1}(\Omega)$. □

Under additional assumptions, the domain of A_0 consists of smoother functions. More precisely, from Theorem 13.5.5 in Appendix II we obtain the following theorem.

Theorem 3.6.2. *Suppose that $\partial\Omega$ is of class C^2. Then*

$$\mathcal{D}(A_0) = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)\,. \tag{3.6.8}$$

The concept of boundary of class C^m is explained in Section 13.5.

Remark 3.6.3. By Proposition 3.4.5 and Corollary 3.4.6, A_0 has unique extensions such that

$$A_0 \in \mathcal{L}(\mathcal{H}_0^1(\Omega), \mathcal{H}^{-1}(\Omega))\,, \qquad A_0 \in \mathcal{L}(H, H_{-1})\,,$$

and these are unitary operators. If, as in Remark 3.4.7, we introduce a different notation $\tilde{A}_0$ for the extension of A_0 to a strictly positive operator on $\tilde{H} = \mathcal{H}^{-1}(\Omega)$, then $\tilde{H}_1 = \mathcal{H}_0^1(\Omega)$ and $\tilde{H}_{\frac{1}{2}} = \mathcal{D}(\tilde{A}_0^{\frac{1}{2}}) = L^2(\Omega)$, with equal norms.

Note that if $f \in \mathcal{H}_0^1(\Omega)$, then $A_0 f$ coincides with $-\Delta f$ calculated in $\mathcal{D}'(\Omega)$ (this follows because $\mathcal{D}(\Omega)$ is dense in $\mathcal{H}_0^1(\Omega)$). By contrast, if $f \in H = L^2(\Omega)$, then $A_0 f$ is, in general, different from $-\Delta f$ calculated in $\mathcal{D}'(\Omega)$. This is because $A_0 f$ is now in the dual of $\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$, and $\mathcal{D}(\Omega)$ is not dense in $\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$. Indeed, if f is a non-zero constant, then $\Delta f = 0$, but $A_0 f$ cannot be zero since $A_0 > 0$.

Remark 3.6.4. Since Ω is bounded, according to Proposition 13.4.12, the embedding $\mathcal{D}(A_0) \subset L^2(\Omega)$ is compact. Thus A_0^{-1} is compact and hence, by Proposition 3.2.12, A_0 is diagonalizable with an orthonormal basis (φ_k) of eigenvectors and the corresponding sequence of eigenvalues (λ_k) satisfies $\lambda_k > 0$ and $\lambda_k \to \infty$.

Example 3.6.5. Let a, $b > 0$ and let $\Omega = [0, a] \times [0, b] \subset \mathbb{R}^2$. We show that (3.6.8) holds also for this domain. It is easy to check that the eigenvalues of A_0 are

$$\lambda_{mn} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right), \tag{3.6.9}$$

with $m, n \in \mathbb{N}$. A corresponding orthonormal basis formed of eigenvectors of A_0 is given by

$$\varphi_{mn}(x, y) = \frac{2}{\sqrt{ab}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \qquad \forall\, m, n \in \mathbb{N}. \tag{3.6.10}$$

It is clear that $\partial\Omega$ is not of class C^2, so we cannot use Theorem 3.6.2 to characterize $\mathcal{D}(A_0)$. However, this domain can be characterized by a direct calculation. Indeed, let us assume that

$$z = \sum_{m,n\in\mathbb{N}} c_{mn}\varphi_{mn} \in \mathcal{D}(A_0).$$

This implies, by using (3.2.4) and the fact that A_0 is diagonalizable, that

$$\sum_{m,n\in\mathbb{N}} (m^2 + n^2)^2 |c_{mn}|^2 < \infty. \tag{3.6.11}$$

For $p \in \mathbb{N}$, we set

$$z_p = \sum_{m,n=1}^{p} c_{mn}\varphi_{mn}.$$

It is clear that

$$\lim_{p\to\infty} \|z - z_p\|_{\mathcal{D}(A_0)} = 0. \tag{3.6.12}$$

On the other hand, by a simple calculation we can check that, for $p, q \in \mathbb{N}$ with $p \leqslant q$ and α_1, $\alpha_2 \in \{0, 1, 2\}$ with $\alpha_1 + \alpha_2 \leqslant 2$,

$$\left\| \frac{\partial^{\alpha_1+\alpha_2}(z_p - z_q)}{\partial x^{\alpha_1}\partial y^{\alpha_2}} \right\|_{L^2}^2 = \sum_{p}^{q} m^{2\alpha_1} n^{2\alpha_2} |c_{mn}|^2.$$

The above relation, combined with (3.6.11), implies that (z_p) is a Cauchy sequence in $\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$. Thus there exists $\tilde{z} \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$ such that (z_p) converges to $\tilde{z}$ with respect to the topology of $\mathcal{H}^2(\Omega)$. Since both convergences in $\mathcal{D}(A_0)$ and in $\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$ imply convergence in $L^2(\Omega)$, it follows that $z = \tilde{z}$ so we have that $z \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$. We have thus shown that $\mathcal{D}(A_0) \subset \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$. The opposite inclusion is obvious, so we conclude that (3.6.8) holds.

Remark 3.6.6. The computations in the above example can be generalized easily to rectangular domains in $\mathbb{R}^n$, to conclude that (3.6.8) holds. We mention that this equality remains valid for more general domains whose boundary is not of class C^2, such as convex polygons $\mathbb{R}^2$. However, in general we only have $\mathcal{D}(A_0) \supset \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$. We refer the reader to Grisvard [77] for a detailed discussion.

Consider Ω to be the hypercube $[0,a]^n$, where $a > 0$. By using the multi-index notation introduced at the beginning of Appendix II, it is not difficult to check that the eigenvalues of A_0 are

$$\lambda_\alpha = \left(\frac{\pi}{a}\right)^2 \sum_{k=1}^n \alpha_k^2 \qquad \forall\, \alpha \in \mathbb{N}^n. \tag{3.6.13}$$

A corresponding orthonormal basis formed of eigenvectors of A_0 is given by

$$\varphi_\alpha(x) = \left(\frac{2}{a}\right)^{\frac{n}{2}} \prod_{k=1}^n \sin\left(\frac{\alpha_k x_k}{a}\right) \qquad \forall\, \alpha \in \mathbb{N}^n,\ x \in \Omega. \tag{3.6.14}$$

Formula (3.6.13) has the following consequence.

Proposition 3.6.7. *Let $n \in \mathbb{N}$, $a > 0$, $\Omega = [0,a]^n$ and let $(\lambda_\alpha)_{\alpha\in\mathbb{N}^n}$ be the eigenvalues of A_0, as given by* (3.6.13)*. For $\omega > 0$ we denote by $d_n(\omega)$ the number of terms of the sequence (λ_α) which are less than or equal to ω. Then*

$$\lim_{\omega\to\infty} \frac{d_n(\omega)}{\omega^{\frac{n}{2}}} = \frac{a^n V_n}{2^n \pi^n}, \tag{3.6.15}$$

where V_n is the volume of the unit ball in $\mathbb{R}^n$.

Proof. According to (3.6.13), $d_n(\omega)$ is the number of points having all the coordinates in $\mathbb{N}$ which are contained in the closed ball of radius $\frac{a\sqrt{\omega}}{\pi}$. We denote by $\mathcal{B}_n(r)$ the part of the closed ball of radius r centered at zero where all the coordinates of the points are non-negative. Clearly the volume of $\mathcal{B}_n(r)$ is $\frac{r^n V_n}{2^n}$. Let $\tilde{d}_n(\omega)$ be the number of points having all the coordinates in $\mathbb{Z}_+$ which are contained in $\mathcal{B}_n\left(\frac{a\sqrt{\omega}}{\pi}\right)$. With each point $\alpha \in \mathbb{Z}_+^n \cap \mathcal{B}_n\left(\frac{a\sqrt{\omega}}{\pi}\right)$ we associate the cube

$$\mathcal{C}_\alpha = [\alpha_1, \alpha_1 + 1] \times [\alpha_2, \alpha_2 + 1] \times \cdots \times [\alpha_n, \alpha_n + 1].$$

It can be seen that the union of these cubes is contained in $\mathcal{B}_n\left(\frac{a\sqrt{\omega}}{\pi} + \sqrt{n}\right)$ and it contains $\mathcal{B}_n\left(\frac{a\sqrt{\omega}}{\pi} - \sqrt{n}\right)$. Therefore we have

$$\frac{V_n}{2^n}\left(\frac{a\sqrt{\omega}}{\pi} - \sqrt{n}\right)^n \leqslant \tilde{d}_n(\omega) \leqslant \frac{V_n}{2^n}\left(\frac{a\sqrt{\omega}}{\pi} + \sqrt{n}\right)^n,$$

which clearly implies that

$$\lim_{\omega\to\infty} \frac{\tilde{d}_n(\omega)}{\omega^{\frac{n}{2}}} = \frac{a^n V_n}{2^n \pi^n}.$$

This and the fact that $\tilde{d}_n(\omega) - n\tilde{d}_{n-1}(\omega) \leqslant d_n(\omega) \leqslant \tilde{d}_n(\omega)$ imply (3.6.15). $\square$

Corollary 3.6.8. *With the assumptions and the notation of Proposition* 3.6.7, *we reorder the eigenvalues of* A_0 *to form an increasing sequence* $(\lambda_k)_{k\in\mathbb{N}}$ *such that each* λ_k *is repeated as many times as its geometric multiplicity. Then*

$$\lim_{k\to\infty} \frac{\lambda_k}{k^{\frac{2}{n}}} = \frac{4\pi^2}{a^2 V_n^{\frac{2}{n}}}. \tag{3.6.16}$$

Proof. By applying Proposition 3.6.7 and the fact that $d_n(\lambda_k) = k$ for every $k \in \mathbb{N}$, we obtain that

$$\lim_{k\to\infty} \frac{k}{\lambda_k^{\frac{n}{2}}} = \frac{a^n V_n}{2^n \pi^n},$$

which easily yields (3.6.16). □

Before the next proposition, recall from Remark 3.6.4 that the Dirichlet Laplacian has compact resolvents, hence it is diagonalizable.

Proposition 3.6.9. *Let* $\Omega \subset \mathbb{R}^n$ *be an open bounded set, let* $-A_0$ *be the Dirichlet Laplacian on* Ω *and let* $(\lambda_k)_{k\in\mathbb{N}}$ *be the eigenvalues of* A_0 *in increasing order, such that each* λ_k *is repeated as many times as its geometric multiplicity. Then*

$$\liminf_{k\to\infty} \frac{\lambda_k}{k^{\frac{2}{n}}} > 0, \qquad \limsup_{k\to\infty} \frac{\lambda_k}{k^{\frac{2}{n}}} < \infty.$$

Proof. By combining (3.4.8) and (3.6.7) we obtain that

$$\lambda_k = \min_{\substack{V \text{ subspace of } \mathcal{H}_0^1(\Omega) \\ \dim V = k}} \max_{z\in V\setminus\{0\}} \frac{\|\nabla z\|_{L^2}^2}{\|z\|_{L^2}^2} \qquad \forall\, k \in \mathbb{N}.$$

Let $a > 0$ be such that Ω is contained in a cube Q_a of side length a. Since any function in $\mathcal{H}_0^1(\Omega)$ can be seen, after extension by zero outside Ω, as a function in $\mathcal{H}_0^1(Q_a)$ (see Lemma 13.4.11), it follows that λ_k is greater than the kth eigenvalue of minus the Dirichlet Laplacian on Q_a. Similarly, if Ω contains a cube Q_b of side length $b > 0$, then λ_k is less than or equal to the kth eigenvalue of minus the Dirichlet Laplacian on Q_b. The conclusion follows now by Corollary 3.6.8. □

Remark 3.6.10. The result in the last proposition is sharpened by Weyl's formula (see, for instance, Zuily [246, p. 174]) which asserts that if Ω is connected, then

$$\lim_{k\to\infty} \frac{\lambda_k}{k^{\frac{2}{n}}} = \frac{4\pi^2}{[V_n \mathrm{Vol}(\Omega)]^{\frac{2}{n}}},$$

where $\mathrm{Vol}(\Omega)$ stands for the n-dimensional volume of Ω.

Remark 3.6.11. Let $A = -A_0$ be the Dirichlet Laplacian on a bounded domain $\Omega \subset \mathbb{R}^n$. After extending A_0 as in Remark 3.6.3, we regard A_0 as a strictly positive (densely defined) operator on $X = \mathcal{H}^{-1}(\Omega)$, so that $\mathcal{D}(A_0) = \mathcal{H}_0^1(\Omega)$. According to

Remark 3.6.4, $A = -A_0$ is diagonalizable with an orthonormal basis of eigenvectors and with negative eigenvalues converging to $-\infty$. According to Proposition 2.6.5, A generates a strongly continuous contraction semigroup $\mathbb{T}$ on X. This semigroup is associated with the heat equation with homogeneous Dirichlet boundary conditions on Ω, and it is called the *heat semigroup*. We have encountered the one-dimensional version of this semigroup in Example 2.6.8. It follows from Proposition 2.6.7 that we have

$$\mathbb{T}_t z \in \mathcal{D}(A^\infty) \subset \mathcal{H}_0^1(\Omega) \qquad \forall\, z \in \mathcal{H}^{-1}(\Omega),\ t > 0.$$

We have $\mathcal{D}(A^\infty) \subset \mathcal{H}^p_{\text{loc}}(\Omega)$ for every $p \in \mathbb{N}$, according to Remark 13.5.6 in Appendix II. According to Remark 13.4.5, it follows that $\mathcal{D}(A^\infty) \subset C^m(\Omega)$ for every $m \in \mathbb{N}$, so that

$$\mathbb{T}_t z \in C^\infty(\Omega) \cap \mathcal{H}_0^1(\Omega) \qquad \forall\, z \in \mathcal{H}^{-1}(\Omega),\ t > 0.$$

3.7 Skew-adjoint operators

Let $A : \mathcal{D}(A) \to X$ be densely defined. A is called *skew-symmetric* if

$$\langle Aw, v\rangle = -\langle w, Av\rangle \qquad \forall\, w, v \in \mathcal{D}(A).$$

It is easy to see that this is equivalent to $G(-A) \subset G(A^*)$, and also to the fact that iA is symmetric. It follows from Proposition 3.2.2 that (still assuming dense $\mathcal{D}(A)$) A is skew-symmetric iff $\operatorname{Re}\langle Az, z\rangle = 0$ for all $z \in \mathcal{D}(A)$. It now becomes obvious that skew-symmetric operators are dissipative. Our interest in skew-symmetric operators stems from the following simple result.

Proposition 3.7.1. *Let A be the generator of an isometric semigroup on X. Then A is skew-symmetric and $\mathbb{C}_0 \subset \rho(A)$.*

Proof. Take $z_0 \in \mathcal{D}(A)$ and define $z(t) = \mathbb{T}_t z_0$ (where $\mathbb{T}$ is the semigroup generated by A). Then a simple computation shows that, for every $t \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2 = 2\operatorname{Re}\langle Az(t), z(t)\rangle.$$

Since $\mathbb{T}$ is isometric, the above expression must be zero. Taking $t = 0$ we obtain that $\operatorname{Re}\langle Az_0, z_0\rangle = 0$ for all $z_0 \in \mathcal{D}(A)$. As remarked earlier, this implies that A is skew-symmetric. Since $\mathbb{T}$ is a contraction semigroup, according to Proposition 3.1.13, A is m-dissipative. Now Theorem 3.1.9 implies that $\mathbb{C}_0 \subset \rho(A)$. □

A densely defined operator A is called *skew-adjoint* if $A^* = -A$ (equivalently, iA is self-adjoint). If $A^* = -A$, then clearly $\sigma(A) \subset i\mathbb{R}$. We shall see in Section 3.8 that A is skew-adjoint iff it is the generator of a unitary group.

Proposition 3.7.2. *For $A : \mathcal{D}(A) \to X$, the following statements are equivalent:*

(a) *Both A and $-A$ are m-dissipative.*
(b) *A is skew-adjoint.*

Proof. Suppose that A and $-A$ are m-dissipative, then $\mathrm{Re}\,\langle Az, z\rangle = 0$ for all $z \in \mathcal{D}(A)$. As remarked at the beginning of this section, this implies that A is skew-symmetric, so that $G(-A) \subset G(A^*)$. Since A and $-A$ are m-dissipative, by Proposition 3.1.10 the same is true for A^* and $-A^*$. Repeating the above argument with A^* instead of A, we obtain that A^* is skew-symmetric, so that $G(-A^*) \subset G(A^{**})$. Since $A^{**} = A$, we obtain that $G(A^*) \subset G(-A)$. This inclusion, combined with the one derived earlier, shows that $-A = A^*$.

Conversely, if A is skew-adjoint, then clearly both A and A^* are dissipative. Since A^* is closed and $A = -A^*$, A is also closed. Thus, by Proposition 3.1.11, A is m-dissipative. By a similar argument, $-A$ is also m-dissipative. □

Proposition 3.7.3. *Suppose that A is skew-symmetric.*

(a) *If both $I + A$ and $I - A$ are onto, then A is skew-adjoint.*
(b) *If A is onto, then A is skew-adjoint and $0 \in \rho(A)$.*

Proof. Part (a) follows from the last proposition, but alternatively it can also be derived from Proposition 3.2.4 (with $s = i$ and $A_0 = iA$). Part (b) follows from Proposition 3.2.4 (with $s = 0$ and $A_0 = iA$). □

Remark 3.7.4. The condition that only one of the operators $I - A$ and $I + A$ is onto would not be sufficient in part (a) of the above proposition. Indeed, consider the space $X = L^2[0, \infty)$ and on the subspace

$$\mathcal{D}(A) = \left\{\phi \in \mathcal{H}^1(0, \infty) \mid \phi(0) = 0\right\}$$

define the skew-symmetric operator $A : \mathcal{D}(A) \to X$ by

$$(A\phi)(x) = -\frac{\mathrm{d}\phi}{\mathrm{d}x}(x) \qquad \forall\, x > 0 .$$

This is the generator of the unilateral right shift, encountered in Example 2.4.5. We can easily check that $I - A$ is onto, so A is m-dissipative. On the other hand, if we consider $g \in L^2[0, \infty)$ defined by $g(x) = e^{-x}$, then the equation $(I + A)z = g$ has no solution in $\mathcal{D}(A)$. Thus, $I + A$ is not onto, so $-A$ is not m-dissipative.

Another consequence of Proposition 3.7.2 is the following.

Corollary 3.7.5. *Let $\mathbb{T}$ be a strongly continuous group of operators on X with generator A. If $\mathbb{T}$ satisfies $\|\mathbb{T}_t\| \leqslant 1$ for all $t \in \mathbb{R}$, then A is skew-adjoint.*

Proof. It follows from Remark 2.7.6 and Proposition 3.1.13 that both A and $-A$ are m-dissipative. Now the statement follows from Proposition 3.7.2. □

In what follows we want to introduce a class of skew-adjoint operators which arise as semigroup generators corresponding to second-order differential equations in a Hilbert space, of the form

$$\ddot{w}(t) + A_0 w(t) = 0, \quad \text{with} \quad A_0 > 0.$$

Many undamped wave and plate equations are of this form. The natural state of such a system is the vector $z(t) = \left[\begin{smallmatrix} w(t) \\ \dot{w}(t) \end{smallmatrix}\right]$. We shall say more about the solutions of such a differential equation at the end of Section 3.8.

Proposition 3.7.6. *Let $A_0 : \mathcal{D}(A_0) \to H$ be a strictly positive operator on the Hilbert space H. The Hilbert space $H_{\frac{1}{2}}$ is as in Section 3.4. Define $X = H_{\frac{1}{2}} \times H$, with the scalar product*

$$\left\langle \begin{bmatrix} w_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ v_2 \end{bmatrix} \right\rangle_X = \langle A_0^{\frac{1}{2}} w_1, A_0^{\frac{1}{2}} w_2 \rangle + \langle v_1, v_2 \rangle.$$

Define a dense subspace of X by $\mathcal{D}(A) = \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}})$ and the linear operator $A : \mathcal{D}(A) \to X$ by

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad \textit{i.e.,} \quad A \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} \psi \\ -A_0 \varphi \end{bmatrix}. \tag{3.7.1}$$

Then A is skew-adjoint on X and $0 \in \rho(A)$. Moreover,

$$X_1 = H_1 \times H_{\frac{1}{2}}, \qquad X_{-1} = H \times H_{-\frac{1}{2}}.$$

Proof. It is easy to see that A is skew-symmetric. The equation

$$A \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \in X$$

is equivalent to the relations $\psi = f \in H_{\frac{1}{2}}$ and $-A_0 \varphi = g \in H$. Since $A_0 > 0$, it is invertible (see Proposition 3.3.2), so that there exists a (unique) $\varphi \in \mathcal{D}(A_0)$ satisfying the last equation. Thus, A is onto. By Proposition 3.7.3, A is skew-adjoint and $0 \in \rho(A)$. It is clear that $\mathcal{D}(A)$, with the norm $\|z\|_1 = \|Az\|$, is $X_1 = H_1 \times H_{\frac{1}{2}}$.

Note that $A^{-1} = \left[\begin{smallmatrix} 0 & -A_0^{-1} \\ I & 0 \end{smallmatrix}\right]$, so that, using Proposition 3.4.5,

$$\left\| \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \right\|_{-1}^2 = \|\varphi\|^2 + \|\psi\|_{-\frac{1}{2}}^2 \qquad \forall \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in X.$$

Taking the completion of X with respect to this norm, we get $X_{-1} = H \times H_{-\frac{1}{2}}$. $\square$

Proposition 3.7.7. *With the notation of Proposition* 3.7.6, $\phi = \left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right] \in \mathcal{D}(A)$ *is an eigenvector of* A, *corresponding to the eigenvalue* $i\mu$ *(where* $\mu \in \mathbb{R}$*), if and only if* φ *is an eigenvector of* A_0, *corresponding to the eigenvalue* μ^2 *and* $\psi = i\mu\varphi$.

Now suppose that A_0 *is diagonalizable, with an orthonormal basis* $(\varphi_k)_{k\in\mathbb{N}}$ *in* H *formed of eigenvectors of* A_0. *Denote by* $\lambda_k > 0$ *the eigenvalue corresponding to* φ_k *and* $\mu_k = \sqrt{\lambda_k}$. *For all* $k \in \mathbb{N}$ *we define* $\varphi_{-k} = -\varphi_k$ *and* $\mu_{-k} = -\mu_k$. *Then* A *is diagonalizable, with the eigenvalues* $i\mu_k$ *corresponding to the orthonormal basis of eigenvectors*

$$\phi_k = \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{i\mu_k}\varphi_k\\ \varphi_k\end{bmatrix} \qquad \forall\, k \in \mathbb{Z}^*\,. \tag{3.7.2}$$

Recall that $\mathbb{Z}^*$ denotes the set of all the non-zero integers. Recall also that if A_0^{-1} is compact, then A_0 is diagonalizable, with an orthonormal basis of eigenvectors and a sequence of positive eigenvalues converging to ∞ (see Proposition 3.2.12).

Proof. Suppose that $\left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right] \in X \setminus \{\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]\}$ is such that $A\left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right] = i\mu\left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right]$. Then, according to the definition of A, we have that $\psi = i\mu\varphi$ and $-A_0\varphi = i\mu\psi$, which implies that $A_0\varphi = \mu^2\varphi$ with $\varphi \neq 0$. Thus, μ^2 is an eigenvalue of A_0 corresponding to the eigenvector φ. Note that $\mu \neq 0$, according to Proposition 3.7.6.

Conversely, if φ is an eigenvector of A_0 corresponding to the eigenvalue μ^2, it follows immediately from the structure of A that $A\left[\begin{smallmatrix}\varphi\\ i\mu\varphi\end{smallmatrix}\right] = i\mu\left[\begin{smallmatrix}\varphi\\ i\mu\varphi\end{smallmatrix}\right]$.

Now suppose that A_0 is diagonalizable, and let λ_k (with $k \in \mathbb{N}$) and φ_k, μ_k (with $k \in \mathbb{Z}^*$) be defined as in the proposition. Then it follows from the first part of the proposition (which we have already proved) that the vectors ϕ_k defined in (3.7.2) are eigenvectors of A. It is also easy to verify that these eigenvectors are an orthonormal set (for the orthogonality of ϕ_k and ϕ_j with $k \neq j$, we have to consider separately the cases $k = -j$ and $k \neq -j$). Denote $\mathcal{B} = \{\phi_k \mid k \in \mathbb{Z}^*\}$. To show that $(\phi_k)_{k\in\mathbb{Z}^*}$ is an orthonormal basis in X, it remains to show that $\mathcal{B}^\perp = \{0\}$ (see Section 1.1). Take $\left[\begin{smallmatrix}f\\ g\end{smallmatrix}\right] \in \mathcal{B}^\perp$. Since $(\varphi_k)_{k\in\mathbb{N}}$ is an orthonormal basis in H, by Proposition 2.5.2, there exist sequences (f_k) and (g_k) in l^2 such that

$$f = \sum_{k\in\mathbb{N}} f_k\varphi_k\,, \qquad g = \sum_{k\in\mathbb{N}} g_k\varphi_k\,.$$

Applying Proposition 3.4.8 to $f \in \mathcal{D}(A_0^{\frac{1}{2}})$, we have that $(\mu_k f_k) \in l^2$ and $A_0^{\frac{1}{2}} f = \sum_{k\in\mathbb{N}} \mu_k f_k\varphi_k$. This implies that for all $k \in \mathbb{Z}^*$,

$$\sqrt{2}\left\langle \begin{bmatrix} f\\ g\end{bmatrix}, \phi_k\right\rangle = i\mu_k\langle f, \varphi_k\rangle + \langle g, \varphi_k\rangle\,.$$

Since $\left[\begin{smallmatrix}f\\ g\end{smallmatrix}\right] \in \mathcal{B}^\perp$, by taking in the last formula $k \in \mathbb{N}$ and then $-k$, we obtain

$$i\mu_k\langle f, \varphi_k\rangle + \langle g, \varphi_k\rangle = 0\,, \qquad -i\mu_k\langle f, \varphi_k\rangle + \langle g, \varphi_k\rangle = 0 \qquad \forall\, k \in \mathbb{N}.$$

This implies that

$$\langle f, \varphi_k \rangle = 0\,, \qquad \langle g, \varphi_k \rangle = 0 \qquad \forall\, k \in \mathbb{N}\,.$$

Thus, $f = g = 0$, so that $\mathcal{B}$ is an orthonormal basis in X. □

Note that the above proposition is a generalization of Examples 2.7.13 and 2.7.15.

3.8 The theorems of Lumer–Phillips and Stone

The main aim of this section is to show that any m-dissipative operator is the generator of a contraction semigroup. For this, we need a certain type of approximation of unbounded operators by bounded ones, called the Yosida approximation.

Definition 3.8.1. Let $A : \mathcal{D}(A) \to X$ satisfy the assumption in Proposition 2.3.4. Then the $\mathcal{L}(X)$-valued function

$$A_\lambda = \lambda A(\lambda I - A)^{-1} = \lambda^2 (\lambda I - A)^{-1} - \lambda I\,, \tag{3.8.1}$$

defined for $\lambda > \lambda_0$, is called the *Yosida approximation* of A.

Notice that if A is the generator of a strongly continuous semigroup on X, or if A is m-dissipative on X, then it satisfies the assumption in the above definition. For generators this was explained after Proposition 2.3.4, while for m-dissipative operators it follows from Proposition 3.1.9.

Remark 3.8.2. The word "approximation" in the name given to A_λ above is justified by the property

$$\lim_{\lambda \to \infty} A_\lambda z = Az \qquad \forall\, z \in \mathcal{D}(A)\,.$$

To see that this is true, notice that $A_\lambda z = \lambda(\lambda I - A)^{-1} Az$ for all $z \in \mathcal{D}(A)$. Now the above limit property follows from Proposition 2.3.4.

Proposition 3.8.3. *Let A be an m-dissipative operator on X and let A_λ, $\lambda > 0$, be its Yosida approximation. Then the following statements hold:*

(i) $\|e^{tA_\lambda}\| \leqslant 1$ *for all* $t \geqslant 0$ *and all* $\lambda > 0$.
(ii) $\|e^{tA_\lambda} z - e^{tA_\mu} z\| \leqslant t \|A_\lambda z - A_\mu z\|$ *for all* $t \geqslant 0$, $\lambda, \mu > 0$ *and* $z \in X$.

Proof. (i) According to (3.8.1) we have

$$e^{tA_\lambda} = e^{\lambda^2 t (\lambda I - A)^{-1}} e^{-\lambda t}\,.$$

This, together with (2.1.2) and (3.1.5), implies that (i) holds.

(ii) Consider t, λ, $\mu > 0$. Since A_λ and A_μ commute, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{e^{\tau t A_\lambda} e^{(1-\tau)tA_\mu} z\right\} = t e^{\tau t A_\lambda} e^{(1-\tau)tA_\mu}(A_\lambda z - A_\mu z)$$

for all $\tau \in [0,1]$ and for all $z \in X$. In particular, it follows from property (i) that

$$\left\|\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{e^{\tau t A_\lambda} e^{(1-\tau)tA_\mu} z\right\}\right\| \leqslant t\,\|A_\lambda z - A_\mu z\| \qquad \forall\, \tau \in [0,1].$$

From here, we obtain (ii) by integration:

$$\|e^{tA_\lambda} z - e^{tA_\mu} z\| = \left\|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{e^{\tau t A_\lambda} e^{(1-\tau)tA_\mu} z\right\}\mathrm{d}\tau\right\| \leqslant t\,\|A_\lambda z - A_\mu z\|\,. \qquad \square$$

The following result is known as the Lumer–Phillips theorem.

Theorem 3.8.4. *For any $A : \mathcal{D}(A) \to X$ the following statements are equivalent:*

(1) *A is the generator of a contraction semigroup on X.*
(2) *A is m-dissipative.*

Proof. The fact that (1) implies (2) was proved in Proposition 3.1.13.

Conversely, let A be m-dissipative and let A_λ be its Yosida approximation. Our aim is to define $\mathbb{T}$ (the semigroup generated by A) by

$$\mathbb{T}_t z = \lim_{n\to\infty} e^{tA_n} z \qquad \forall\, z \in X. \tag{3.8.2}$$

For this, first we consider $w \in \mathcal{D}(A)$. By part (ii) of Proposition 3.8.3 we have

$$\left\|e^{tA_m} w - e^{tA_n} w\right\| \leqslant t\|A_m w - A_n w\| \qquad \forall\, m, n \in \mathbb{N},\ t \geqslant 0. \tag{3.8.3}$$

Using Remark 3.8.2 it follows that the sequence $(e^{tA_n} w)$ is a Cauchy sequence in X, for every $t \geqslant 0$. Thus, we can define $\mathbb{T}_t w$ (for $w \in \mathcal{D}(A)$ and $t \geqslant 0$) as the limit of this Cauchy sequence. From statement (i) of Proposition 3.8.3 it follows that

$$\|\mathbb{T}_t w\| \leqslant \|w\| \qquad \forall\, w \in \mathcal{D}(A).$$

Since $\mathcal{D}(A)$ is dense in X, it follows that (for every $t \geqslant 0$) $\mathbb{T}_t$ can be extended to an operator in $\mathcal{L}(X)$, also denoted by $\mathbb{T}_t$, and we have $\|\mathbb{T}_t\| \leqslant 1$. Taking limits in (3.8.3) as $m \to \infty$, we obtain that for $w \in \mathcal{D}(A)$,

$$\left\|\mathbb{T}_t w - e^{tA_n} w\right\| \leqslant t\|Aw - A_n w\| \qquad \forall\, n \in \mathbb{N},\ t \geqslant 0. \tag{3.8.4}$$

Now we show that the limit in (3.8.2) holds uniformly on bounded intervals, for every $z \in X$. Let $z \in X$ and $w \in \mathcal{D}(A)$. We use the decomposition

$$\|\mathbb{T}_t z - e^{tA_n} z\| \leqslant \|\mathbb{T}_t(z - w)\| + \|\mathbb{T}_t w - e^{tA_n} w\| + \|e^{tA_n}(w - z)\|\,.$$

For a fixed z, by choosing $w \in \mathcal{D}(A)$ such that $\|z - w\| \leqslant \frac{\varepsilon}{3}$, the first and the last terms on the right-hand side above become $\leqslant \frac{\varepsilon}{3}$ (we have used statement (i) of Proposition 3.8.3 again). Once such a w has been chosen, for every bounded interval $J \subset [0, \infty)$ we can find (according to (3.8.4) and Remark 3.8.2) an index $N \in \mathbb{N}$ such that for $t \in J$ and $n \geqslant N$, the middle term on the right-hand side above becomes $\leqslant \frac{\varepsilon}{3}$. Thus, given a bounded interval $J \subset [0, \infty)$, we can find $N \in \mathbb{N}$ such that $\|\mathbb{T}_t z - e^{tA_n} z\| \leqslant \varepsilon$ holds for all $t \in J$ and for all $n \geqslant N$, which is the uniform convergence property claimed earlier.

The uniform convergence of (3.8.2) on bounded intervals implies that the functions $t \to \mathbb{T}_t z$ are continuous (for every $z \in X$); i.e., the family $\mathbb{T} = (\mathbb{T}_t)_{t \geqslant 0}$ is strongly continuous. The properties

$$\mathbb{T}_{t+\tau} = \mathbb{T}_t \mathbb{T}_\tau \quad \forall t, \tau \geqslant 0 \quad \text{and} \quad \mathbb{T}_0 = I$$

follow from the corresponding properties of e^{tA_n}, by taking limits. Thus, we have shown that $\mathbb{T}$ is a contraction semigroup on X.

It remains to be shown that the generator of $\mathbb{T}$ is A. For each $z \in \mathcal{D}(A)$ we have, using Remark 2.1.7 applied to A_λ,

$$\mathbb{T}_t z - z = \lim_{\lambda \to \infty} e^{tA_\lambda} z - z = \lim_{\lambda \to \infty} \int_0^t e^{\sigma A_\lambda} A_\lambda z \, \mathrm{d}\sigma = \int_0^t \mathbb{T}_\sigma A z \, \mathrm{d}\sigma .$$

Denote the generator of $\mathbb{T}$ by $\widetilde{A}$, so that $\widetilde{A}$ is m-dissipative. If we divide both sides of the above equation by t and take limits as $t \to 0$, we obtain that $z \in \mathcal{D}(\widetilde{A})$ and $\widetilde{A} z = Az$. Thus, $\widetilde{A}$ is a dissipative extension of A. Since A was assumed to be m-dissipative, this implies that $\widetilde{A} = A$. $\square$

Proposition 3.8.5. *Let $A : \mathcal{D}(A) \to X$, $A \leqslant 0$. Then A generates a strongly continuous semigroup $\mathbb{T}$ on X. For all $t \geqslant 0$ we have $\mathbb{T}_t \geqslant 0$ and*

$$\|\mathbb{T}_t\| = e^{-mt}, \quad \text{where} \quad -m = \max \sigma(A) .$$

Proof. If $-m = \max \sigma(A)$, then $A = -mI - A_0$ where $A_0 \geqslant 0$ and $0 \in \sigma(A_0)$. (The fact that $A_0 \geqslant 0$ follows from Proposition 3.3.3.) According to Proposition 3.3.5 and the Lumer–Phillips theorem, $-A_0$ generates a contraction semigroup $\mathbb{T}^0$ on X. Since $0 \in \sigma(-A_0)$, we have the growth bound $\omega_0(\mathbb{T}^0) = 0$. According to Remark 2.2.16 we have $r(\mathbb{T}^0_t) = 1$ for all $t \geqslant 0$. Since $r(\mathbb{T}^0_t) \leqslant \|\mathbb{T}^0_t\|$, this implies that $\|\mathbb{T}^0_t\| = 1$ for all $t \geqslant 0$. The operator A generates the semigroup $\mathbb{T}_t = e^{-mt} \mathbb{T}^0_t$, which implies that $\|\mathbb{T}_t\| = e^{-mt}$ for all $t \geqslant 0$. Proposition 2.8.5 implies that $\mathbb{T}^*_t = \mathbb{T}_t$. Since $\mathbb{T}_t = \mathbb{T}^2_{t/2} = \mathbb{T}^*_{t/2} \mathbb{T}_{t/2}$, it follows that $\mathbb{T}_t \geqslant 0$. $\square$

Bibliographic notes. Theorem 3.8.4 is a basic tool for establishing that the Cauchy problem for certain linear systems of equations is well posed. It is due to E. Hille, K. Yosida, G. Lumer and R. Phillips (in various versions) in the period 1957–1961 and it is known as the *Lumer–Phillips theorem*, based on [162]. A related but more complicated theorem is the *Hille–Yosida theorem*, which gives necessary and sufficient conditions for a densely defined linear operator on a Banach space X to be the generator of a strongly continuous semigroup $\mathbb{T}$ on X satisfying the growth estimate (2.1.4). The conditions are that every $s \in \mathbb{C}$ with $\operatorname{Re} s > \omega$ belongs to $\rho(A)$ and for every such s,

$$\|(sI - A)^{-n}\| \leqslant \frac{M_\omega}{(\operatorname{Re} s - \omega)^n} \qquad \forall\, n \in \mathbb{N}. \tag{3.8.5}$$

It is enough to verify that $s \in \rho(A)$ and (3.8.5) holds for all real $s > \omega$. We omit the proof of this theorem, because it is not needed in this book.

Using Proposition 3.1.9, the Lumer–Phillips theorem may be regarded as a particular case of the Hille–Yosida theorem. Going in the opposite direction, it is not difficult to obtain the Hille–Yosida theorem from the Lumer–Phillips theorem (the version for Banach spaces). This approach to prove the Hille–Yosida theorem is adopted in Pazy [182, around p. 20]. We mention that the terminology is not universally agreed upon: what we (and many others) call the Lumer–Phillips theorem is called by some authors the Hille–Yosida theorem.

An important result, the *theorem of Stone* given below, characterizes the generators of unitary groups. It can be proven using the Lumer–Phillips theorem, as we do it here. Actually, it was published by M.H. Stone in 1932, many years before the paper of Lumer and Phillips [162], and the original proof used the spectral theory of self-adjoint operators, as in Rudin [195, p. 360].

Theorem 3.8.6. *For any $A : \mathcal{D}(A) \to X$ the following statements are equivalent:*

(1) *A is the generator of a unitary group on X.*
(2) *A is skew-adjoint.*

Proof. Assume that A is the generator of a unitary group $\mathbb{T}$ on X. We introduce the inverse group $\mathbb{S}$, as in Remark 2.7.6, then according to the same remark the generator of $\mathbb{S}$ is $-A$. But from the definition of a unitary group it follows that $\mathbb{S}$ is the adjoint group of $\mathbb{T}$. According to Proposition 2.8.5 we obtain that $-A = A^*$. An alternative way to see that (1) implies (2) is to use Corollary 3.7.5.

Conversely, suppose that A is skew-adjoint. Then according to Proposition 3.7.2, both A and $-A$ are m-dissipative, hence they both generate semigroups of contractions, denoted $\mathbb{T}$ and $\mathbb{S}$. We extend the family $\mathbb{T}$ to $\mathbb{R}$ by putting $\mathbb{T}_{-t} = \mathbb{S}_t$ for all $t > 0$. By Proposition 2.7.8, this extended $\mathbb{T}$ is a strongly continuous group on X, so that $\mathbb{S}_t = (\mathbb{T}_t)^{-1}$. On the other hand, $-A = A^*$, so that by Proposition 2.8.5 we have $\mathbb{S}_t = \mathbb{T}_t^*$. This shows that $\mathbb{T}_t$ is unitary for all $t \geqslant 0$. □

We present an application of Stone's theorem to certain second-order differential equations on a Hilbert space H.

Proposition 3.8.7. *We use the notation of Proposition* 3.7.6*. Then A generates a unitary group on $X = H_{\frac{1}{2}} \times H$.*

If $w_0 \in H_1$ and $v_0 \in H_{\frac{1}{2}}$, then the initial value problem

$$\ddot{w}(t) + A_0 w(t) = 0, \qquad w(0) = w_0, \qquad \dot{w}(0) = v_0, \tag{3.8.6}$$

has a unique solution

$$w \in C([0,\infty); H_1) \cap C^1([0,\infty); H_{\frac{1}{2}}) \cap C^2([0,\infty); H), \tag{3.8.7}$$

and this solution satisfies

$$\|w(t)\|_{\frac{1}{2}}^2 + \|\dot{w}(t)\|^2 = \|w_0\|_{\frac{1}{2}}^2 + \|v_0\|^2 \qquad \forall\, t \geqslant 0. \tag{3.8.8}$$

Proof. If we denote $z(t) = \left[\begin{smallmatrix} w(t) \\ \dot{w}(t) \end{smallmatrix}\right]$, then w satisfies (3.8.6) and (3.8.7) iff z satisfies the initial value problem $\dot{z}(t) = Az(t)$, $z(0) = z_0$, where $z_0 = \left[\begin{smallmatrix} w_0 \\ v_0 \end{smallmatrix}\right] \in \mathcal{D}(A)$ and

$$z \in C([0,\infty); X_1) \cap C^1([0,\infty); X). \tag{3.8.9}$$

According to Proposition 3.7.6, A is skew-adjoint on X. According to Stone's theorem, A generates a unitary group on X. We know from Proposition 2.3.5 that the initial value problem $\dot{z}(t) = Az(t)$, $z(0) = z_0$, has a unique solution satisfying (3.8.9). Thus, we have proved the existence of a unique solution w of (3.8.6) which satisfies (3.8.7). The energy identity (3.8.8) is a consequence of the fact that the semigroup generated by A is unitary. $\square$

In particular, if we take $A_0 = -\Delta$, where Δ is the Dirichlet Laplacian from Section 3.6, then (3.8.6) becomes the wave equation with Dirichlet boundary conditions and Proposition 3.8.7 becomes an existence and uniqueness result for the solutions of this wave equation; see Proposition 7.1.1.

3.9 The wave equation with boundary damping

In this section we show, as an application of the Lumer–Philips theorem, that the wave equation, with a Dirichlet boundary condition on a part of the boundary and with a dissipative condition on the remaining part of the boundary, defines a contraction semigroup on an appropriate Hilbert space. Our approach follows closely the presentation in Komornik and Zuazua [132]. Other papers which study well-posedness and other issues for the same system are Malinen and Staffans [165], Rodriguez-Bernal and Zuazua [192], and Weiss and Tucsnak [235].

Notation and preliminaries. We denote by $v \cdot w$ the bilinear product of $v, w \in \mathbb{C}^n$ ($n \in \mathbb{N}$), defined by $v \cdot w = v_1 w_1 + \cdots + v_n w_n$, and by $|\cdot|$ the Euclidean norm on $\mathbb{C}^n$. The set $\Omega \subset \mathbb{R}^n$ is supposed bounded, connected and with a Lipschitz boundary $\partial\Omega$. We assume that Γ_0, Γ_1 are open subsets of $\partial\Omega$ such that

$$\operatorname{clos}\Gamma_0 \cup \operatorname{clos}\Gamma_1 = \partial\Omega, \qquad \Gamma_0 \cap \Gamma_1 = \emptyset, \qquad \Gamma_0 \neq \emptyset.$$

Let $\mathcal{H}^1_{\Gamma_0}(\Omega)$ be the space of all those functions in $\mathcal{H}^1(\Omega)$ which vanish on Γ_0. This space is presented in more detail in Appendix II (Section 13.6). According to Theorem 13.6.9, the Poincaré inequality holds for Ω and Γ_0; i.e., there exists a $c > 0$ such that

$$\int_\Omega |f(x)|^2 \mathrm{d}x \leqslant c^2 \int_\Omega |(\nabla f)(x)|^2 \mathrm{d}x \qquad \forall\, f \in \mathcal{H}^1_{\Gamma_0}(\Omega).$$

This implies that $\mathcal{H}^1_{\Gamma_0}(\Omega)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{\mathcal{H}^1_{\Gamma_0}(\Omega)} = \int_\Omega \nabla f \cdot \nabla \overline{g} \,\mathrm{d}x \qquad \forall\, f,\, g \in \mathcal{H}^1_{\Gamma_0}(\Omega),$$

and that the corresponding norm is equivalent to the restriction to $\mathcal{H}^1_{\Gamma_0}(\Omega)$ of the usual norm in $\mathcal{H}^1(\Omega)$. This implies in turn that the space

$$X = \mathcal{H}^1_{\Gamma_0}(\Omega) \times L^2(\Omega),$$

endowed with the inner product

$$\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \right\rangle = \int_\Omega \nabla f \cdot \nabla \overline{\varphi} \,\mathrm{d}x + \int_\Omega g \overline{\psi} \,\mathrm{d}x \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in X, \tag{3.9.1}$$

is a Hilbert space. The induced norm on X, which we simply denote by $\|\cdot\|$, is equivalent to the restriction to X of the usual norm on $\mathcal{H}^1(\Omega) \times L^2(\Omega)$.

For $f \in \mathcal{H}^1_{\Gamma_0}(\Omega)$ we cannot define the Neumann trace $\frac{\partial f}{\partial \nu}$ on Γ_1, in the sense of the trace theorems in Section 13.6. However, for $f \in \mathcal{H}^1_{\Gamma_0}(\Omega)$ with $\Delta f \in L^2(\Omega)$ and for $h \in L^2(\Gamma_1)$ we can define the equality $\frac{\partial f}{\partial \nu}|_{\Gamma_1} = h$ in a weak sense by

$$\langle \Delta f, \varphi\rangle_{L^2(\Omega)} + \langle \nabla f, \nabla \varphi\rangle_{[L^2(\Omega)]^n} = \langle h, \varphi\rangle_{L^2(\Gamma_1)} \qquad \forall\, \varphi \in \mathcal{H}^1_{\Gamma_0}(\Omega). \tag{3.9.2}$$

The above definition clearly coincides with the usual one if f is smooth enough (in $\mathcal{H}^2(\Omega)$). If $\partial\Gamma_0$ and $\partial\Gamma_1$ have surface measure zero in $\partial\Omega$ and f and h satisfy (3.9.2), then h is uniquely determined by f. Indeed, in this case, the traces of functions $\varphi \in \mathcal{H}^1_{\Gamma_0}(\Omega)$ on Γ_1 are dense in $L^2(\Gamma_1)$, as follows from Remark 13.6.14.

Finally, we assume that $b \in L^\infty(\Gamma_1)$ is real valued. The equations of the system considered in this section are

$$\begin{cases} \ddot{z}(x,t) = \Delta z(x,t) & \text{on } \Omega \times [0,\infty), \\ z(x,t) = 0 & \text{on } \Gamma_0 \times [0,\infty), \\ \frac{\partial}{\partial \nu} z(x,t) + b^2(x)\dot{z}(x,t) = 0 & \text{on } \Gamma_1 \times [0,\infty), \\ z(x,0) = z_0(x), \quad \dot{z}(x,0) = w_0(x) & \text{on } \Omega. \end{cases} \tag{3.9.3}$$

The functions z_0 and w_0 are the initial state of the system. The part Γ_0 of the boundary is just reflecting waves, while on the portion Γ_1 we have a *dissipative* boundary condition. This terminology can be justified by a simple formal calculation. More precisely, if we assume that z is a smooth enough solution of (3.9.3), then simple integrations by parts show that for every $t \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla z(\cdot,t)\|^2_{[L^2(\Omega)]^n} + \|\dot z(\cdot,t)\|^2_{L^2(\Omega)}\right) = -2\int_{\Gamma_1} b^2|\dot z(\cdot,t)|^2\,\mathrm{d}\sigma. \tag{3.9.4}$$

Therefore the function $t \mapsto \|z(\cdot,t)\|^2_{[L^2(\Omega)]^n} + \|\dot z(\cdot,t)\|^2_{L^2(\Omega)}$, which in many applications is the total energy of the system, is non-increasing.

To transform the above formal analysis into a rigorous one, we introduce the space $\mathcal{D}(A) \subset X$ formed of those $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{H}^1_{\Gamma_0}(\Omega) \times \mathcal{H}^1_{\Gamma_0}(\Omega)$ such that $\Delta f \in L^2(\Omega)$ and

$$\langle \Delta f, \varphi\rangle_{L^2(\Omega)} + \langle \nabla f, \nabla\varphi\rangle_{[L^2(\Omega)]^n} = -\langle b^2 g, \varphi\rangle_{L^2(\Gamma_1)} \qquad \forall\, \varphi \in \mathcal{H}^1_{\Gamma_0}(\Omega). \tag{3.9.5}$$

As explained a little earlier, (3.9.5) means that, in a weak sense, $\frac{\partial f}{\partial\nu}|_{\Gamma_1} + b^2 g = 0$. Moreover, if $\partial\Gamma_0$ and $\partial\Gamma_1$ have measure zero in $\partial\Omega$, then $b^2 g$ is determined by f.

The above definition of $\mathcal{D}(A)$ takes an easier-to-understand form if we make much stronger assumptions on the sets Ω, Γ_0 and Γ_1.

Proposition 3.9.1. *Assume that $\partial\Omega$ is of class C^2,* clos $\Gamma_0 = \Gamma_0$*,* clos $\Gamma_1 = \Gamma_1$ *and $b \in C^1(\partial\Omega)$. Then*

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in \left[\mathcal{H}^2(\Omega) \cap \mathcal{H}^1_{\Gamma_0}(\Omega)\right] \times \mathcal{H}^1_{\Gamma_0}(\Omega) \;\middle|\; \frac{\partial f}{\partial\nu}|_{\Gamma_1} = -b^2 g|_{\Gamma_1} \right\}, \tag{3.9.6}$$

where $\frac{\partial f}{\partial\nu}|_{\Gamma_1}$ and $g|_{\Gamma_1}$ are taken in the sense of the trace theorems from Section 13.6.

Proof. Let $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{D}(A)$. We know from Remark 13.6.15 that $g|_{\Gamma_1} \in \mathcal{H}^{\frac12}(\Gamma_1)$, so that $-b^2 g|_{\Gamma_1} \in \mathcal{H}^{\frac12}(\Gamma_1)$. According to Proposition 13.6.16, there exists a unique $\tilde f \in \mathcal{H}^2(\Omega) \cap \mathcal{H}^1_{\Gamma_0}(\Omega)$ such that

$$\Delta\tilde f = \Delta f \ \text{ in } \ L^2(\Omega), \qquad \frac{\partial\tilde f}{\partial\nu}|_{\Gamma_1} = -b^2 g|_{\Gamma_1}.$$

Taking the inner product of the first formula above with $\varphi \in \mathcal{H}^1_{\Gamma_0}(\Omega)$ and using Remark 13.7.3, it follows that

$$\langle \Delta f, \varphi\rangle_{L^2(\Omega)} + \langle \nabla\tilde f, \nabla\varphi\rangle_{[L^2(\Omega)]^n} = -\langle b^2 g, \varphi\rangle_{L^2(\Gamma_1)} \qquad \forall\, \varphi \in \mathcal{H}^1_{\Gamma_0}(\Omega).$$

Comparing the above formula with (3.9.5) it follows that

$$\langle \nabla\tilde f, \nabla\varphi\rangle_{[L^2(\Omega)]^n} = \langle \nabla f, \nabla\varphi\rangle_{[L^2(\Omega)]^n} \qquad \forall\, \varphi \in \mathcal{H}^1_{\Gamma_0}(\Omega),$$

so that $f = \tilde f \in \mathcal{H}^2(\Omega)$ and $\frac{\partial f}{\partial\nu}|_{\Gamma_1} = -b^2 g|_{\Gamma_1}$. $\square$

The operator $A : \mathcal{D}(A) \to X$ is defined by

$$A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \Delta f \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A). \tag{3.9.7}$$

The main result of this section is the following.

Proposition 3.9.2. *The operator A defined above is m-dissipative.*

Proof. We first note that from (3.9.1) and (3.9.7) we obtain that

$$\left\langle A \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} f \\ g \end{bmatrix} \right\rangle = \langle \nabla g, \nabla f \rangle_{[L^2(\Omega)]^n} + \langle \Delta f, g \rangle_{L^2(\Omega)} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A).$$

Using (3.9.5) with $\varphi = g$ it follows that

$$\operatorname{Re} \left\langle A \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} f \\ g \end{bmatrix} \right\rangle = - \|b\, g\|^2_{L^2(\Gamma_1)} \leqslant 0 \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A),$$

so that A is dissipative. To show that A is m-dissipative, we prove that $I - A$ is onto. For this we take $\left[\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right] \in X$ and we prove the existence of $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{D}(A)$ such that $A \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] = \left[\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right]$. First note that, by the Riesz representation theorem, for every $\left[\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right] \in X$ there exists a unique $f \in V$ such that

$$\begin{aligned} &\langle \nabla f, \nabla \varphi \rangle_{[L^2(\Omega)]^n} + \langle f, \varphi \rangle_{L^2(\Omega)} + \langle b^2 f, \varphi \rangle_{L^2(\Gamma_1)} \\ &\qquad = \langle \xi + \eta, \varphi \rangle_{L^2(\Omega)} + \langle b^2 \xi, \varphi \rangle_{L^2(\Gamma_1)} \qquad \forall\, \varphi \in \mathcal{H}^1_{\Gamma_0}(\Omega). \end{aligned} \tag{3.9.8}$$

Taking $\varphi = \overline{\psi}$, with $\psi \in \mathcal{D}(\Omega)$, it follows that

$$\int_\Omega (\nabla f \cdot \nabla \psi + f \psi)\, \mathrm{d}x = \int_\Omega (\xi + \eta) \psi\, \mathrm{d}x \qquad \forall\, \psi \in \mathcal{D}(\Omega),$$

so that in $\mathcal{D}'(\Omega)$ we have

$$\Delta f = f - \xi - \eta \in L^2(\Omega). \tag{3.9.9}$$

Substituting the above formula in (3.9.8) and setting

$$g = f - \xi, \tag{3.9.10}$$

we obtain that $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ satisfies (3.9.5) which, combined with (3.9.9), implies that $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{D}(A)$. Moreover, using (3.9.7), (3.9.9) and (3.9.10) we see that $(I - A) \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] = \left[\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right]$, so that $I - A$ is onto. Thus A is m-dissipative. $\square$

We say that z is a strong solution of (3.9.3) if

$$\begin{bmatrix} z \\ \dot{z} \end{bmatrix} \in C([0, \infty); \mathcal{D}(A)), \tag{3.9.11}$$

and the first equation in (3.9.3) holds in $C([0, \infty); L^2(\Omega))$.

As a consequence of Proposition 3.9.2, we obtain the following result.

Corollary 3.9.3. *For every* $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \mathcal{D}(A)$, *the initial and boundary value problem* (3.9.3) *admits a unique strong solution. Moreover, the energy estimate* (3.9.4) *holds for every* $t \geqslant 0$.

Proof. We know from the last proposition that A is m-dissipative, so that, by applying the Lumer–Phillips theorem (Theorem 3.8.4) it follows that A is the generator of a contraction semigroup $\mathbb{T}$ on X. We denote as usual by X_1 the space $\mathcal{D}(A)$ endowed with the graph norm. We set, for every $t \geqslant 0$, $\left[\begin{smallmatrix} z(t) \\ w(t) \end{smallmatrix}\right] = \mathbb{T}_t \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right]$. According to Proposition 2.3.5 it follows that

$$\begin{bmatrix} z \\ w \end{bmatrix} \in C([0,\infty), X_1) \cap C^1([0,\infty), X), \tag{3.9.12}$$

$$\begin{bmatrix} \dot z(t) \\ \dot w(t) \end{bmatrix} = A \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \quad \text{for} \quad t \geqslant 0, \qquad \begin{bmatrix} z(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ w_0 \end{bmatrix}. \tag{3.9.13}$$

Using (3.9.7) it follows that $w(t) = \dot z(t)$, so that we have (3.9.11) and the first equation in (3.9.3) holds in $C([0,\infty); L^2(\Omega))$. We have thus shown that for every $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in X$ there exists a strong solution of (3.9.3) which is given by

$$\begin{bmatrix} z(t) \\ \dot z(t) \end{bmatrix} = \mathbb{T}_t \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \qquad \forall\, t \geqslant 0. \tag{3.9.14}$$

To show that this solution is unique, we note that if z is a strong solution of (3.9.3), then, denoting $\dot z(t) = w(t)$, we have that $\left[\begin{smallmatrix} z(t) \\ w(t) \end{smallmatrix}\right]$ satisfies (3.9.12), (3.9.13) so that, according to Proposition 2.3.5, z satisfies (3.9.14).

We still have to prove (3.9.4). A direct calculation combined with the fact that $\ddot z(t) = \Delta z(t)$ gives

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} &\left(\|\nabla z(\cdot,t)\|^2_{[L^2(\Omega)]^n} + \|\dot z(\cdot,t)\|^2_{L^2(\Omega)} \right) \\ &= \mathrm{Re}\, \left(\langle \nabla z(t), \nabla \dot z(t) \rangle_{[L^2(\Omega)]^n} + \langle \dot z(t), \Delta z(t) \rangle_{L^2(\Omega)} \right) \qquad \forall\, t \geqslant 0. \end{aligned}$$

Using (3.9.5) with $f = z(t)$ and $\varphi = \dot z(t)$ we obtain (3.9.4). $\square$

Chapter 4

Control and Observation Operators

Notation. Throughout this chapter, U, X and Y are complex Hilbert spaces which are identified with their duals. $\mathbb{T}$ is a strongly continuous semigroup on X, with generator $A : \mathcal{D}(A) \to X$ and growth bound $\omega_0(\mathbb{T})$. Recall from Section 2.10 that X_1 is $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, where $\beta \in \rho(A)$ is fixed, while X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1}z\|$. Remember that we use the notation A and $\mathbb{T}_t$ also for the extension of the original generator to X and for the extension of the original semigroup to X_{-1}. Recall also that X_1^d is $\mathcal{D}(A^*)$ with the norm $\|z\|_1^d = \|(\overline{\beta} I - A^*)z\|$ and X_{-1}^d is the completion of X with respect to the norm $\|z\|_{-1}^d = \|(\overline{\beta} I - A^*)^{-1}z\|$. Recall that X_{-1} is the dual of X_1^d with respect to the pivot space X.

Let $u, v \in L^2_{\mathrm{loc}}([0,\infty); U)$ and let $\tau \geqslant 0$. Then the *τ-concatenation of u and v*, $u \underset{\tau}{\diamondsuit} v$ is the function in $L^2_{\mathrm{loc}}([0,\infty); U)$ defined by

$$(u \underset{\tau}{\diamondsuit} v)(t) = \begin{cases} u(t) & \text{for} \quad t \in [0,\tau), \\ v(t-\tau) & \text{for} \quad t \geqslant \tau. \end{cases}$$

For $u \in L^2_{\mathrm{loc}}([0,\infty); U)$ and $\tau \geqslant 0$, the truncation of u to $[0,\tau]$ is denoted by $\mathbf{P}_\tau u$. This function is regarded as an element of $L^2([0,\infty); U)$ which is zero for $t > \tau$. Equivalently, $\mathbf{P}_\tau u = u \underset{\tau}{\diamondsuit} 0$. For every $\tau > 0$, $\mathbf{P}_\tau$ is an operator of norm 1 on $L^2([0,\infty); U)$. We denote by $\mathbf{S}_\tau$ the operator of right shift by τ on $L^2_{\mathrm{loc}}([0,\infty); U)$, so that $(\mathbf{S}_\tau u)(t) = u(t-\tau)$ for $t > \tau$, and $(\mathbf{S}_\tau u)(t) = 0$ for $t \in [0,\tau]$. Thus,

$$(u \underset{\tau}{\diamondsuit} v)(t) = \mathbf{P}_\tau u + \mathbf{S}_\tau v.$$

For any open interval J, the spaces $\mathcal{H}^1(J; U)$ and $\mathcal{H}^2(J; U)$ are defined as at the beginning of Chapter 2. $\mathcal{H}^1_{\mathrm{loc}}((0,\infty); U)$ is the space of those functions on $(0,\infty)$ whose restriction to $(0,n)$ is in $\mathcal{H}^1((0,n); U)$, for every $n \in \mathbb{N}$. The space $\mathcal{H}^2_{\mathrm{loc}}((0,\infty); U)$ is defined similarly. Recall that $\mathbb{C}_\alpha$ is the half-plane where $\operatorname{Re} s > \alpha$.

4.1 Solutions of non-homogeneous differential equations

The state trajectories z of a linear time-invariant system are defined as the solutions of a non-homogeneous differential equation of the form $\dot{z}(t) = Az(t) + Bu(t)$, where u is the input function. For this reason, we should clarify what we mean by a solution of such a differential equation, and then give some basic existence and uniqueness results. In this section, the operator B is not important, so that in our discussion we shall replace $Bu(t)$ by $f(t)$, and we call f the *forcing function.*

Definition 4.1.1. Consider the differential equation

$$\dot{z}(t) = Az(t) + f(t), \tag{4.1.1}$$

where $f \in L^1_{\rm loc}([0,\infty); X_{-1})$. A *solution of* (4.1.1) *in* X_{-1} is a function

$$z \in L^1_{\rm loc}([0,\infty); X) \cap C([0,\infty); X_{-1})$$

which satisfies the following equations in X_{-1}:

$$z(t) - z(0) = \int_0^t [Az(\sigma) + f(\sigma)]\,{\rm d}\sigma \qquad \forall\, t \in [0,\infty). \tag{4.1.2}$$

The above concept could also be called a "strong solution of (4.1.1) in X_{-1}", because (4.1.2) implies that z is absolutely continuous with values in X_{-1} and (4.1.1) holds for almost every $t \geqslant 0$, with the derivative computed with respect to the norm of X_{-1}. Equation (4.1.1) does not necessarily have a solution in the above sense.

Remark 4.1.2. We could also define the concept of a "weak solution of (4.1.1) in X_{-1}", by requiring instead of (4.1.2) that for every $\varphi \in X_1^d$ and every $t \geqslant 0$,

$$\langle z(t) - z(0), \varphi\rangle_{X_{-1},X_1^d} = \int_0^t \left[\langle z(\sigma), A^*\varphi\rangle_X + \langle f(\sigma), \varphi\rangle_{X_{-1},X_1^d}\right] {\rm d}\sigma\,.$$

However, it is easy to see that this is an equivalent concept to the concept of solution defined earlier. For this reason, we just use the term "solution in X_{-1}".

Sometimes it is convenient to use the above equivalent definition of a solution of (4.1.1). Sometimes it is also convenient to do this without identifying X with its dual X'. This can be done in the framework of Remark 2.10.11.

Remark 4.1.3. If $f \in L^1_{\rm loc}([0,\infty); X)$, then the concept of a solution of (4.1.1) in X can be defined similarly, by replacing everywhere in Definition 4.1.1 X_{-1} by X and X by X_1. This concept of a solution appears often in the literature. Similarly, we could introduce solutions of (4.1.1) in X_{-2} (this space was introduced in Section 2.10). It is easy to see that if z is a solution of (4.1.1) in X, then it is also a solution of (4.1.1) in X_{-1} (and any solution in X_{-1} is also a solution in X_{-2}). For our purposes, the most useful concept is the one we introduced in Definition 4.1.1.

Proposition 4.1.4. *With the notation of Definition* 4.1.1*, suppose that z is a solution of* (4.1.1) *in X_{-1} and denote $z_0 = z(0)$. Then z is given by*

$$z(t) = \mathbb{T}_t z_0 + \int_0^t \mathbb{T}_{t-\sigma} f(\sigma) \mathrm{d}\sigma . \tag{4.1.3}$$

In particular, for every $z_0 \in X$ there exists at most one solution in X_{-1} of (4.1.1) *which satisfies the initial condition $z(0) = z_0$.*

Proof. For $t > 0$ and $\varphi \in \mathcal{D}(A^{*2})$ fixed, introduce the function $g : [0,t] \to \mathbb{C}$ by

$$g(\sigma) = \langle \mathbb{T}_{t-\sigma} z(\sigma), \varphi \rangle_{X_{-1}, X_1^d} .$$

Moving $\mathbb{T}_{t-\sigma}$ to the right side of the above duality pairing and using the fact that the function $\sigma \to \mathbb{T}^*_{t-\sigma}\varphi$ is in $C^1([0,t], X_1^d)$, we see that g is absolutely continuous and its derivative is given, for almost every $\sigma \in [0,t]$, by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\sigma} g(\sigma) &= \langle Az(\sigma) + f(\sigma), \mathbb{T}^*_{t-\sigma}\varphi \rangle_{X_{-1}, X_1^d} - \langle z(\sigma), A^* \mathbb{T}^*_{t-\sigma}\varphi \rangle_{X_{-1}, X_1^d} \\ &= \langle f(\sigma), \mathbb{T}^*_{t-\sigma}\varphi \rangle_{X_{-1}, X_1^d} = \langle \mathbb{T}_{t-\sigma} f(\sigma), \varphi \rangle_{X_{-1}, X_1^d} . \end{aligned}$$

Integrating from 0 to t we obtain

$$g(t) - g(0) = \left\langle \int_0^t \mathbb{T}_{t-\sigma} f(\sigma) \mathrm{d}\sigma, \varphi \right\rangle_{X_{-1}, X_1^d} .$$

By the density of $\mathcal{D}(A^{*2})$ in X_1^d, we obtain the desired formula:

$$z(t) - \mathbb{T}_t z(0) = \int_0^t \mathbb{T}_{t-\sigma} f(\sigma) \mathrm{d}\sigma . \qquad \square$$

Definition 4.1.5. With the notation of Definition 4.1.1, the X_{-1}-valued function z defined in (4.1.3) is called the *mild solution* of (4.1.1), corresponding to the initial state $z_0 \in X$ and the forcing function $f \in L^1_{\mathrm{loc}}([0,\infty; X_{-1})$.

In the last proposition we have shown that every solution of (4.1.1) in X_{-1} is a mild solution of (4.1.1). The converse of this statement is not true. However, the following theorem shows that for forcing functions of class $\mathcal{H}^1$, the mild solution of (4.1.1) is actually a solution of (4.1.1) in X_{-1}, and moreover this solution is a continuous X-valued function.

Theorem 4.1.6. *If $z_0 \in X$ and $f \in \mathcal{H}^1_{\mathrm{loc}}((0,\infty); X_{-1})$, then* (4.1.1) *has a unique solution in X_{-1}, denoted z, that satisfies $z(0) = z_0$. Moreover, this solution is such that*

$$z \in C([0,\infty); X) \cap C^1([0,\infty); X_{-1}),$$

and it satisfies (4.1.1) *in the classical sense, at every $t \geqslant 0$.*

Note that from the above theorem it follows immediately that

$$Az + f \in C([0,\infty); X_{-1}).$$

Proof. Let $(\mathcal{S}_t)_{t\geqslant 0}$ be the unilateral left shift semigroup on $L^2([0,\infty); X_{-1})$ (see Example 2.3.7 for the scalar case $X = \mathbb{C}$). The generator of $\mathcal{S}$ is the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$, with domain $\mathcal{H}^1((0,\infty); X_{-1})$. We introduce the forcing function to state operators

$$\Phi_\tau \in \mathcal{L}(L^2([0,\infty); X_{-1}), X_{-1})$$

defined for all $\tau \geqslant 0$ by

$$\Phi_\tau f = \int_0^\tau \mathbb{T}_{\tau-\sigma} f(\sigma)\,\mathrm{d}\sigma.$$

Then the mild solution z of (4.1.1) is given by $z(t) = \mathbb{T}_t z_0 + \Phi_t f$. It is a routine task to verify that on $\mathcal{X} = X_{-1} \times L^2([0,\infty); X_{-1})$ the operators

$$\mathcal{T}_\tau = \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ 0 & \mathcal{S}_\tau \end{bmatrix}$$

form a strongly continuous semigroup, and the generator of this semigroup is

$$\mathcal{A}\begin{bmatrix} z_0 \\ f \end{bmatrix} = \begin{bmatrix} Az_0 + f(0) \\ \frac{\mathrm{d}f}{\mathrm{d}x} \end{bmatrix}, \qquad \mathcal{D}(\mathcal{A}) = X \times \mathcal{H}^1((0,\infty); X_{-1}).$$

The graph norm on the space $\mathcal{X}_1 = \mathcal{D}(\mathcal{A})$ turns out to be equivalent to the usual product norm of $X \times \mathcal{H}^1((0,\infty); X_{-1})$. Thus, we shall use this product norm on $\mathcal{X}_1$.

First we prove the theorem for $f \in \mathcal{H}^1((0,\infty); U)$. Choose $\left[\begin{smallmatrix} z_0 \\ f \end{smallmatrix}\right] \in \mathcal{D}(\mathcal{A})$ and define $q(t) = \mathcal{T}_t \left[\begin{smallmatrix} z_0 \\ f \end{smallmatrix}\right]$. We know from Proposition 2.3.5 that q satisfies

$$q \in C([0,\infty); \mathcal{X}_1) \cap C^1([0,\infty); \mathcal{X}).$$

The first component of q is the mild solution z of (4.1.1), corresponding to z_0 and f. Therefore,

$$z \in C([0,\infty); X) \cap C^1([0,\infty); X_{-1}).$$

We want to show that z is a solution of (4.1.1) in X_{-1}. According to Remark 2.1.7 we have

$$\mathcal{T}_t \begin{bmatrix} z_0 \\ f \end{bmatrix} - \begin{bmatrix} z_0 \\ f \end{bmatrix} = \mathcal{A} \int_0^t \mathcal{T}_\sigma \begin{bmatrix} z_0 \\ f \end{bmatrix} \mathrm{d}\sigma$$

for every $t \geqslant 0$. Looking at the first component only, we obtain that

$$z(t) - z_0 = \int_0^t [Az(\sigma) + f(\sigma)]\,\mathrm{d}\sigma.$$

Since this holds for all $t \geqslant 0$, z is indeed a solution of (4.1.1) in X_{-1}. Differentiating the above equation in X_{-1}, we obtain that z satisfies (4.1.1) at every $t \geqslant 0$. Thus, we have proved the theorem for the special case when $f \in \mathcal{H}^1((0,\infty); U)$.

Now let us consider $z_0 \in X$, $f \in \mathcal{H}^1_{\rm loc}((0,\infty);X_{-1})$ and let z be the corresponding mild solution of (4.1.1). Choose $\tau > 0$. It will be enough to prove that the restriction of z to $[0,\tau]$, denoted $\mathbf{P}_\tau z$, has the desired properties, i.e.,

$$\mathbf{P}_\tau z \in C([0,\tau];X) \cap C^1([0,\tau];X_{-1}),$$
$$z(t) - z_0 = \int_0^t [Az(\sigma) + f(\sigma)]\,{\rm d}\sigma \qquad \forall\, t \in [0,\tau].$$

On $[\tau,\infty)$ we modify f such that $f \in \mathcal{H}^2([0,\infty);X_{-1})$. Since $\mathbf{P}_\tau z$ depends only on z_0 and on $\mathbf{P}_\tau f$, z does not change on $[0,\tau]$. Thus, $\mathbf{P}_\tau z$ has the desired properties listed earlier, due to the special case of the theorem proved earlier. □

Remark 4.1.7. The last theorem remains valid for forcing functions f that satisfy $f(t) - f(0) = \int_0^t v(\sigma)\,{\rm d}\sigma$ for every $t \geqslant 0$, where $v \in L^1_{\rm loc}([0,\infty);X_{-1})$. The proof is similar, using a semigroup acting on the Banach space $X_{-1} \times L^1([0,\infty);X_{-1})$.

Remark 4.1.8. Let $z_0 \in X_{-1}$, $f \in L^1_{\rm loc}([0,\infty);X_{-1})$ and let z be the corresponding mild solution of (4.1.1) (i.e., given by (4.1.3)). Then z satisfies (4.1.2), still as an equality in X_{-1}, but with the integration carried out in X_{-2}.

Indeed, we know from the last theorem that (4.1.2) holds if $z_0 \in X$ and $f \in \mathcal{H}^1((0,\infty);X_{-1})$ (with the integration carried out in X_{-1}). Since both sides (as elements of X_{-2}) depend continuously on z_0 (as an element of X_{-1}) and on f (as an element of $L^1_{\rm loc}([0,\infty);X_{-1})$) and since $\mathcal{H}^1((0,\infty);X_{-1})$ is dense in $L^1_{\rm loc}([0,\infty);X_{-1})$, it follows that (4.1.2) holds as an equality in X_{-2}. But clearly the left-hand side is in X_{-1}, so that in fact we have an equality in X_{-1}, as claimed.

An easy consequence of the statement that we have just proved is that every mild solution of (4.1.1) corresponding to $z_0 \in X_{-1}$ and $f \in L^1_{\rm loc}([0,\infty);X_{-1})$ is a solution of this equation in X_{-2}.

Remark 4.1.9. For $f \in L^1_{\rm loc}([0,\infty);X_{-1})$ the *Laplace transform* of f and its domain (the set of points $s \in \mathbb{C}$ where $\hat{f}(s)$ exists) are defined in Appendix I (around (12.4.5)). If z is the mild solution of (4.1.1) corresponding to $z_0 \in X$ and f, then its Laplace transform is

$$\hat{z}(s) = (sI - A)^{-1}\left[z(0) + \hat{f}(s)\right],$$

and this exists at all the points $s \in \mathbb{C}$ for which ${\rm Re}\, s > \omega_0(\mathbb{T})$ and $\hat{f}(s)$ exists (and possibly also for all s in a larger half-plane). This follows from Remark 4.1.8, applying the Laplace transformation to (4.1.2).

4.2 Admissible control operators

The concept of an admissible control operator is motivated by the study of the solutions of the differential equation $\dot{z}(t) = Az(t)+Bu(t)$, where $u \in L^2_{\mathrm{loc}}([0,\infty);U)$, $z(0) \in X$ and $B \in \mathcal{L}(U, X_{-1})$. We would like to study those operators B for which all the mild solutions z of this equation (with u and $z(0)$ as described) are continuous X-valued functions. Such operators B will be called admissible.

Let $B \in \mathcal{L}(U, X_{-1})$ and $\tau \geqslant 0$. We define $\Phi_\tau \in \mathcal{L}(L^2([0,\infty);U), X_{-1})$ by

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma} Bu(\sigma)\,\mathrm{d}\sigma. \tag{4.2.1}$$

We are interested in these operators because they appear in (4.1.3) if we take $f = Bu$. It is clear that we could have defined Φ_τ such that $\Phi_\tau \in \mathcal{L}(L^2([0,\tau];U), X_{-1})$, but we wanted to avoid later difficulties which would occur if the domain of Φ_τ depended on τ. It is easy to see that $\Phi_\tau = \Phi_\tau \mathbf{P}_\tau$ (causality) and that for every $t, \tau \geqslant 0$,

$$\Phi_{\tau+t}\,(u \underset{\tau}{\diamondsuit} v) = \mathbb{T}_t\,\Phi_\tau u + \Phi_t v. \tag{4.2.2}$$

The latter property is called the *composition property.*

Definition 4.2.1. The operator $B \in \mathcal{L}(U; X_{-1})$ is called an *admissible control operator* for $\mathbb{T}$ if for some $\tau > 0$, $\operatorname{Ran} \Phi_\tau \subset X$.

Note that if B is admissible, then in (4.2.1) (with $t = \tau$) we integrate in X_{-1}, but the integral is in X, a dense subspace of X_{-1}.

The operator B (as in the above definition) is called *bounded* if $B \in \mathcal{L}(U, X)$ (and unbounded otherwise). Obviously, every bounded B is admissible for $\mathbb{T}$.

Proposition 4.2.2. *Suppose that $B \in \mathcal{L}(U, X_{-1})$ is admissible; i.e.,* $\operatorname{Ran} \Phi_\tau \subset X$ *holds for a specific $\tau > 0$. Then for every $t \geqslant 0$ we have*

$$\Phi_t \in \mathcal{L}(L^2([0,\infty);U), X).$$

Proof. Choose $\beta \in \rho(A)$ and define $B_0 = (\beta I - A)^{-1}B$. Then $B_0 \in \mathcal{L}(U, X)$ and

$$\Phi_\tau\, u = (\beta I - A) \int_0^\tau \mathbb{T}_{\tau-\sigma}\, B_0\, u(\sigma)\,\mathrm{d}\sigma,$$

which shows that Φ_τ is closed. By the closed-graph theorem, Φ_τ is bounded.

Let $t \in [0, \tau)$. We rewrite (4.2.2) with $u = 0$ and with $\tau - t$ in place of τ as follows: $\Phi_\tau(0 \underset{\tau-t}{\diamondsuit} v) = \Phi_t v$. This shows that $\Phi_t \in \mathcal{L}(L^2([0,\infty);U), X)$.

The identity (4.2.2) with $t = \tau$ implies that $\Phi_{2\tau}$ is bounded. By induction, Φ_t is bounded for all t of the form $t = 2^n\tau$, where $n \in \mathbb{N}$. Combining this with what we proved in the previous paragraph, we obtain that Φ_t is bounded for all $t \geqslant 0$. $\square$

The operators Φ_t as in the above proposition are called the *input maps* corresponding to (A, B). B can be recovered from them by the following formula:

$$B\mathrm{v} = \lim_{t \to 0} \frac{1}{t} \Phi_t \mathrm{v} \qquad \forall \, \mathrm{v} \in U \,, \tag{4.2.3}$$

where we have used the notation v also for the constant function equal to v, defined for all $t \geqslant 0$. (The above limit is taken in X_{-1}.) The proof of (4.2.3) is easy, using the fact that $\mathbb{T}$ is strongly continuous on X_{-1} (see Proposition 2.10.4).

Remark 4.2.3. By a *step function* on $[0, \tau]$ (or a piecewise constant function) we mean a function that is constant on each interval of a partition of $[0, \tau]$ into finitely many intervals. We have the following equivalent characterization of admissible control operators: $B \in \mathcal{L}(U, X_{-1})$ is admissible iff, for some $\tau > 0$, there exists a $K_\tau > 0$ such that for every step function $v : [0, \tau] \to U$,

$$\|\Phi_\tau v\|_X \leqslant K_\tau \|v\|_{L^2} \,. \tag{4.2.4}$$

Indeed, if v is a step function, then $\Phi_\tau v \in X$ (regardless if B is admissible), as it follows from Proposition 2.1.6 (with X_{-1} in place of X). If (4.2.4) holds, then, by the density of step functions in $L^2([0, \tau]; U)$ (see Section 12.5), Φ_τ is bounded, so that B is admissible. The converse statement follows from Proposition 4.2.2.

Proposition 4.2.4. *Suppose that B is an admissible control operator for $\mathbb{T}$. Then the function*

$$\varphi(t, u) = \Phi_t u$$

is continuous on the product $[0, \infty) \times L^2([0, \infty); U)$.

Proof. Taking in (4.2.2) $u = 0$ and taking the supremum of the norm over all $v \in L^2([0, \infty); U)$ with $\|v\| = 1$ we get, denoting $T = \tau + t$,

$$\|\Phi_t\| \leqslant \|\Phi_T\| \quad \text{for} \;\; t \leqslant T \,, \tag{4.2.5}$$

so that $\|\Phi_t\|$ is non-decreasing.

First we prove the continuity of $\varphi(t, u)$ with respect to the time t, so for the time being let $u \in L^2([0, \infty); U)$ be fixed and let

$$f(t) = \Phi_t u \,.$$

Inequality (4.2.5), together with causality ($\Phi_t = \Phi_t \mathbf{P}_t$), implies that

$$\|f(t)\| \leqslant \|\Phi_1\| \cdot \|\mathbf{P}_t u\| \qquad \forall \, t \in [0, 1] \,.$$

Obviously $\|\mathbf{P}_t u\| \to 0$ for $t \to 0$, so that $\lim_{t \to 0} f(t) = 0$. The right continuity of f in any point $\tau > 0$ now follows easily from the composition property (4.2.2).

To prove the left continuity of f in $\tau > 0$ we take a sequence (ε_n) with $\varepsilon_n \in [0, \tau]$ and $\varepsilon_n \to 0$ and we define $u_n(t) = u(\varepsilon_n + t)$, so that $u_n \in L^2([0, \infty); U)$ and $u_n \to u$. We have $u = u \underset{\varepsilon_n}{\diamondsuit} u_n$, so that according to (4.2.2),

$$\Phi_{\varepsilon_n + (\tau - \varepsilon_n)} u \;=\; \mathbb{T}_{\tau - \varepsilon_n} \Phi_{\varepsilon_n} u + \Phi_{\tau - \varepsilon_n} u_n \,.$$

From here

$$\Phi_\tau u - \Phi_{\tau - \varepsilon_n} u \;=\; \mathbb{T}_{\tau - \varepsilon_n} \Phi_{\varepsilon_n} u + \Phi_{\tau - \varepsilon_n} (u_n - u) \,,$$

which yields

$$\|\Phi_\tau u - \Phi_{\tau - \varepsilon_n} u\| \leqslant M \cdot \|f(\varepsilon_n)\| + \|\Phi_\tau\| \cdot \|u_n - u\| \,,$$

where M is a bound for $\|\mathbb{T}_t\|$ on $[0, \tau]$. Since $f(\varepsilon_n) \to 0$, the left continuity of f in any point $\tau > 0$ is now also proved.

The joint continuity of φ follows now easily from the decomposition

$$\Phi_t v - \Phi_\tau u \;=\; \Phi_t (v - u) + (\Phi_t - \Phi_\tau) u \,,$$

where $(t, v) \to (\tau, u)$. □

The following proposition shows that if B is admissible and $u \in L^2_{\text{loc}}([0,\infty); U)$, then the initial value problem associated with the equation $\dot{z}(t) = Az(t) + Bu(t)$ has a unique solution in X_{-1}, in the sense of Definition 4.1.1.

Proposition 4.2.5. *Assume that $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$. Then for every $z_0 \in X$ and every $u \in L^2_{\text{loc}}([0, \infty); U)$, the initial value problem*

$$\dot{z}(t) \;=\; Az(t) + Bu(t) \,, \qquad z(0) \;=\; z_0 \,, \tag{4.2.6}$$

has a unique solution in X_{-1}. This solution is given by

$$z(t) \;=\; \mathbb{T}_t z_0 + \Phi_t u \tag{4.2.7}$$

and it satisfies

$$z \in C([0, \infty); X) \cap \mathcal{H}^1_{\text{loc}}((0, \infty); X_{-1}) \,.$$

Proof. With B, z_0 and u as in the proposition, define the function z by (4.2.7). According to our concept of a solution of (4.2.6) in X_{-1}, we have to show that $z \in L^1_{\text{loc}}([0, \infty); X) \cap C([0, \infty); X_{-1})$ and it satisfies (4.1.2) with $f = Bu$, i.e.,

$$z(t) - z(0) \;=\; \int_0^t [Az(\sigma) + Bu(\sigma)] \, \mathrm{d}\sigma \qquad \forall\, t \in [0, \infty) \,, \tag{4.2.8}$$

with the integration carried out in X_{-1}. According to Remark 4.1.8 the above equality holds in X_{-1}, with the integration carried out in X_{-2}. It follows from Proposition 4.2.4 that $z \in C([0, \infty); X)$. Hence, the terms of (4.2.8) are in fact in X and what we integrate is in $L^2_{\text{loc}}([0, \infty); X_{-1})$, so that we may consider the integration to be done in X_{-1}. Thus, z is a solution of (4.2.6). It also follows that $z \in \mathcal{H}^1_{\text{loc}}(0, \infty; X_{-1})$. The uniqueness of z follows from Proposition 4.1.4. □

Remark 4.2.6. The above result implies the following: With the assumptions of Proposition 4.2.5, for every $z_0 \in X$ and every $u \in L^2_{\rm loc}([0,\infty);U)$ there exists a unique $z \in C([0,\infty);X)$ such that, for every $t \geqslant 0$,

$$\langle z(t) - z_0, \psi\rangle_X \;=\; \int_0^t \left[\langle z(\sigma), A^*\psi\rangle_X + \langle u(\sigma), B^*\psi\rangle_U\right] {\rm d}\sigma \qquad \forall\, \psi \in \mathcal{D}(A^*).$$

Sometimes it is more convenient not to identify X with its dual X'. Then $A^* \in \mathcal{L}(X_1^d, X')$ and $B^* \in \mathcal{L}(X_1^d, U)$, where X_1^d is as in Remark 2.10.11, and the inner product in X has to be replaced with the duality pairing between X and X'.

Example 4.2.7. Take $X = L^2[0,\infty)$ and let $\mathbb{T}$ be the unilateral right shift semigroup on X (i.e., $\mathbb{T}_t z_0 = \mathbf{S}_t z_0$), with generator

$$A \,=\, -\frac{\rm d}{{\rm d}x}\,, \qquad \mathcal{D}(A) \,=\, \mathcal{H}^1_0(0,\infty)$$

(recall that $\mathcal{H}^1_0(0,\infty)$ consists of those $\varphi \in \mathcal{H}^1(0,\infty)$ for which $\varphi(0) = 0$). Then $\mathcal{D}(A^*) = \mathcal{H}^1(0,\infty)$ and X_{-1} is the dual of $\mathcal{H}^1(0,\infty)$ with respect to the pivot space X (see Section 2.9 for the concept of duality with a pivot). We have encountered this semigroup (and its dual) in Examples 2.4.5, 2.8.7 and 2.10.7.

We take $U = \mathbb{C}$, so that $\mathcal{L}(U, X_{-1})$ can be identified with X_{-1}. For every $\alpha \geqslant 0$ we define δ_α (the "delta function at α") as an element of X_{-1} by

$$\langle \varphi, \delta_\alpha\rangle_{X_1^d, X_{-1}} \,=\, \varphi(\alpha) \qquad \forall\, \varphi \in X_1^d = \mathcal{H}^1(0,\infty).$$

Clearly, $\mathbb{T}_t \delta_\alpha = \delta_{\alpha+t}$. We take the control operator $B = \delta_0$. Then it is not difficult to check that

$$(\Phi_t u)(x) \,=\, \begin{cases} u(t-x) & \text{for } \; x \in [0,t]\,, \\ \;0 & \text{for } \; x \geqslant t\,. \end{cases}$$

Intuitively, we can imagine the system described by $\dot z(t) = Az(t) + Bu(t)$ as an infinite conveyor belt moving to the right, with information entering at its left end and being transported along the belt with unity speed. It is clear that Ran $\Phi_t \subset X$, so that B is admissible. In fact, we have $\|\Phi_t\|_{\mathcal{L}(L^2[0,\infty),X)} = 1$ for all $t > 0$. We shall reformulate this system as a boundary control system in Example 10.1.9.

Let $B \in \mathcal{L}(U, X_{-1})$. We introduce the space

$$Z \,=\, X_1 + (\beta I - A)^{-1} BU \,=\, (\beta I - A)^{-1}(X + BU)\,, \tag{4.2.9}$$

where $\beta \in \rho(A)$ (Z does not depend on the choice of β). The norm on Z is defined by regarding Z as a factor space of $X \times U$:

$$\|z\|_Z^2 \,=\, \inf\left\{\|x\|^2 + \|{\rm v}\|^2 \mid x \in X,\, {\rm v} \in U,\, z = (\beta I - A)^{-1}(x + B{\rm v})\right\}\,,$$

so that Z is a Hilbert space, continuously embedded in X. On the space $X + BU$ we consider a similar norm, but omitting the factor $(\beta I - A)^{-1}$. Clearly Z may be regarded as the image of $X + BU$ through the isomorphism $(\beta I - A)^{-1}$.

Lemma 4.2.8. *Let $B \in \mathcal{L}(U, X_{-1})$ be an admissible control operator for $\mathbb{T}$. Then for any $u \in \mathcal{H}^1((0,T);U)$ with $u(0) = 0$, the solution z of*

$$\dot{z} = Az + Bu \tag{4.2.10}$$

with $z(0) = 0$ is such that

$$z \in C([0,T]; Z) \cap C^1([0,T]; X).$$

Proof. Let $u \in \mathcal{H}^1((0,T);U)$ with $u(0) = 0$ and denote by w the solution of

$$\dot{w} = Aw + B\dot{u}, \qquad w(0) = 0.$$

As B is an admissible control operator we have that $w \in C([0,T]; X)$. Moreover, it is easily checked that the function z defined by $z(t) = \int_0^t w(s)\,\mathrm{d}s$ satisfies (4.2.10). Since the solution of (4.2.10) with $z(0) = 0$ is unique, we obtain

$$z(t) = \int_0^t w(s)\,\mathrm{d}s,$$

which obviously yields that

$$z \in C^1([0,T]; X). \tag{4.2.11}$$

On the other hand, (4.2.10) gives

$$(\beta I - A)z(t) = \beta z(t) - \dot{z}(t) + Bu(t) \qquad \forall\, t \in [0,T]. \tag{4.2.12}$$

Since $\beta z - \dot{z} + Bu \in C([0,T], X + BU)$, relation (4.2.12) with $\beta \in \rho(A)$ implies

$$z \in C([0,T]; Z). \tag{4.2.13}$$

From (4.2.11) and (4.2.13) we clearly obtain the conclusion of the lemma. □

Remark 4.2.9. In Lemma 4.2.8 we may replace the condition that B is admissible with the condition that $u \in \mathcal{H}^2((0,T);U)$ (we still assume that $u(0) = 0$). The conclusion remains the same, and the proof is also the same, except that now, to show that $w \in C([0,T]; X)$, we use Theorem 4.1.6 instead of the admissibility of B.

Proposition 4.2.10. *Let $B \in \mathcal{L}(U, X_{-1})$ be an admissible control operator for $\mathbb{T}$. If $z_0 \in X$ and $u \in \mathcal{H}^1_{\mathrm{loc}}((0,\infty);U)$ are such that $Az_0 + Bu(0) \in X$, then the solution z of* (4.2.6) *satisfies*

$$z \in C([0,\infty); Z) \cap C^1([0,\infty); X).$$

Proof. We decompose $z = z_n + z_c$, where z_n satisfies

$$\dot{z}_n = Az_n + B[u - u(0)], \qquad z_n(0) = 0,$$

and z_c satisfies

$$\dot{z}_c = Az_c + Bu(0), \qquad z_c(0) = z_0.$$

It follows from Lemma 4.2.8 that $z_n \in C([0,\infty); Z) \cap C^1([0,\infty); X)$. It is easy to see (using Remark 2.1.7) that for every $t \geqslant 0$,

$$Az_c(t) = \mathbb{T}_t [Az_0 + Bu(0)] - Bu(0). \tag{4.2.14}$$

This shows that $Az_c \in C([0,\infty); X + BU)$, whence $z_c \in C([0,\infty); Z)$. We also see from (4.2.14) that $\dot{z}_c(t) = \mathbb{T}_t[Az_0 + Bu(0)]$, whence $z_c \in C^1([0,\infty); X)$. □

In Proposition 4.2.10 we may replace the condition that B is admissible with the condition that $u \in \mathcal{H}^2_{\text{loc}}((0,\infty); U)$.

Proposition 4.2.11. *If $B \in \mathcal{L}(U, X_{-1})$, $z_0 \in X$ and $u \in \mathcal{H}^2_{\text{loc}}((0,\infty); U)$ are such that $Az_0 + Bu(0) \in X$, then the solution z of* (4.2.6) *satisfies*

$$z \in C([0,\infty); Z) \cap C^1([0,\infty); X).$$

The proof is very similar to the proof of Proposition 4.2.10, except that now we use Remark 4.2.9 in place of Lemma 4.2.8.

4.3 Admissible observation operators

We now introduce the concept of an admissible observation operator, which will turn out to be the dual of the concept of an admissible control operator.

Let $C \in \mathcal{L}(X_1, Y)$. We are interested in the output functions y generated by the system

$$\begin{cases} \dot{z}(t) &= Az(t), \qquad z(0) = z_0, \\ y(t) &= Cz(t), \end{cases}$$

where $z_0 \in X_1$ and $t \geqslant 0$. According to Proposition 2.3.5, the initial value problem $\dot{z}(t) = Az(t)$, $z(0) = z_0$, has the unique solution $z(t) = \mathbb{T}_t z_0$. This motivates the introduction of the operators from z_0 to the truncated output $\mathbf{P}_\tau y$:

$$(\Psi_\tau z_0)(t) = \begin{cases} C\,\mathbb{T}_t\, z_0 & \text{for} \quad t \in [0,\tau], \\ 0 & \text{for} \quad t > \tau. \end{cases} \tag{4.3.1}$$

We shall regard these operators as elements of $\mathcal{L}(X_1, L^2([0,\infty); Y))$. Clearly, we could just as well define Ψ_τ such that $\Psi_\tau \in \mathcal{L}(X_1, L^2([0,\tau]; Y))$, but we want to avoid later difficulties which would occur if the range space of Ψ_τ depended on τ.

It is easy to see that $\Psi_\tau = \mathbf{P}_\tau \Psi_\tau$ and that for every $t, \tau \geqslant 0$,

$$\Psi_{\tau+t}\, z_0 = \Psi_\tau\, z_0 \underset{\tau}{\Diamond} \Psi_t\, \mathbb{T}_\tau z_0. \tag{4.3.2}$$

We shall call this formula the *dual composition property*. We shall see in the proof of Theorem 4.5.5 that (4.3.2) is indeed the dual counterpart of (4.2.2).

Definition 4.3.1. The operator $C \in \mathcal{L}(X_1, Y)$ is called an *admissible observation operator* for $\mathbb{T}$ if for some $\tau > 0$, Ψ_τ has a continuous extension to X.

Equivalently, $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$ iff, for some $\tau > 0$, there exists a constant $K_\tau \geqslant 0$ such that

$$\int_0^\tau \|C\mathbb{T}_t z_0\|_Y^2 \mathrm{d}t \leqslant K_\tau^2 \|z_0\|_X^2 \qquad \forall\, z_0 \in \mathcal{D}(A). \tag{4.3.3}$$

The operator C (as in the above definition) is called *bounded* if it can be extended such that $C \in \mathcal{L}(X, Y)$ (and unbounded otherwise). Obviously, every bounded C is admissible for $\mathbb{T}$. If Y is finite dimensional and C is closed (as a densely defined operator from X to Y), then it is bounded (this follows from Remark 2.8.3). Usually, C is not closed and it also has no closed extension.

If C is an admissible observation operator for $\mathbb{T}$, then we denote the (unique) extension of Ψ_τ to X by the same symbol. It is now clear that the norm of the extended operator, $\|\Psi_\tau\|$, is the smallest constant K_τ for which (4.3.3) holds. The following result is similar to Proposition 4.2.2.

Proposition 4.3.2. *Suppose that $C \in \mathcal{L}(X_1, Y)$ is admissible; i.e., Ψ_τ has a continuous extension to X for a specific $\tau > 0$. Then for every $t \geqslant 0$ we have*

$$\Psi_t \in \mathcal{L}(X, L^2([0,\infty); Y)).$$

Proof. If $t < \tau$, then from $\Psi_t = \mathbf{P}_t \Psi_\tau$ we see that Ψ_t is bounded, by which we mean that $\Psi_t \in \mathcal{L}(X, L^2([0,\infty); Y))$. If we take $t = \tau$ in (4.3.2), then we obtain that $\Psi_{2\tau}$ is bounded. By induction, we see that Ψ_t is bounded for all t of the form $t = 2^n\tau$, where $n \in \mathbb{N}$. Combining all these facts, we obtain that Ψ_t is bounded for all $t \geqslant 0$, as claimed in the proposition. $\square$

The operators Ψ_t as in the above proposition are called the *output maps* corresponding to (A, C). C can be recovered from them as follows: For any $\tau > 0$,

$$Cz_0 = (\Psi_\tau z_0)(0) \qquad \forall\, z_0 \in X_1.$$

Indeed, this follows from the continuity of $\Psi_\tau z_0$ on the interval $[0, \tau]$, which in turn is due to the strong continuity of $\mathbb{T}$ on X_1 (see Proposition 2.3.5). If we regard $\Psi_\tau z_0$ as an element of $L^2([0,\infty); Y)$, then a point evaluation at zero is not defined, but we can rewrite the formula in a valid form as follows:

$$Cz_0 = \lim_{t \to 0} \frac{1}{t} \int_0^t (\Psi_\tau z_0)(\sigma) \mathrm{d}\sigma \qquad \forall\, z_0 \in X_1. \tag{4.3.4}$$

Note that this is now similar to formula (4.2.3).

We now examine $\|\Psi_t\|$ as a function of t. An obvious observation is that $\|\Psi_t\|$ is non-decreasing. More information is in the following proposition.

Proposition 4.3.3. *With the notation of the previous proposition, let $\omega \in \mathbb{R}$ and $M \geqslant 1$ be such that $\|\mathbb{T}_t\| \leqslant Me^{\omega t}$, for all $t \geqslant 0$ (see Section 2.1).*

(1) *If $\omega > 0$, then there exists $K \geqslant 0$ such that $\|\Psi_t\| \leqslant Ke^{\omega t}$, for all $t \geqslant 0$.*
(2) *If $\omega = 0$, then there exists $K \geqslant 0$ such that $\|\Psi_t\| \leqslant K(1+t)^{\frac{1}{2}}$, for all $t \geqslant 0$.*
(3) *If $\omega < 0$, then there exists $K \geqslant 0$ such that $\|\Psi_t\| \leqslant K$, for all $t \geqslant 0$.*

Proof. It is easy to see that for any $z_0 \in X$ and any $n \in \mathbb{N}$,

$$\|\Psi_n z_0\|^2 = \|\Psi_1 z_0\|^2 + \|\Psi_1 \mathbb{T}_1 z_0\|^2 + \cdots + \|\Psi_1 \mathbb{T}_{n-1} z_0\|^2,$$

whence

$$\|\Psi_n z_0\| \leqslant \|\Psi_1\| \left(1 + M^2 e^{2\omega} + \cdots + M^2 e^{2\omega(n-1)}\right)^{\frac{1}{2}} \|z_0\|. \tag{4.3.5}$$

For $\omega > 0$ it follows that for all $t \in [n-1, n]$,

$$\begin{aligned}\|\Psi_t\| \leqslant \|\Psi_n\| \leqslant \|\Psi_1\| M \left(\frac{e^{2\omega n}-1}{e^{2\omega}-1}\right)^{\frac{1}{2}} &\leqslant \|\Psi_1\| M \frac{e^{\omega}}{\sqrt{e^{2\omega}-1}}\, e^{\omega(n-1)} \\ &\leqslant Ke^{\omega t}, \quad \text{where} \quad K = \|\Psi_1\| M \frac{e^{\omega}}{\sqrt{e^{2\omega}-1}}.\end{aligned}$$

For $\omega = 0$ we see from (4.3.5) that for all $t \in [n-1, n]$,

$$\|\Psi_t\| \leqslant \|\Psi_n\| \leqslant \|\Psi_1\| M n^{\frac{1}{2}} \leqslant K(1+t)^{\frac{1}{2}}.$$

For $\omega < 0$ we see again from (4.3.5) that $\|\Psi_n\|$ is bounded. □

We regard $L^2_{\rm loc}([0,\infty); Y)$ as a Fréchet space with the seminorms being the L^2 norms on the intervals $[0, n]$, $n \in \mathbb{N}$. (This means that in $L^2_{\rm loc}([0,\infty); Y)$ we have $y_k \to 0$ iff $\|y_k\|_{L^2[0,n]} \to 0$ for every $n \in \mathbb{N}$.) Let $C \in \mathcal{L}(X_1, Y)$ be an admissible observation operator for $\mathbb{T}$. Then it is easy to see that there exists a continuous operator $\Psi : X \to L^2_{\rm loc}([0,\infty); Y)$ such that

$$(\Psi z_0)(t) = C\mathbb{T}_t z_0 \qquad \forall\, z_0 \in \mathcal{D}(A),\ t \geqslant 0. \tag{4.3.6}$$

The operator Ψ is completely determined by (4.3.6), because $\mathcal{D}(A)$ is dense in X. We call Ψ the *extended output map* of (A, C). Clearly,

$$\mathbf{P}_\tau \Psi = \Psi_\tau \qquad \forall\, \tau \geqslant 0.$$

It follows from (4.3.2) that the extended output map satisfies the functional equation

$$\Psi z_0 = \Psi_\tau z_0 \underset{\tau}{\diamondsuit} \Psi \mathbb{T}_\tau z_0. \tag{4.3.7}$$

Proposition 4.3.4. *If C and Ψ are as above, then for every $z_0 \in \mathcal{D}(A)$ we have $\Psi z_0 \in \mathcal{H}^1_{\mathrm{loc}}((0,\infty);Y)$ and for every $t \geqslant 0$,*

$$C\mathbb{T}_t z_0 = Cz_0 + \int_0^t (\Psi A z_0)(\sigma)\,\mathrm{d}\sigma .$$

Proof. Take $z_0 \in \mathcal{D}(A^2)$, so that

$$(\Psi A z_0)(t) = C\mathbb{T}_t A z_0 \qquad \forall\, t \geqslant 0. \tag{4.3.8}$$

The derivative of $\mathbb{T}_t z_0$, as an X_1-valued function of t, is $\mathbb{T}_t A z_0$, see Proposition 2.10.4, so that $\int_0^t C\mathbb{T}_\sigma A z_0\,\mathrm{d}\sigma = C\mathbb{T}_t z_0 - Cz_0$. Thus, integrating both sides of (4.3.8), we obtain the desired formula for $z_0 \in \mathcal{D}(A^2)$. Since $\mathcal{D}(A^2)$ is dense in X_1 and C is bounded on X_1, the formula in the proposition must be true for all $z_0 \in X_1$. □

Remark 4.3.5. Assume that $\mathbb{T}$ is exponentially stable and $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$. Denote by Ψ the extended output map of (A, C). Then

$$\Psi \in \mathcal{L}(X, L^2([0,\infty);Y)).$$

Indeed, it follows from part (3) of Proposition 4.3.3 that there exists $K \geqslant 0$ such that $\|\Psi_t\| \leqslant K$ for all $t \geqslant 0$. Take $z_0 \in X$. By taking the limit of $\|\Psi_t z_0\|_{L^2}$ (as $t \to \infty$), we obtain that $\Psi z_0 \in L^2([0,\infty);Y)$ and $\|\Psi z_0\|_{L^2} \leqslant K\|z_0\|$.

Proposition 4.3.6. *Let $C \in \mathcal{L}(X_1, Y)$ be an admissible observation operator for $\mathbb{T}$ and let Ψ be the extended output map of (A, C). For each $\alpha \in \mathbb{R}$ we define $\Psi^\alpha : X \to L^2_{\mathrm{loc}}([0,\infty);Y)$ by*

$$(\Psi^\alpha z_0)(t) = e^{-\alpha t}(\Psi z_0)(t).$$

Then for every $\alpha > \omega_0(\mathbb{T})$ we have

$$\Psi^\alpha \in \mathcal{L}(X, L^2([0,\infty);Y)).$$

Proof. Let $\alpha > \omega_0(\mathbb{T})$. Introduce the operator semigroup $\mathbb{T}^\alpha$ generated by $A - \alpha I$. Its growth bound is $\omega_0(\mathbb{T}^\alpha) = \omega_0(\mathbb{T}) - \alpha < 0$. Hence, there exist $\omega < 0$ and $M \geqslant 0$ such that

$$\|\mathbb{T}^\alpha_t\| \leqslant M e^{\omega t} \qquad \forall\, t \geqslant 0.$$

Clearly C is admissible for $\mathbb{T}^\alpha$. The extended output map of $(A - \alpha I, C)$ is exactly Ψ^α. According to Remark 4.3.5, we have $\Psi^\alpha \in \mathcal{L}(X, L^2([0,\infty);Y))$. □

Theorem 4.3.7. *Let $C \in \mathcal{L}(X_1, Y)$ be an admissible observation operator for $\mathbb{T}$ and let Ψ be the extended output map of (A, C). Then for every $z_0 \in X$ and every $s \in \mathbb{C}$ with $\operatorname{Re} s > \omega_0(\mathbb{T})$, the function $t \mapsto e^{-st}(\Psi z_0)(t)$ is in $L^1([0,\infty);Y)$, so that the Laplace transform of Ψz_0 exists at s. This Laplace transform is given by*

$$\widehat{(\Psi z_0)}(s) = C(sI - A)^{-1} z_0 .$$

Moreover, for every $\alpha > \omega_0(\mathbb{T})$ there exists $K_\alpha \geqslant 0$ such that

$$\|C(sI - A)^{-1}\| \leqslant \frac{K_\alpha}{\sqrt{\operatorname{Re} s - \alpha}} \qquad \forall\, s \in \mathbb{C}_\alpha . \tag{4.3.9}$$

Proof. For $s \in \mathbb{C}$ with $\operatorname{Re} s > \omega_0(\mathbb{T})$, choose $\alpha \in (\omega_0(\mathbb{T}), \operatorname{Re} s)$ and denote $\varepsilon = \operatorname{Re} s - \alpha$ (so that $\varepsilon > 0$). According to Proposition 4.3.6, we have $\Psi^\alpha \in \mathcal{L}(X, L^2([0,\infty); Y))$. Using the Cauchy–Schwarz inequality we have

$$\begin{aligned}\|\widehat{(\Psi z_0)}(s)\| &\leqslant \int_0^\infty |e^{-st}| \cdot \|(\Psi z_0)(t)\| \,\mathrm{d}t = \int_0^\infty |e^{-\varepsilon t}| \cdot \|e^{-\alpha t}(\Psi z_0)(t)\| \,\mathrm{d}t \\ &\leqslant \|e^{-\varepsilon \cdot}\|_{L^2} \cdot \|\Psi^\alpha z_0\|_{L^2} \leqslant \frac{K_\alpha}{\sqrt{\varepsilon}} \|z_0\| . \end{aligned} \tag{4.3.10}$$

This implies that for any fixed s with $\operatorname{Re} s > \omega_0(\mathbb{T})$, $\widehat{(\Psi z_0)}(s)$ defines a bounded linear operator from X to Y.

For every $z_0 \in \mathcal{D}(A)$, $t \mapsto \mathbb{T}_t z_0$ is a continuous X_1-valued function. Since $C \in \mathcal{L}(X_1, Y)$, we obtain from Proposition 2.3.1 that

$$\widehat{(\Psi z_0)}(s) = \int_0^\infty e^{-st} C \mathbb{T}_t z_0 \,\mathrm{d}t = C(sI - A)^{-1} z_0 .$$

The left- and right-hand sides above have continuous extensions to X, so that their equality remains valid for all $z_0 \in X$, as claimed. Combining this fact with (4.3.10), we get the estimate in the theorem. □

The last theorem gives an upper bound for $\|(C(sI - A)^{-1}\|$ for large values of $\operatorname{Re} s$, but it gives no information at all about the size of $\|(C(sI - A)^{-1}\|$ for $\operatorname{Re} s$ close to $\omega_0(A)$. However, such information is sometimes needed. The following simple proposition provides such an upper bound for contraction semigroups. The estimate given is valid for all $s \in \mathbb{C}_0$, but what is important here is the region of small positive $\operatorname{Re} s$. For large $\operatorname{Re} s$, Theorem 4.3.7 gives a stronger estimate.

Proposition 4.3.8. *Assume that $\mathbb{T}$ is a contraction semigroup and $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$. Then there exists $K \geqslant 0$ such that*

$$\|C(sI - A)^{-1}\| \leqslant K \left(1 + \frac{1}{\operatorname{Re} s}\right) \qquad \forall\, s \in \mathbb{C}_0 .$$

Proof. Clearly $\omega_0(A) \leqslant 0$. Take $s = \lambda + i\omega \in \mathbb{C}_0$, so that $\lambda > 0$. Denote $s_1 = 1 + i\omega$, then according to the resolvent identity (see Remark 2.2.5) we have

$$C(sI - A)^{-1} = C(s_1 I - A)^{-1} \left[I + (1 - \lambda)(sI - A)^{-1} \right] .$$

According to Theorem 4.3.7 with $\alpha = \frac{1}{2}$, there exists $k = \sqrt{2} \cdot K_{\frac{1}{2}}$ such that for all s_1 as above, $\|C(s_1 I - A)^{-1}\| \leqslant k$ (k is independent of ω). Thus,

$$\|C(sI - A)^{-1}\| \leqslant k \left[1 + |1 - \lambda| \cdot \|(sI - A)^{-1}\| \right] \qquad \forall\, s \in \mathbb{C}_0 .$$

We know from Proposition 3.1.13 that A is m-dissipative. This implies, according to Proposition 3.1.9, that we have $\|(sI - A)^{-1}\| \leqslant 1/\mathrm{Re}\, s$, for all $s \in \mathbb{C}_0$. Substituting this into the previous estimate, we obtain

$$\|C(sI - A)^{-1}\| \leqslant k \left(1 + \frac{|1-\lambda|}{\lambda}\right) \qquad \forall\, s \in \mathbb{C}_0,\ \lambda = \mathrm{Re}\, s\,.$$

From here it is easy to obtain the estimate in the proposition, with $K = 2k$. $\square$

4.4 The duality between the admissibility concepts

In this section we show that the concept of admissible observation operator is dual to the concept of admissible control operator. This duality allows us to translate many statements into dual statements which might be easier to prove or understand.

If $B \in \mathcal{L}(U, X_{-1})$, then, using the duality between X_1^d and X_{-1} (see Section 2.10) and identifying U with its dual, we have $B^* \in \mathcal{L}(X_1^d, U)$. The adjoint of Φ_τ from (4.2.1), which is in $\mathcal{L}(X_1^d, L^2([0,\infty); U))$, can be expressed using B^* as follows.

Proposition 4.4.1. *If $B \in \mathcal{L}(U, X_{-1})$, then for every $\tau > 0$ and every $z_0 \in X_1^d$,*

$$(\Phi_\tau^* z_0)(t) \;=\; \begin{cases} B^* \mathbb{T}_{\tau-t}^* z_0 & \text{for} \;\; t \in [0,\tau]\,, \\ 0 & \text{for} \;\; t > \tau\,. \end{cases} \tag{4.4.1}$$

If B is an admissible control operator for $\mathbb{T}$, so that Φ_τ can also be regarded as an operator in $\mathcal{L}(L^2([0,\infty); U), X)$, then its adjoint in $\mathcal{L}(X, L^2([0,\infty); U))$ is given, for $z_0 \in \mathcal{D}(A^*)$, by the same formula (4.4.1).

Proof. For every $z_0 \in X_1^d$ and $u \in L^2([0,\infty); U)$ we have

$$\begin{aligned} \langle \Phi_\tau u, z_0 \rangle_{X_{-1}, X_1^d} &= \int_0^\tau \langle \mathbb{T}_{\tau-\sigma} B u(\sigma), z_0 \rangle_{X_{-1}, X_1^d} \,\mathrm{d}\sigma \\ &= \int_0^\tau \left\langle u(\sigma), B^* \mathbb{T}_{\tau-\sigma}^* z_0 \right\rangle_U \mathrm{d}\sigma \;=\; \langle u, v \rangle_{L^2([0,\infty); U)}\,, \end{aligned}$$

where v is the function on the right-hand side of (4.4.1). This implies (4.4.1).

Now assume that B is admissible and regard Φ_τ as a bounded operator from $L^2([0,\infty); U)$ to X. Then, because of the equality

$$\langle \Phi_\tau u, z_0 \rangle_X \;=\; \langle \Phi_\tau u, z_0 \rangle_{X_{-1}, X_1^d}\,,$$

formula (4.4.1) gives the restriction of $\Phi_\tau^* z_0$ to $\mathcal{D}(A^*)$. $\square$

Remark 4.4.2. Let us denote by Ψ_τ^d the output maps corresponding to the semigroup $\mathbb{T}^*$ with the observation operator B^* (defined similarly as for $\mathbb{T}$ and C, see Section 4.3). Recall the time-reflection operators $\boldsymbol{\mathfrak{R}}_\tau$ introduced in Section 1.4. Then Proposition 4.4.1 shows that (without assuming admissibility)

$$\Phi_\tau^* z_0 = \boldsymbol{\mathfrak{R}}_\tau \Psi_\tau^d z_0 \qquad \forall\, z_0 \in \mathcal{D}(A^*),\ \tau \geqslant 0, \tag{4.4.2}$$

as in Proposition 1.4.3. If B is admissible, then we have a choice between regarding Φ_τ as an element of $\mathcal{L}(L^2([0,\infty);U),X)$ or of $\mathcal{L}(L^2([0,\infty);U),X_{-1})$. Proposition 4.4.1 tells us that regardless of the choice, the above formula holds.

Theorem 4.4.3. *Suppose that $B \in \mathcal{L}(U, X_{-1})$. Then B is an admissible control operator for $\mathbb{T}$ if and only if B^* is an admissible observation operator for $\mathbb{T}^*$. If B is admissible, then*

$$\|\Phi_\tau^* z_0\| = \|\Psi_\tau^d z_0\| \qquad \forall\, z_0 \in X,\ \tau \geqslant 0,$$

where Ψ_τ^d (with $\tau \geqslant 0$) are the output maps of $\mathbb{T}^*$ and B^*.

Proof. Suppose that B is an admissible control operator for $\mathbb{T}$ and for some $\tau > 0$, let $\Phi_\tau \in \mathcal{L}(L^2([0,\infty);U),X)$ be the operator from (4.2.1). Clearly $\Phi_\tau^* \in \mathcal{L}(X, L^2([0,\infty);U))$. Since (4.4.2) holds for all $z_0 \in \mathcal{D}(A^*)$, $\mathbb{T}^*$ and B^* satisfy condition (4.3.3) with $K_\tau = \|\Phi_\tau\|$. It follows that B^* is an admissible observation operator for $\mathbb{T}^*$, i.e., $\Psi_\tau^d \in \mathcal{L}(X, L^2([0,\infty);U))$. The equality of norms claimed in the theorem follows easily from (4.4.2), since $\boldsymbol{\mathfrak{R}}_\tau$ has no influence on the norm.

To prove the converse implication, assume that B^* is admissible for $\mathbb{T}^*$. Then for every $\tau > 0$ there exists $K_\tau \geqslant 0$ such that for all $z_0 \in \mathcal{D}(A^*)$, $\|\Psi_\tau^d z_0\| \leqslant K_\tau \|z_0\|$. Take a *step function* $v : [0,\tau] \to U$, then according to (4.4.2),

$$\langle \Phi_\tau v, z_0 \rangle_{X_{-1}, X_1^d} = \langle v, \boldsymbol{\mathfrak{R}}_\tau \Psi_\tau^d z_0 \rangle_{L^2}.$$

Since for any step function v we have $\Phi_\tau v \in X$ (see Remark 4.2.3), we obtain

$$|\langle \Phi_\tau v, z_0 \rangle_X| \leqslant \|v\|_{L^2} \cdot K_\tau \cdot \|z_0\|_X \qquad \forall\, z_0 \in \mathcal{D}(A^*).$$

This implies that $\|\Phi_\tau v\|_X \leqslant K_\tau \|v\|_{L^2}$ holds for every step function $v : [0,\tau] \to U$. Thus, the admissibility criterion in Remark 4.2.3 is satisfied. □

Example 4.4.4. We describe a system that is dual to the one discussed in Example 4.2.7. Take $X = L^2[0,\infty)$ and let $\mathbb{T}$ be the unilateral left shift semigroup on X (i.e., $\mathbb{T}_t z_0 = \mathbf{S}_t^* z_0$), with generator

$$A = \frac{\mathrm{d}}{\mathrm{d}x}, \qquad \mathcal{D}(A) = \mathcal{H}^1(0,\infty).$$

Then the adjoint semigroup is the unilateral right shift semigroup, so that $\mathcal{D}(A^*) = \mathcal{H}_0^1(0,\infty)$. Thus, X_{-1} is the dual of $\mathcal{H}_0^1(0,\infty)$ with respect to the pivot space

X, which is denoted $\mathcal{H}^{-1}(0,\infty)$. (We have met this semigroup (and its dual) in Examples 2.3.7, 2.4.5 and 2.8.7.) We take $Y = \mathbb{C}$ and define $C \in \mathcal{L}(X_1, Y)$ by $C\varphi = \varphi(0)$. (With the notation of Example 4.2.7, we have $C = \delta_0$.) Then it is not difficult to check that for every $z \in \mathcal{D}(A)$,

$$(\Psi z)(t) = z(t).$$

By continuous extension, this formula remains valid for every $z \in L^2[0,\infty)$.

Intuitively, we can imagine that the information is being transported to the left on an infinite conveyor belt, and the information that reaches the left end of the belt becomes the output. It is clear that $\Psi = I$ is bounded from X to $L^2[0,\infty)$, so that C is admissible. The operators A and C defined in this example are the adjoints of A and B defined in Example 4.2.7.

The duality theorem (Theorem 4.4.3) permits us to translate results about the admissible control operators into results about admissible observation operators, or the other way round. For example, we have the following from Proposition 4.3.3.

Proposition 4.4.5. *Let $\omega \in \mathbb{R}$ and $M \geqslant 1$ be such that $\|\mathbb{T}_t\| \leqslant Me^{\omega t}$, for all $t \geqslant 0$. Let $B \in \mathcal{L}(U, X_{-1})$ be an admissible control operator for $\mathbb{T}$.*

(1) *If $\omega > 0$, then there exists $K \geqslant 0$ such that $\|\Phi_t\| \leqslant Ke^{\omega t}$, for all $t \geqslant 0$.*
(2) *If $\omega = 0$, then there exists $K \geqslant 0$ such that $\|\Phi_t\| \leqslant K(1+t)^{\frac{1}{2}}$, for all $t \geqslant 0$.*
(3) *If $\omega < 0$, then there exists $K \geqslant 0$ such that $\|\Phi_t\| \leqslant K$, for all $t \geqslant 0$.*

From Theorem 4.3.7 we obtain by duality the following proposition (note that it is not exactly a mirror image of Theorem 4.3.7, because certain parts of this theorem are difficult to translate into the control context).

Proposition 4.4.6. *Let $B \in \mathcal{L}(U, X_{-1})$ be an admissible control operator for $\mathbb{T}$. Then for every $\alpha > \omega_0(\mathbb{T})$ there exists $K_\alpha \geqslant 0$ such that*

$$\|(sI - A)^{-1}B\| \leqslant \frac{K_\alpha}{\sqrt{\operatorname{Re} s - \alpha}} \qquad \forall\, s \in \mathbb{C}_\alpha .$$

4.5 Two representation theorems

When we introduced the concept of an admissible observation operator in Section 4.3, we have assumed that $C \in \mathcal{L}(X_1, Y)$. It is legitimate to ask if this is not introducing an artificial constraint into our theory. Maybe for some semigroup $\mathbb{T}$ we could find a dense $\mathbb{T}$-invariant subspace $W \subset X$, other than $\mathcal{D}(A)$, and an operator $\tilde{C} : W \to Y$ which is admissible in a similar (but clearly more general) sense; i.e., for some $\tau > 0$ there exists $K_\tau > 0$ such that

$$\int_0^\tau \|\tilde{C}\mathbb{T}_t z_0\|^2 \mathrm{d}t \leqslant K_\tau^2 \|z_0\|^2 \qquad \forall\, z_0 \in W .$$

In this case $\tilde{C}$ is meaningful as an observation operator for $\mathbb{T}$, but it does not fit into the framework developed in Section 4.3. Such an observation operator would give rise, using the obvious generalization of Proposition 4.3.2, to a family of bounded operators $\Psi_\tau \in \mathcal{L}(X, L^2([0,\infty);Y))$ (the output maps corresponding to $(A,\tilde{C})$) which would again satisfy the dual composition property (4.3.2).

The answer is that it is indeed easy to find observation operators defined on spaces other than $\mathcal{D}(A)$, and which are admissible in this more general sense; see Example 4.5.4 below. However, such observation operators will not lead to any new family of output maps. Indeed, any family of output maps $\Psi_\tau \in \mathcal{L}(X, L^2([0,\infty);Y))$ satisfying (4.3.2) is generated by a unique admissible observation operator $C \in \mathcal{L}(X_1, Y)$. Thus, we may start with $\tilde{C} : W \to Y$, but if this $\tilde{C}$ is admissible for $\mathbb{T}$, then we can find an equivalent $C \in \mathcal{L}(X_1, Y)$. (Here, C being equivalent to $\tilde{C}$ means that they give rise to the same output maps.) This is a consequence of the first representation theorem in this section, Theorem 4.5.3 below.

Lemma 4.5.1. *Suppose that $(\Psi_\tau)_{\tau\geqslant 0}$ is a family of operators in $\mathcal{L}(X, L^2([0,\infty);Y))$ that satisfies the dual composition property* (4.3.2) *and $\Psi_0 = 0$.*

Then for every $\tau, T \geqslant 0$ with $\tau \leqslant T$ we have $\mathbf{P}_\tau \Psi_T = \Psi_\tau$. Moreover, for every $\omega > 0$ with $\omega > \omega_0(\mathbb{T})$ there exists $K \geqslant 0$ such that

$$\|\Psi_t\| \leqslant K e^{\omega t} \qquad \forall\, t \geqslant 0. \tag{4.5.1}$$

Proof. Taking $t = 0$ in (4.3.2) we see that $\mathbf{P}_\tau \Psi_\tau = \Psi_\tau$. Using this and again (4.3.2) with $T = \tau + t$, we obtain that indeed $\mathbf{P}_\tau \Psi_T = \Psi_\tau$.

Now notice that Proposition 4.3.3 remains true for the family $(\Psi_\tau)_{\tau\geqslant 0}$, with the same proof. Hence, for ω as described, we can find K such that (4.5.1) holds. □

Remark 4.5.2. The above lemma implies that there exists a unique operator $\Psi \in \mathcal{L}(X, L^2_{\mathrm{loc}}([0,\infty);Y))$ such that

$$\Psi_\tau = \mathbf{P}_\tau \Psi \qquad \forall\, \tau \geqslant 0. \tag{4.5.2}$$

Indeed, we may define Ψ using limits in the Fréchet space $L^2_{\mathrm{loc}}([0,\infty);Y)$:

$$\Psi z_0 = \lim_{\tau\to\infty} \Psi_\tau z_0 \qquad \forall\, z_0 \in X.$$

This operator Ψ is like the extended output map introduced in the previous section, but here we do not know (yet) that Ψ is determined by an operator C as in (4.3.6). Moreover, it follows from (4.3.2) and (4.5.2) that Ψ satisfies (4.3.7).

Theorem 4.5.3. *Suppose that $(\Psi_\tau)_{\tau\geqslant 0}$ is a family of bounded operators from X to $L^2([0,\infty);Y)$ that satisfies* (4.3.2) *and $\Psi_0 = 0$.*

Then there is a unique admissible $C \in \mathcal{L}(X_1, Y)$ such that for every $\tau \geqslant 0$,

$$(\Psi_\tau z_0)(t) = C\,\mathbb{T}_t z_0 \qquad \forall\, z_0 \in \mathcal{D}(A),\ t \in [0,\tau]. \tag{4.5.3}$$

Proof. We define the extended output map Ψ as in Remark 4.5.2. Let, for any $s \in \mathbb{C}$ with $\operatorname{Re} s > \omega$, the operator $\Lambda_s : X \to Y$ be defined by the Laplace integral

$$\Lambda_s z = \int_0^\infty e^{-st}(\Psi z)(t)\,\mathrm{d}t \qquad \forall\, z \in X\,.$$

We have to check that this definition is correct, i.e., the above integral converges absolutely. We have, using (4.5.1) and (4.5.2) and denoting $\lambda = \operatorname{Re} s$,

$$\begin{aligned}\int_0^\infty \|e^{-st}(\Psi z)(t)\|\,\mathrm{d}t &= \sum_{n=1}^\infty \int_{n-1}^n e^{-\lambda t}\|(\Psi z)(t)\|\,\mathrm{d}t \\ &\leqslant e^\lambda \sum_{n=1}^\infty e^{-\lambda n}\|\Psi_n z\| \\ &\leqslant K e^\lambda \sum_{n=1}^\infty e^{-(\lambda-\omega)n}\|z\|\,.\end{aligned}$$

(We have used above that on $[n-1,n]$, the L^1-norm is smaller than or equal the L^2-norm.) Thus we have got that for $\operatorname{Re} s > \omega$, Λ_s is well defined and, moreover, $\Lambda_s \in \mathcal{L}(X,Y)$.

The functional equation (4.3.7) implies that for every $z \in X$ and every $\tau > 0$,

$$\begin{aligned}\Lambda_s z &= \int_0^\tau e^{-st}(\Psi z)(t)\,\mathrm{d}t + \int_\tau^\infty e^{-st}(\Psi\,\mathbb{T}_\tau z)(t-\tau)\,\mathrm{d}t \\ &= \int_0^\tau e^{-st}(\Psi z)(t)\,\mathrm{d}t + e^{-s\tau}\,\Lambda_s\,\mathbb{T}_\tau\,z\,.\end{aligned}$$

Rearranging, we have

$$\frac{1}{\tau}\int_0^\tau e^{-st}(\Psi z)(t)\,\mathrm{d}t \;=\; \frac{1-e^{-s\tau}}{\tau}\,\Lambda_s z - e^{-s\tau}\Lambda_s\,\frac{\mathbb{T}_\tau\,z - z}{\tau}\,. \tag{4.5.4}$$

For $x \in D(A)$ the right-hand side of (4.5.4) converges as $\tau \to 0$, so the left-hand side has to converge too. Moreover, the limit does not depend on s, because of the simple fact that (Ψz being in $L^1_{\mathrm{loc}}([0,\infty);Y)$)

$$\lim_{\tau \to 0}\left[\frac{1}{\tau}\int_0^\tau e^{-st}(\Psi z)(t)\,\mathrm{d}t - \frac{1}{\tau}\int_0^\tau(\Psi z)(t)\,\mathrm{d}t\right] \;=\; 0\,. \tag{4.5.5}$$

Let us denote, for every $z \in \mathcal{D}(A)$,

$$Cz \;=\; \lim_{\tau \to 0}\frac{1}{\tau}\int_0^\tau(\Psi z)(t)\,\mathrm{d}t\,.$$

Then (4.5.4) and (4.5.5) imply that for every $z \in \mathcal{D}(A)$,

$$Cz \;=\; s\Lambda_s z - \Lambda_s A z\,, \tag{4.5.6}$$

and since $A \in \mathcal{L}(X_1, X)$, we get that $C \in \mathcal{L}(X_1, Y)$. Denoting $w = (sI - A)z$, (4.5.6) can be written in the form

$$\Lambda_s w = C(sI - A)^{-1} w, \tag{4.5.7}$$

which holds for every $w \in X$, because $sI - A$ maps $\mathcal{D}(A)$ onto X.

Let $\mathcal{Y}$ be the space of those strongly measurable functions $y : [0, \infty) \to Y$ whose Laplace integral is absolutely convergent for $\operatorname{Re} s > \omega$ (we identify functions which are equal almost everywhere). We have seen at the beginning of the proof that $\Psi w \in \mathcal{Y}$ for any $w \in X$. On the other hand, for $z \in \mathcal{D}(A)$, the function $\eta_z : [0, \infty) \to Y$ defined by $\eta_z(t) = C\mathbb{T}_t z$ belongs to $\mathcal{Y}$. This follows from the fact that $\mathbb{T}$ is a strongly continuous semigroup on X_1, having the same growth bound as on X, and $C \in \mathcal{L}(X_1, Y)$. Since the Laplace transformation is one-to-one on $\mathcal{Y}$ (see the comments on the generalization of Proposition 12.4.5 in Section 12.5), it follows from (4.5.7) that $\Psi z = \eta_z$, i.e., (4.5.3) holds. The uniqueness of C is obvious. □

Remember that at the beginning of this section we have introduced a more general concept of an admissible observation operator for $\mathbb{T}$ (defined on a dense $\mathbb{T}$-invariant subspace of X). We have called two such observation operators equivalent if they give rise to the same output maps. The following example shows that it may happen that observation operators having domains whose intersection is zero are equivalent. The same example shows that even if two equivalent observation operators have the same domain, they do not have to coincide on it.

Example 4.5.4. Let X be the closed subspace of $L^2[0, 2\pi]$ defined by

$$X = \left\{ z \in L^2[0, 2\pi] \;\middle|\; \int_0^{2\pi} z(x)\,\mathrm{d}x = 0 \right\}.$$

Let $\mathbb{T}$ be the periodic left shift group on X (this is similar to the operator group discussed in Example 2.7.12); i.e.,

$$(\mathbb{T}_t z)(x) = z(x + t - k \cdot 2\pi) \quad \text{for} \quad k \cdot 2\pi \leqslant x + t < (k+1) \cdot 2\pi .$$

The space $\mathcal{D}(A)$ consists of all $z \in \mathcal{H}^1(0, 2\pi)$ such that

$$\int_0^{2\pi} z(x)\mathrm{d}x = 0, \qquad z(0) = z(2\pi).$$

By a step function on $[0, 2\pi]$ we mean a function constant on each of a finite set of non-overlapping intervals covering $[0, 2\pi]$. Let W_1 be the vector space of step functions contained in X, let $W_2 = W_1$ and let W_3 be the vector space of trigonometric polynomials contained in X (i.e., any function in W_3 is a finite linear combination of the functions $\sin nx$ and $\cos nx$, where $n \in \mathbb{N}$). For $i \in \{1, 2, 3\}$, let $C_i : W_i \to \mathbb{C}$ be defined by

$$C_1 z = z(0)$$

(i.e., the value of z on the first interval of constancy),

$$C_2 z = z(2\pi)$$

(i.e., the value of z on the last interval of constancy) and

$$C_3 z = z(0).$$

Then for C_1, C_2 and C_3 are admissible and equivalent, despite the facts that C_1 and C_2 do not coincide on their (common) domain and $W_1 \cap W_3 = \{0\}$. The unique admissible observation operator $C \in \mathcal{L}(X_1, \mathbb{C})$ that is equivalent to C_1, C_2 and C_3 is given by $Cz = z(0) = z(2\pi)$.

Let us now state the dual version of the problem discussed at the beginning of this section. When we introduced the concept of an admissible control operator in Section 4.2, we have assumed that $B \in \mathcal{L}(U, X_{-1})$. It is legitimate to ask if this is not overly restrictive. Maybe for some semigroup $\mathbb{T}$ we could find a Hilbert space V other than X_{-1}, such that X is a dense subspace of V, $\mathbb{T}$ has a continuous extension to an operator semigroup acting on V, and an operator $\tilde{B} : U \to V$ which is admissible in the sense that for some $\tau > 0$,

$$\int_0^\tau \mathbb{T}_{t-\sigma} \tilde{B} u(\sigma) \mathrm{d}\sigma \in X \qquad \forall\, u \in L^2([0,\tau]; U).$$

In this case $\tilde{B}$ is meaningful as a control operator for $\mathbb{T}$, but it does not fit into the framework developed in Section 4.2. Such a control operator would give rise, using the obvious generalization of Proposition 4.2.2, to a family of bounded operators $\Phi_\tau \in \mathcal{L}(L^2([0,\infty); U), X)$ (the input maps corresponding to $(A, \tilde{B})$) which would again satisfy the composition property (4.2.2).

The answer is of course similar to the one in the case of admissible observation operators. It is indeed easy to find control operators whose range is in spaces other than X_{-1}, and which are admissible in this more general sense. However, such control operators will not lead to any new family of input maps. Indeed, any family of input maps $\Phi_\tau \in \mathcal{L}(L^2([0,\infty); U), X)$ satisfying (4.2.2) is generated by a unique admissible control operator $B \in \mathcal{L}(U, X_{-1})$. This is a consequence of our second representation theorem given below.

Theorem 4.5.5. *Suppose that $(\Phi_\tau)_{\tau \geqslant 0}$ is a family of bounded operators from $L^2([0,\infty); U)$ to X that satisfies the composition property* (4.2.2).

Then there is a unique admissible $B \in \mathcal{L}(U, X_{-1})$ such that for every $\tau \geqslant 0$,

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{t-\sigma} B u(\sigma)\, \mathrm{d}\sigma \qquad \forall\, u \in L^2([0,\infty); U). \tag{4.5.8}$$

Proof. Taking $t = \tau = 0$ in (4.2.2) we see that $\Phi_0 = 0$. Recall the time-reflection operators $\boldsymbol{Я}_\tau$ introduced in Section 1.4 and denote, for every $\tau \geqslant 0$,

$$\Psi_\tau = \boldsymbol{Я}_\tau \Phi_\tau^*. \tag{4.5.9}$$

Let us rewrite (4.2.2) (for $t, \tau \geqslant 0$ fixed) in the form

$$\Phi_{t+\tau}\,[\mathbf{P}_\tau\ \mathbf{S}_\tau]\begin{bmatrix}u\\ v\end{bmatrix} \;=\; [\mathbb{T}_t\Phi_\tau\ \Phi_t]\begin{bmatrix}u\\ v\end{bmatrix} \qquad \forall\, u, v \in L^2([0,\infty); U)\,.$$

Eliminating u, v and taking adjoints, we obtain that

$$\begin{bmatrix}\mathbf{P}_\tau\\ \mathbf{S}_\tau^*\end{bmatrix}\Phi_{t+\tau}^* \;=\; \begin{bmatrix}\Phi_\tau^*\mathbb{T}_t^*\\ \Phi_t^*\end{bmatrix}.$$

Multiplying both sides with $\begin{bmatrix}\mathbf{P}_\tau & \mathbf{S}_\tau\end{bmatrix}$, we obtain

$$\Phi_{t+\tau}^* \;=\; \mathbf{P}_\tau\Phi_\tau^*\mathbb{T}_t^* + \mathbf{S}_\tau\Phi_t^*\,,$$

whence

$$\Phi_{t+\tau}^* z_0 \;=\; \Phi_\tau^*\mathbb{T}_t^* z_0 \underset{\tau}{\diamondsuit} \Phi_t^* z_0 \qquad \forall\, z_0 \in X\,.$$

Applying $\boldsymbol{\mathcal{R}}_{t+\tau}$ to both sides and using the elementary identity

$$\boldsymbol{\mathcal{R}}_{t+\tau}(u \underset{\tau}{\diamondsuit} v) = \boldsymbol{\mathcal{R}}_t v \underset{t}{\diamondsuit} \boldsymbol{\mathcal{R}}_\tau u\,,$$

we obtain

$$\boldsymbol{\mathcal{R}}_{t+\tau}\Phi_{t+\tau}^* z_0 \;=\; \boldsymbol{\mathcal{R}}_t\Phi_t^* z_0 \underset{t}{\diamondsuit} \boldsymbol{\mathcal{R}}_\tau\Phi_\tau^*\mathbb{T}_t^* z_0 \qquad \forall\, z_0 \in X\,,$$

which is the same as $\Psi_{t+\tau} z_0 \;=\; \Psi_t z_0 \underset{t}{\diamondsuit} \Psi_\tau\mathbb{T}_t^* z_0$, for all $z_0 \in X$. This is the dual composition property (4.3.2), with $\mathbb{T}^*$ in place of $\mathbb{T}$ and with the roles of τ and t reversed. We denote, as usual, $X_1^d = \mathcal{D}(A^*)$, with the graph norm. It follows from Theorem 4.5.3 that there exists a unique $C \in \mathcal{L}(X_1^d, U)$ such that

$$(\Psi_\tau z_0)(t) \;=\; \begin{cases} C\,\mathbb{T}_t^* z_0 & \text{for } \ t \in [0,\tau] \\ 0 & \text{for } \ t > \tau \end{cases} \qquad \forall\, z_0 \in \mathcal{D}(A^*)\,.$$

Define $B \in \mathcal{L}(U, X_{-1})$ by $B = C^*$. Then from $\Phi_\tau^* = \boldsymbol{\mathcal{R}}_\tau\Psi_\tau$ (a consequence of (4.5.9)) we obtain that Φ_τ^* is given by (4.4.1). According to Proposition 4.4.1, Φ_τ^* is the same as the adjoint of Φ_τ as defined by (4.5.8). Hence, Φ_τ is given by (4.5.8). $\square$

4.6 Infinite-time admissibility

Assume that B is an admissible control operator for $\mathbb{T}$. Remember from (4.2.5) that $\|\Phi_\tau\|$ is a non-decreasing function of τ. It is worthwhile to examine when this function remains bounded. In the latter case, B is called infinite-time admissible.

Definition 4.6.1. An operator $B \in \mathcal{L}(U, X_{-1})$ is called an *infinite-time admissible control operator for* $\mathbb{T}$ if there is a $K \geqslant 0$ such that

$$\|\Phi_\tau\|_{\mathcal{L}(L^2([0,\infty);U),X)} \leqslant K \qquad \forall\, \tau \geqslant 0\,. \tag{4.6.1}$$

Obviously, every infinite-time admissible control operator for $\mathbb{T}$ is an admissible control operator for $\mathbb{T}$. It follows from part (3) of Proposition 4.4.5 that if $\mathbb{T}$ is exponentially stable and B is an admissible control operator for $\mathbb{T}$, then B is infinite-time admissible. The control operator from Example 4.2.7 is infinite-time admissible, but the semigroup in this example is not exponentially stable (it is isometric).

If B is infinite-time admissible, then we define the bounded operator Φ_∞^- from
$L^2((-\infty,0];U)$ to X by

$$\Phi_\infty^- u = \lim_{T\to\infty} \int_{-T}^0 \mathbb{T}_{-\sigma} B u(\sigma)\,\mathrm{d}\sigma\,. \tag{4.6.2}$$

The above limit exists, because for $0 < \tau < T$,

$$\left\| \int_{-T}^{-\tau} \mathbb{T}_{-\sigma} B u(\sigma)\,\mathrm{d}\sigma \right\|^2 \leqslant K^2 \int_{-T}^{-\tau} \|u(\sigma)\|^2\,\mathrm{d}\sigma\,,$$

where K is as in (4.6.1). The intuitive interpretation of Φ_∞^- is that it gives the state $z(0)$ if the state trajectory z (defined for $t \leqslant 0$) satisfies $\dot z(t) = Az(t) + Bu(t)$ (in X_{-1}), $u(t)$ is the input signal for $t \leqslant 0$ and if $\lim_{t\to-\infty} z(t) = 0$. Thus, Φ_∞^- allows us to solve something similar to a Cauchy problem on the interval $(-\infty, 0]$. We call the operator Φ_∞^- the *extended input map* of (A, B).

It is tempting to write $\Phi_\infty^- u$ as an integral from $-\infty$ to 0, but this would be wrong in general (the function we would like to integrate is not necessarily Bochner integrable on $(-\infty, 0]$, see Section 12.5 for the concepts). It is easy to see that

$$\|\Phi_\infty^-\| = \lim_{\tau\to\infty} \|\Phi_\tau\|\,.$$

We denote by $BC([0,\infty); X)$ the Banach space of bounded and continuous X-valued functions defined on $[0,\infty)$, with the supremum norm.

Remark 4.6.2. If B is an admissible control operator for $\mathbb{T}$, then for every $T > 0$ and for every $u \in L^2([0,T];U)$, the function $z(t) = \Phi_t u$ satisfies

$$\|z\|_{C([0,T];X)} \leqslant \|\Phi_T\| \cdot \|u\|_{L^2([0,T];U)}\,. \tag{4.6.3}$$

If B is infinite-time admissible, then the above estimate implies

$$\|z\|_{BC([0,\infty);X)} \leqslant \|\Phi_\infty^-\| \cdot \|u\|_{L^2([0,\infty);U)}.$$

We also have the following converse statement: If for every $u \in L^2([0,\infty);U)$ the function $z(t) = \Phi_t u$ is bounded (on $[0,\infty)$), then B is infinite-time admissible. This follows from the uniform boundedness theorem applied to the operators Φ_τ.

Recall the time-reflection operators $\boldsymbol{\mathfrak{R}}_\tau$ introduced in Section 1.4. In addition, we introduce the *infinite time-reflection operator* $\boldsymbol{\mathfrak{R}}$ which acts on any function u defined on $\mathbb{R}$ by $(\boldsymbol{\mathfrak{R}}u)(t) = u(-t)$. Thus, $\boldsymbol{\mathfrak{R}}L^2((-\infty,0];U) = L^2([0,\infty);U)$.

Proposition 4.6.3. *Suppose that $B \in \mathcal{L}(U, X_{-1})$ is an infinite-time admissible control operator for $\mathbb{T}$. We denote by Ψ^d the extended output map of (A^*, B^*). Then $\Psi^d \in \mathcal{L}(X, L^2([0,\infty);U))$ and*

$$\Phi_\infty^- \boldsymbol{\mathfrak{R}} = (\Psi^d)^*. \tag{4.6.4}$$

Proof. As in Remark 4.4.2, we denote by Ψ_τ^d the output maps of (A^*, B^*). The definition of the operator Φ_∞^- can be rewritten in the form

$$\Phi_\infty^- u = \lim_{\tau\to\infty} \Phi_\tau \boldsymbol{\mathfrak{R}}_\tau \boldsymbol{\mathfrak{R}} u.$$

We rewrite (4.4.2) (using continuous extension to X) in the form

$$\boldsymbol{\mathfrak{R}}_\tau \Phi_\tau^* = \Psi_\tau^d \qquad \forall\, \tau \geqslant 0.$$

This can be rewritten equivalently as

$$\langle z_0, \Phi_\tau \boldsymbol{\mathfrak{R}}_\tau u\rangle = \langle \Psi_\tau^d z_0, u\rangle \qquad \forall\, z_0 \in X\,,\ u \in L^2([0,\infty);U)\,,\ \tau \in [0,\infty). \tag{4.6.5}$$

It follows from the uniform boundedness of the operators Φ_τ and from Theorem 4.4.3 that $\Psi^d \in \mathcal{L}(X, L^2([0,\infty);U))$. Using that $u = \boldsymbol{\mathfrak{R}}^2 u$ and taking limits in (4.6.5) as $\tau \to \infty$, we obtain the desired formula. □

Definition 4.6.4. Let $C \in \mathcal{L}(X_1, Y)$. We say that C is an *infinite-time admissible observation operator for* $\mathbb{T}$ if there exists a $K > 0$ such that

$$\int_0^\infty \|C\mathbb{T}_t z_0\|_Y^2 \,\mathrm{d}t \leqslant K^2 \|z_0\|_X^2 \qquad \forall\, z_0 \in \mathcal{D}(A). \tag{4.6.6}$$

Clearly the above condition is equivalent to the requirement that C is admissible and the operators Ψ_τ from (4.3.1) (extended to X) are uniformly bounded by K. It is also clear that C is infinite-time admissible iff

$$\Psi \in \mathcal{L}(X, L^2([0,\infty);Y)),$$

where Ψ is the extended output map from (4.3.6). As we already noted in Remark 4.3.5, if $\mathbb{T}$ is exponentially stable and C is an admissible observation operator for $\mathbb{T}$, then C is infinite-time admissible.

The simplest example of an infinite-time admissible observation operator corresponding to a semigroup that is not exponentially stable is the point observation of a left shift semigroup, as described in Example 4.4.4.

Remark 4.6.5. It follows from Theorem 4.4.3 that $B \in \mathcal{L}(U, X_{-1})$ is an infinite-time admissible control operator for the semigroup $\mathbb{T}$ iff B^* is an infinite-time admissible observation operator for the adjoint semigroup $\mathbb{T}^*$.

We have the following simpler version of Theorem 4.3.7.

Proposition 4.6.6. *Let $C \in \mathcal{L}(X_1, Y)$ be an infinite-time admissible observation operator for $\mathbb{T}$, so that* (4.6.6) *holds. Then*

$$\|C(sI - A)^{-1}\| \leqslant \frac{K}{\sqrt{2\mathrm{Re}\, s}} \qquad \forall\, s \in \mathbb{C}_0$$

in the following sense: The function $C(sI - A)^{-1}$, originally defined on some right half-plane in $\rho(A)$, has an analytic continuation to $\mathbb{C}_0$ that satisfies the estimate.

The proof is similar to that of Theorem 4.3.7, but simpler: the boundedness of Ψ is already known, and now we take $\alpha = 0$. For any $z \in X$, the analytic continuation of $C(sI - A)^{-1}z$ is the Laplace transform of Ψz. The dual version of this proposition should be obvious, and we refrain from stating it.

Remark 4.6.7. Replacing in Proposition 4.6.3 A^* with A and B^* with C and then using the definition (4.6.2) of Φ_∞^-, we obtain the following formula:

$$\Psi^* u = \lim_{\tau \to \infty} \int_0^\tau \mathbb{T}_t^* C^* u(t)\, \mathrm{d}t \qquad \forall\, u \in L^2([0, \infty); Y).$$

4.7 Remarks and bibliographical notes on Chapter 4

General remarks. The area of admissible control and observation operators has probably reached maturity, and an excellent survey paper on it is Jacob and Partington [112] (the paper Jacob, Partington and Pott [116] also has good survey value). A systematic presentation of admissibility is available in Staffans [209, Chapter 10].

Sections 4.1 and 4.2. We cannot trace the origin of the material in Section 4.1. Relavant material can be found in many books such as Lions and Magenes [157], Pazy [182], Staffans [209]. We have used Malinen et al. [167] and Weiss [228].

To our knowledge, the first paper to formulate the concept of an admissible control operator (with scalar input function) was Ho and Russell [100]. Soon afterwards, the admissibility assumption, formulated in the same abstract framework as in this book (but not under this name) has been an important ingredient in the theory of neutral systems developed in Salamon [202]. This assumption was also present in the first systematic treatment of well-posed linear systems in the papers Salamon [203, 204]. It should be noted that long before the emergence of the abstract concept of admissibility, systems described by either PDEs with boundary control or by delay differential equations that have unbounded control operators, have been analyzed without using the concept of a control operator. For example, the paper Lasiecka, Lions and Triggiani [143] is essentially a paper on admissibility for the boundary control of the wave equation, but without using control operators. Already in the 1970s there were various admissible control operator concepts available that were suitable mainly for analytic semigroups; see Curtain and Pritchard [38, Chapter 8], Lasiecka [142], Pritchard and Wirth [184] and Washburn [225]. Most of the material (and the terminology and notation) in Section 4.2 is based on [228].

Sections 4.3 and 4.4. Admissible observation operators in the sense defined here have appeared for the first time in Salamon [202], as far as we know. Other relevant early references are Curtain and Pritchard [38, Chapter 8], Dolecki and Russell [51], Pritchard and Wirth [184]. Admissible observation operators appeared as an ingredient of the theory of well-posed linear systems in Salamon [203, 204], Weiss [232, 231] and many later papers.

A systematic study of admissible observation operators was undertaken in Weiss [229], and most of the material in these two sections is based on [229]. The exceptions are: Proposition 4.4.6 has appeared in Weiss [230, Proposition 2.3] (and the estimate (4.3.9) is its dual counterpart). Theorem 4.3.8 is new, as far as we know.

The duality between the theory of observation and control (of which admissibility is just one aspect) has been known for a long time and we cannot trace its origins, but at least in the infinite-dimensional context, Dolecki and Russell [51] deserve some of the credit. Duality has been used extensively in Lions [156]. However, there are many facts and problems in each theory (observation and control) that do not have a natural dual counterpart. This is reflected in this book: our Sections 4.2 and 4.3 are not mirror images. Duality becomes more problematic when we work with Banach spaces and L^p functions – see [209, 229] for discussions.

Section 4.5. The representation theorem for output maps (Theorem 4.5.3) appeared in [204, 229] (actually, in [229] X and Y were Banach spaces and the output functions were required to be in L^p_{loc}). The dual representation theorem for input maps (Theorem 4.5.5) appeared in [204, 228]. (Actually, in [228] X and U were Banach spaces and the input functions were in L^p_{loc}, $p < \infty$. The proof was direct, not by duality. One of the other results in [228] is that if X is reflexive and $p = 1$, then any admissible B is bounded.) The paper [228] considered also the Banach space of all the admissible control operators for given U, X and $\mathbb{T}$, with the natural norm that makes this space complete.

Section 4.6. Infinite-time admissible observation operators (with $Y = \mathbb{C}$) have been formally introduced in Grabowski [74], but the infinite-time admissibility condition has been present already in Grabowski [73]. Infinite-time admissible control operators were considered in Hansen and Weiss [89]. All of these papers were mainly concerned with the particular case when the semigroup is diagonal, and they all considered the connection between infinite-time admissibility and a Lyapunov equation.

Concepts related to admissibility. An important part of [229, 231, 232] that is not considered in this book is the study of two extensions of an admissible observation operator C, defined as follows:

$$C_L z = \lim_{\tau \to 0} C \frac{1}{\tau} \int_0^\tau \mathbb{T}_t z \, \mathrm{d}t, \qquad C_\Lambda z = \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} z .$$

Each of these operators has the "natural" domain; i.e., the space of those $z \in X$ for which the limit defining the operator converges. C_Λ is an extension of C_L. If we replace in (4.3.1) C by C_L, then the formula becomes valid (for almost every t) for every initial state $z_0 \in X$. More importantly, a similar simplification is true for the formula giving the input-output map of a well-posed system (see [231], Staffans and Weiss [210]), and the extensions of C are also useful to express the generating operators of closed-loop systems obtained from well-posed systems via output feedback (see [232]). The papers [229, 232] studied also the invariance of C_L and C_Λ under certain perturbations of the semigroup.

The following concept has been introduced in Rebarber and Weiss [188]: Let $\mathbb{T}$ be a strongly continuous semigroup on the Hilbert space X, with generator A, and let $B \in \mathcal{L}(U, X_{-1})$. The *degree of unboundedness* of B, denoted by $\alpha(B)$, is the infimum of those $\alpha \geqslant 0$ for which there exist positive constants δ, ω such that

$$\|(\lambda I - A)^{-1} B\|_{\mathcal{L}(U,X)} \leqslant \frac{\delta}{\lambda^{1-\alpha}} \qquad \forall\, \lambda \in (\omega, \infty) . \tag{4.7.1}$$

It is clear from Proposition 4.4.6 that for any admissible $B \in \mathcal{L}(U, X_{-1})$ we have $\alpha(B) \leqslant \frac{1}{2}$, and if B is bounded, then $\alpha(B) = 0$.

If $C \in \mathcal{L}(X_1, Y)$, then the *degree of unboundedness* of C, denoted by $\alpha(C)$, is defined similarly as $\alpha(B)$ (with $C(sI - A)^{-1}$ in place of $(sI - A)^{-1}B$). We have $\alpha(C) = \alpha(C^*)$, where C^* is regarded as a control operator for $\mathbb{T}^*$. This concept is sometimes useful to establish the well-posedness or the regularity of systems.

Chapter 5

Testing Admissibility

This chapter is devoted to results which can help to determine if an observation operator or a control operator is admissible for an operator semigroup. We use the same notation as listed at the beginning of Chapter 4.

5.1 Gramians and Lyapunov inequalities

Suppose that C is an admissible observation operator for $\mathbb{T}$. As usual, we denote by Ψ_τ are the output maps corresponding to (A, C) $(\tau \geqslant 0)$. For each $\tau \geqslant 0$, we define the *observability Gramian* $Q_\tau \in \mathcal{L}(X)$ by

$$Q_\tau = \Psi_\tau^* \Psi_\tau .$$

If C is infinite-time admissible and Ψ is the extended output map of (A, C), then (as explained after Definition 4.6.4) $\Psi \in \mathcal{L}(X, L^2([0, \infty); Y))$. In this case, the operator $Q \in \mathcal{L}(X)$ defined by

$$Q = \Psi^* \Psi$$

is called the *infinite-time observability Gramian* of (A, C). We have encountered the finite-dimensional version of these concepts in Section 1.5.

We introduce some stability concepts for strongly continuous semigroups. $\mathbb{T}$ is called *uniformly bounded* if $\sup_{t \geqslant 0} \|\mathbb{T}_t\| < \infty$. $\mathbb{T}$ is called *weakly stable* if

$$\lim_{t \to \infty} \langle \mathbb{T}_t z, q \rangle = 0 \qquad \forall\, z, q \in X .$$

$\mathbb{T}$ is called *strongly stable* if

$$\lim_{t \to \infty} \|\mathbb{T}_t z\| = 0 \qquad \forall\, z \in X .$$

The above stability properties are related to Lyapunov inequalities and to infinite-time admissibility, as the following theorem shows.

Theorem 5.1.1. *Let $C \in \mathcal{L}(X_1, Y)$. The following four statements are equivalent:*

(a) *C is infinite-time admissible for $\mathbb{T}$.*

(b) *There exists an operator $Q \in \mathcal{L}(X)$ such that for any $z \in \mathcal{D}(A)$,*

$$Qz = \lim_{\tau \to \infty} \int_0^\tau \mathbb{T}_t^* C^* C \mathbb{T}_t z \, \mathrm{d}t. \tag{5.1.1}$$

(c) *There exist operators $\Pi \in \mathcal{L}(X), \Pi \geqslant 0$, which satisfy the following equation in X_{-1}^d:*

$$A^*\Pi z + \Pi A z = -C^*Cz \qquad \forall\, z \in \mathcal{D}(A). \tag{5.1.2}$$

(*Equivalently,* $2\mathrm{Re}\,\langle \Pi z, Az\rangle = -\|Cz\|^2$ *for all* $z \in \mathcal{D}(A)$.)

(d) *There exist operators $\Pi \in \mathcal{L}(X), \Pi \geqslant 0$, which satisfy the inequality*

$$2\mathrm{Re}\,\langle \Pi z, Az\rangle \leqslant -\|Cz\|^2 \qquad \forall\, z \in \mathcal{D}(A). \tag{5.1.3}$$

Moreover, if C is infinite-time admissible, then the following statements hold:

(1) *Q from* (5.1.1) *is the infinite-time observability Gramian of* (A, C).

(2) *Q satisfies* (5.1.2).

(3) *Q is the smallest positive solution of* (5.1.3) (*hence, also of* (5.1.2)).

(4) *We have* $\lim_{t\to\infty} Q^{\frac{1}{2}}\mathbb{T}_t z = 0$ *for every* $z \in X$. (*In particular, if* $Q > 0$ *then, $\mathbb{T}$ is strongly stable.*)

(5) *If $\mathbb{T}$ is strongly stable, then Q is the unique self-adjoint solution of* (5.1.2).

(6) *If $\mathbb{T}$ is uniformly bounded and* $\mathrm{Ker}\, Q = \{0\}$, *then $\mathbb{T}$ is weakly stable.*

Equation (5.1.2) is called a *Lyapunov equation*, and (5.1.3) is a *Lyapunov inequality*. Note that (5.1.1) can also be written as $Qz = \lim_{\tau\to\infty} Q_\tau z$.

Proof. First we shall prove that (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).
(a) $\Rightarrow$ (b): Assume that (a) holds. We define $Q = \Psi^*\Psi$, so that $Q \in \mathcal{L}(X)$. Then Remark 4.6.7 implies that Q is given by (5.1.1), so that (b) holds.
(b) $\Rightarrow$ (a): Assume that $Q \in \mathcal{L}(X)$ satisfies (5.1.1) (this formula determines Q since $\mathcal{D}(A)$ is dense in X). For any $z \in \mathcal{D}(A)$ and $\tau \geqslant 0$,

$$\|\Psi_\tau z\|^2 = \langle Q_\tau z, z\rangle = \left\langle \int_0^\tau \mathbb{T}_t^* C^* C \mathbb{T}_t z \, \mathrm{d}t, z \right\rangle \leqslant \langle Qz, z\rangle,$$

which shows that the operators Ψ_τ (with $\tau \geqslant 0$) are uniformly bounded.
(b) $\Rightarrow$ (c): Let $Q \in \mathcal{L}(X)$ be defined by (5.1.1). We show that (5.1.2) is satisfied for $\Pi = Q$. Let $z, w \in \mathcal{D}(A^2)$ and for $t \geqslant 0$ define $f(t) = \langle C\mathbb{T}_t z, C\mathbb{T}_t w\rangle$. Then f is continuously differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = \langle C\mathbb{T}_t Az, C\mathbb{T}_t w\rangle + \langle C\mathbb{T}_t z, C\mathbb{T}_t Aw\rangle\,.$$

Integrating both sides on $[0, \tau]$ gives

$$f(\tau) - f(0) = \left\langle \int_0^\tau \mathbb{T}_t^* C^* C \mathbb{T}_t A z \, \mathrm{d}t, w \right\rangle + \left\langle \int_0^\tau \mathbb{T}_t^* C^* C \mathbb{T}_t z \, \mathrm{d}t, Aw \right\rangle. \quad (5.1.4)$$

Since $Az \in \mathcal{D}(A)$, by (b) each of the above integrals converges (in X) as $\tau \to \infty$. Hence $\lim_{\tau \to \infty} f(\tau)$ also exists. Since by (b) the integral $\int_0^\tau f(t) \mathrm{d}t$ has a finite limit as $\tau \to \infty$, we must have $f(\tau) \to 0$ as $\tau \to \infty$. We then let $\tau \to \infty$ in (5.1.4) to find that

$$\langle QAz, w \rangle + \langle Qz, Aw \rangle = - \langle Cz, Cw \rangle.$$

Since $\mathcal{D}(A^2)$ is dense in X_1, by continuity the above equality remains valid for all $z, w \in \mathcal{D}(A)$. This implies that Q satisfies (5.1.2).

(c) $\Rightarrow$ (d): Assume that $\Pi \in \mathcal{L}(X)$, $\Pi \geqslant 0$ and Π satisfies (5.1.2). Take the duality pairing of the terms of (5.1.2) with z, and by simple manipulations we obtain

$$2\mathrm{Re} \langle \Pi z, Az \rangle = - \|Cz\|^2 \qquad \forall\, z \in \mathcal{D}(A).$$

Obviously this implies (d).

(d) $\Rightarrow$ (a): Assume that $\Pi \in \mathcal{L}(X)$, $\Pi \geqslant 0$ and Π satisfies (5.1.3). For all $z \in X$ and $t \in [0, \infty)$, we define $E_t(z)$ by $E_t(z) = \langle \Pi \mathbb{T}_t z, \mathbb{T}_t z \rangle$. Then $E_t(z) \geqslant 0$ and for every fixed $z \in \mathcal{D}(A)$, $E_t(z)$ is a continuously differentiable function of t. Using (5.1.3) we derive that for every $z \in \mathcal{D}(A)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} E_t(z) = 2\mathrm{Re} \langle \Pi \mathbb{T}_t z, A \mathbb{T}_t z \rangle \leqslant - \|C \mathbb{T}_t z\|^2 \leqslant 0, \quad (5.1.5)$$

so that $E_t(z)$ is non-increasing. Since $E_t(z)$ is a continuous function of z, from the density of $\mathcal{D}(A)$ in X we conclude that for any $z \in X$, $E_t(z)$ is non-increasing. This can be written in the following form: for $0 \leqslant \tau \leqslant t$,

$$\mathbb{T}_t^* \Pi \mathbb{T}_t \leqslant \mathbb{T}_\tau^* \Pi \mathbb{T}_\tau.$$

We know that any non-increasing positive operator-valued function has a strong limit; see Lemma 12.3.2 in Appendix I. Thus, there exists $\Pi_\infty \in \mathcal{L}(X)$, $\Pi_\infty \geqslant 0$, such that for all $z \in X$,

$$\lim_{t \to \infty} \mathbb{T}_t^* \Pi \mathbb{T}_t z = \Pi_\infty z \qquad (\text{in } X). \quad (5.1.6)$$

It is clear that $0 \leqslant \Pi_\infty \leqslant \Pi$. Integrating (5.1.5) on $[0, \infty)$, we get that for $z \in X_1$,

$$\langle \Pi z, z \rangle - \langle \Pi_\infty z, z \rangle \geqslant \int_0^\infty \|C \mathbb{T}_t z\|^2 \mathrm{d}t = \|\Psi z\|^2. \quad (5.1.7)$$

From here we see that Ψ is bounded, so that (a) holds.

In what follows we assume that C is infinite-time admissible and we prove (1)–(6). Statement (1) has been already proved when we proved that (a) $\Rightarrow$ (b). Statement (2) has been already proved when we proved (b) $\Rightarrow$ (c).

We prove statement (3). We have seen earlier that Q satisfies (5.1.2). If $\Pi \in \mathcal{L}(X)$, $\Pi \geqslant 0$ and (5.1.3) holds, then by (5.1.7) (using that $\|\Psi z\|^2 = \langle Qz, z\rangle$) we have that for all $z \in \mathcal{D}(A)$,

$$\langle Qz, z\rangle \leqslant \langle \Pi z, z\rangle - \langle \Pi_\infty z, z\rangle . \tag{5.1.8}$$

By continuity, this remains true for all $z \in X$, so that $Q \leqslant \Pi$, as claimed in (3).

To prove (4), we take $\Pi = Q$ in (5.1.6) and (5.1.8) and obtain $\Pi_\infty = 0$. By (5.1.6) this implies $\lim_{t\to\infty} \langle Q\mathbb{T}_t z, \mathbb{T}_t z\rangle = 0$ for any $z \in X$, which implies (4).

To prove (5), assume that $\mathbb{T}$ is strongly stable and $\Pi = \Pi^*$ is a solution of (5.1.2). Define again $E_t(z) = \langle \Pi\mathbb{T}_t z, \mathbb{T}_t z\rangle$. Then by the argument in the proof of (d) $\Rightarrow$ (a), the equality version of (5.1.5) holds:

$$\frac{\mathrm{d}}{\mathrm{d}t} E_t(z) \,=\, 2\mathrm{Re}\,\langle \Pi\mathbb{T}_t\, z, A\mathbb{T}_t\, z\rangle \,=\, -\,\|C\mathbb{T}_t\, z\|^2 \,\leqslant\, 0 .$$

From the strong stability of $\mathbb{T}$ we have $\lim_{t\to\infty} E_t(z) = 0$. We obtain a version of (5.1.7) by integration:

$$\langle \Pi z, z\rangle \,=\, \int_0^\infty \|C\mathbb{T}_t z\|^2\,\mathrm{d}t \,=\, \|\Psi z\|^2_{L^2([0,\infty);Y)} .$$

This shows that $\langle \Pi z, z\rangle = \langle Qz, z\rangle$ for all $z \in X$, whence $\Pi = Q$.

To prove (6), denote $V = \mathrm{Ran}\, Q^{\frac{1}{2}}$, then V is dense in X (because clos V is the orthogonal complement of Ker $Q^{\frac{1}{2}} = \mathrm{Ker}\, Q = \{0\}$, see (1.1.7)). It follows from statement (4) of the theorem that for any $z \in X$ and any $v \in V$, $\lim_{t\to\infty}\langle \mathbb{T}_t z, v\rangle = 0$. Let $z, q \in X$ be fixed. We claim that for any $\varepsilon > 0$ we can find $T \geqslant 0$ such that $\langle \mathbb{T}_t z, q\rangle \leqslant \varepsilon$ for each $t \geqslant T$. Indeed, let $v \in V$ be such that $\langle \mathbb{T}_t z, q - v\rangle \leqslant \frac{\varepsilon}{2}$ for all $t \geqslant 0$ (this is possible by the uniform boundedness of $\mathbb{T}$). Now if T is such that $\langle \mathbb{T}_t z, v\rangle \leqslant \frac{\varepsilon}{2}$ for all $t \geqslant T$, then T is the desired number. The existence of such a T for any $\varepsilon > 0$ means that $\langle \mathbb{T}_t z, q\rangle \to 0$. □

Example 5.1.2. Let (A, C) be as in Example 4.4.4, so that $\mathbb{T}$ is the left shift semigroup on $X = L^2[0,\infty)$, $Y = \mathbb{C}$ and $Cz = z(0)$ for each $z \in \mathcal{D}(A) = \mathcal{H}^1(0,\infty)$. Then it is clear that C is infinite-time admissible and $Q = I$. We have that $\mathbb{T}$ is strongly stable, as claimed in statement (4) of Theorem 5.1.1. The operator Q is the unique solution of (5.1.2), according to statement (5) of the theorem.

If instead we look at the adjoint semigroup $\mathbb{T}^*$ with $C = 0$ (which happens to be the restriction of the earlier C to $\mathcal{D}(A^*)$), then obviously $Q = 0$, but any multiple of the identity I satisfies the Lyapunov equation (5.1.2). $\mathbb{T}^*$ is only weakly stable.

As an application of Theorem 5.1.1 we give a simple sufficient condition for admissibility for semigroups generated by negative operators.

Proposition 5.1.3. *Let $A : \mathcal{D}(A) \to X$ be self-adjoint and $A \leqslant 0$. Define $X_{\frac{1}{2}}$ as the completion of $\mathcal{D}(A)$ with respect to the norm*

$$\|z\|_{\frac{1}{2}}^2 = \langle (I - A)z, z\rangle \qquad \forall\, z \in \mathcal{D}(A).$$

If $C \in \mathcal{L}(X_{\frac{1}{2}}, Y)$, then C is an admissible observation operator for the semigroup $\mathbb{T}$ (of positive operators) generated by A on X.

Proof. The fact that A generates a contraction semigroup of positive operators on X has been shown in Proposition 3.8.5. The space $X_{\frac{1}{2}}$ is similar to the space $H_{\frac{1}{2}}$ discussed in detail in Section 3.4, if we take $H = X$ and $A_0 = I - A$. In particular, we know that $H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}})$ and $A_0^{\frac{1}{2}}$ is an isomorphism from $X_{\frac{1}{2}}$ to X. Our boundedness assumption on C means that there exists $K \geqslant 0$ such that

$$\|Cz\|^2 \leqslant K^2 \langle (I - A)z, z\rangle \qquad \forall\, z \in \mathcal{D}(A).$$

If we denote $\Pi = \frac{K^2}{2} I$, then this can be written as

$$2\mathrm{Re}\,\langle \Pi z, (A - I)z\rangle \leqslant -\|Cz\|^2 \qquad \forall\, z \in \mathcal{D}(A),$$

which is like (5.1.3), but with $A - I$ in place of A. Theorem 5.1.1 implies that C is an infinite-time admissible observation operator for the semigroup generated by $A - I$. Hence, C is an admissible observation operator for $\mathbb{T}$. $\square$

We will show at the end of Section 5.3 that for $A \leqslant 0$, the condition in Proposition 5.1.3 is not necessary for an observation operator to be admissible.

Example 5.1.4. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and put $H = L^2(\Omega)$. Let A be the Dirichlet Laplacian on Ω, as defined in Section 3.6, so that $-A$ is a strictly positive densely defined operator on H. Its domain is $\mathcal{D}(A) = \{\phi \in \mathcal{H}_0^1(\Omega) \mid \Delta\phi \in L^2(\Omega)\}$. According to Proposition 3.6.1 we have $H_{\frac{1}{2}} = \mathcal{D}((-A)^{\frac{1}{2}}) = \mathcal{H}_0^1(\Omega)$, with the norm $\|z\|_{\frac{1}{2}} = \|\nabla z\|_{L^2}$. We know from Remark 3.6.11 that A generates a strongly continuous and diagonalizable semigroup $\mathbb{T}$ on H, called the heat semigroup.

Let $Y = L^2(\Omega)$ (the output space), $b \in L^\infty(\Omega; \mathbb{C}^n)$ and $c \in L^\infty(\Omega)$. We define $C \in \mathcal{L}(H_{\frac{1}{2}}, Y)$ as follows:

$$Cz = b \cdot \nabla z + cz.$$

According to Proposition 5.1.3, C is an admissible observation operator $\mathbb{T}$.

In terms of PDEs, this means that for every $\tau \geqslant 0$ there exists $K_\tau > 0$ with the following property: If z is the solution of the heat equation

$$\begin{aligned}
\frac{\partial z}{\partial t}(x,t) &= \Delta z(x,t), && x \in \Omega,\ t \geqslant 0,\\
z(x,t) &= 0, && x \in \partial\Omega,\ t \geqslant 0,\\
z(x,0) &= z_0(x), && x \in \Omega,
\end{aligned}$$

where $z_0 \in \mathcal{H}_0^1(\Omega)$, $\Delta z_0 \in L^2(\Omega)$, then

$$\int_0^\tau \int_\Omega |b \cdot \nabla z + cz|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant K_\tau^2 \|z_0\|_{L^2}^2 .$$

We shall see an application of this example in Section 10.8.

5.2 Admissible control operators for left-invertible semigroups

Consider the initial value problem

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z_0 ,$$

with $B \in \mathcal{L}(U, X_{-1})$, $z_0 \in X_{-1}$ and $u \in L^2_{\mathrm{loc}}([0,\infty); U)$ (which is contained in $L^1_{\mathrm{loc}}([0,\infty); U)$). Let $z(t) = \mathbb{T}_t z_0 + \Phi_t u$ be the mild solution of this problem (see Definition 4.1.5), which is an X_{-1}-valued continuous function of time. We know from Remark 4.1.9 that the Laplace transform of z is given, at the points $s \in \mathbb{C}$ where $\hat{u}(s)$ exists and $\operatorname{Re} s > \omega_0(\mathbb{T})$, by

$$\hat{z}(s) = (sI - A)^{-1} z_0 + (sI - A)^{-1} B\hat{u}(s). \tag{5.2.1}$$

Thus, taking $z_0 = 0$ we see that $\hat{u}$ gets multiplied with $(sI - A)^{-1}B$, which is an analytic $\mathcal{L}(U, X)$-valued function on the half-plane where $\operatorname{Re} s > \omega_0(\mathbb{T})$. The usual terminology is to call $(sI - A)^{-1}B$ the *transfer function* from u to z.

An important topic in the theory of admissibility is to give necessary and/or sufficient conditions for the admissibility of B in terms of the transfer function mentioned above. We have already seen a necessary condition for admissibility in terms of the function $(sI - A)^{-1}B$ in Propositions 4.4.6. Now we turn our attention to left-invertible operator semigroups, to give a simple sufficient condition for admissibility. (Left-invertible semigroups have been introduced in Section 2.7.)

Lemma 5.2.1. *Suppose that $\mathbb{T}$ is left-invertible.*

If $z \in X_{-1}$ and $t > 0$ are such that $\mathbb{T}_t z \in X$, then $z \in X$.

Proof. For all $n \in \mathbb{N}$ we define $I_n \in \mathcal{L}(X_{-1})$ by

$$I_n z = n \int_0^{\frac{1}{n}} \mathbb{T}_t z \,\mathrm{d}t \qquad \forall\, z \in X_{-1} .$$

These operators are approximations of the identity: We know from Proposition 2.1.6, applied to the extended semigroup $\mathbb{T}$ acting on X_{-1}, that for every $z \in X_{-1}$ we have $I_n z \in X$ and $\lim I_n z = z$ (in X_{-1}). Similarly, we know that if $z \in X$, then $I_n z \in \mathcal{D}(A)$ and $\lim I_n z = z$ (in X).

Now assume that $z \in X_{-1}$ and $t > 0$ are such that $\mathbb{T}_t z \in X$. We claim that $(I_n z)$ is a Cauchy sequence in X. Indeed, since $\mathbb{T}_t$ is left-invertible, there exists $m > 0$ such that $\|\mathbb{T}_t z\| \geqslant m\|z\|$ for all $z \in X$; see the beginning of Section 2.7. It follows that

$$\begin{aligned}\|I_n z - I_m z\| &\leqslant \frac{1}{m}\|\mathbb{T}_t(I_n z - I_m z)\| \\ &= \frac{1}{m}\|I_n \mathbb{T}_t z - I_m \mathbb{T}_t z\|.\end{aligned}$$

Since $(I_n \mathbb{T}_t z)$ is convergent in X (to $\mathbb{T}_t z$), we see that indeed $(I_n z)$ is a Cauchy sequence in X. Since X is complete, this sequence has a limit $z_0 \in X$ (and hence $\lim I_n z = z_0$ also in X_{-1}). But the same sequence has the limit z in X_{-1}. Since the limit in X_{-1} must be unique, it follows that $z = z_0 \in X$. □

Theorem 5.2.2. *Suppose that $\mathbb{T}$ is left-invertible and let $B \in \mathcal{L}(U, X_{-1})$. If for some $\alpha > \omega_0(\mathbb{T})$ we have*

$$\sup_{\operatorname{Re} s = \alpha} \|(sI - A)^{-1}B\|_{\mathcal{L}(U,X)} < \infty,$$

then B is an admissible control operator for $\mathbb{T}$.

Proof. Under the assumptions of the theorem, first we prove that for some $M \geqslant 0$,

$$\|(sI - A)^{-1}B\|_{\mathcal{L}(U,X)} \leqslant M \qquad \forall\, s \in \mathbb{C}_\alpha. \tag{5.2.2}$$

For this, the argument is similar to the one in the proof of Proposition 4.3.8. Take $s = \lambda + i\omega \in \mathbb{C}_\alpha$, so that $\lambda > \alpha$. Denote $s_1 = \alpha + i\omega$, then according to the resolvent identity (see Remark 2.2.5) we have

$$(sI - A)^{-1}B = \left[I + (\alpha - \lambda)(sI - A)^{-1}\right](s_1 I - A)^{-1}B.$$

According to our assumption, there exists $k \geqslant 0$ such that for all s_1 as above, $\|(s_1 I - A)^{-1}B\| \leqslant k$ (k is independent of ω). Thus,

$$\|(sI - A)^{-1}B\| \leqslant k\left[1 + (\lambda - \alpha)\cdot\|(sI - A)^{-1}\|\right] \qquad \forall\, s \in \mathbb{C}_\alpha.$$

According to Corollary 2.3.3 there exists $M_\alpha \geqslant 1$ (independent of $s = \lambda + i\omega$) such that

$$\|(sI - A)^{-1}\| \leqslant \frac{M_\alpha}{\lambda - \alpha}.$$

Substituting this into the previous estimate, we obtain that indeed (5.2.2) holds.

Introduce the shifted semigroup $\tilde{\mathbb{T}}$ by $\tilde{\mathbb{T}}_t = e^{-\alpha t}\mathbb{T}_t$ (this semigroup is exponentially stable and its generator is $A - \alpha I$). For all $t \geqslant 0$ define the input maps corresponding to $\tilde{\mathbb{T}}$ and B by

$$\tilde{\Phi}_t u = \int_0^t \tilde{\mathbb{T}}_{t-\sigma} B u(\sigma)\,\mathrm{d}\sigma \qquad \forall\, u \in L^2([0,\infty); U).$$

If we define $z_u(t) = \tilde{\Phi}_t u$, then, as explained at the beginning of this section, z_u is an X_{-1}-valued continuous function of t. According to (5.2.1),

$$\widehat{z_u} = ((s+\alpha)I - A)^{-1} B\hat{u}(s) \qquad \forall\, s \in \mathbb{C}_0 .$$

By the Paley–Wiener theorem (the version in Section 12.5), we have $\hat{u} \in \mathcal{H}^2(\mathbb{C}_0; U)$. According to (5.2.2) we obtain that $\widehat{z_u} \in \mathcal{H}^2(\mathbb{C}_0; X)$ (its norm is $\leqslant M\|\hat{u}\|_{\mathcal{H}^2}$). Using again the Paley–Wiener theorem we obtain that $z_u \in L^2([0,\infty); X)$. In particular, this means that $z_u(t) \in X$ for almost every $t \geqslant 0$.

Take $u \in L^2([0,1]; U)$ and extend u to all of $L^2([0,\infty); U)$ by putting $u(t) = 0$ for $t > 1$. According to the composition property (4.2.2),

$$z_u(t) = \tilde{\mathbb{T}}_{t-1} \tilde{\Phi}_1 u \qquad \forall\, t > 1 .$$

According to our earlier conclusion that $z_u(t) \in X$ for almost every $t \geqslant 0$, we can find $t > 1$ such that $z_u(t) \in X$. Since $\mathbb{T}$ is left-invertible, Lemma 5.2.1 implies that $\tilde{\Phi}_1 u \in X$. This means that B is admissible for $\tilde{\mathbb{T}}$, and hence also for $\mathbb{T}$. □

Corollary 5.2.3. *Suppose that $\mathbb{T}$ is left-invertible and let $B \in \mathcal{L}(U, X_{-1})$. If for every $\mathrm{v} \in U$ we have that $B\mathrm{v} \in \mathcal{L}(\mathbb{C}, X_{-1})$ is an admissible control operator for $\mathbb{T}$, then B is an admissible control operator for $\mathbb{T}$.*

Proof. Choose $\alpha > \omega_0(\mathbb{T})$. It follows from Proposition 4.4.6 that $(sI - A)^{-1}B\mathrm{v}$ (regarded as an X-valued function of s) is bounded on the vertical line where $\operatorname{Re} s = \alpha$. It follows from the uniform boundedness theorem that $(sI - A)^{-1}B$ (regarded as an $\mathcal{L}(U, X)$-valued function of s) is bounded on the same vertical line. According to Theorem 5.2.2, B is an admissible control operator for $\mathbb{T}$. □

By duality (i.e., using Theorem 4.4.3) we obtain from Theorem 5.2.2 the following.

Corollary 5.2.4. *Suppose that $\mathbb{T}$ is right-invertible and let $C \in \mathcal{L}(X_1, Y)$. If for some $\alpha > \omega_0(\mathbb{T})$ we have*

$$\sup_{\operatorname{Re} s = \alpha} \|C(sI - A)^{-1}\|_{\mathcal{L}(X,Y)} < \infty ,$$

then C is an admissible observation operator for $\mathbb{T}$.

This corollary can be proved also directly (i.e., not from Theorem 5.2.2) and we give an alternative proof, because it is elegant.

Proof. Assume, without loss of generality, that $\mathbb{T}$ is exponentially stable and $\alpha = 0$. First we show by an argument similar to the first half of the proof of Theorem 5.2.2 that in fact we have

$$\sup_{s \in \mathbb{C}_0} \|C(sI - A)^{-1}\|_{\mathcal{L}(X,Y)} = \mu < \infty .$$

The exponential stability of $\mathbb{T}$ implies (by the Paley–Wiener theorem from Section 12.5) that for every $z_0 \in X$, the function $g(s) = ((s+1)I - A)^{-1} z_0$ is in $\mathcal{H}^2(\mathbb{C}_0; X)$ and

$$\|g\|_{\mathcal{H}^2(\mathbb{C}_0;X)} \leqslant \kappa \|z_0\| .$$

Combining the last two estimates, we see that the function f, defined on $\mathbb{C}_0$ by

$$f(s) = C(sI - A)^{-1}((s+1)I - A)^{-1} z_0,$$

is in $\mathcal{H}^2(\mathbb{C}_0; Y)$ and its norm is $\leqslant \mu\kappa\|z_0\|$. By the resolvent identity we have $f(s) = C(sI - A)^{-1} z_0 - C((s+1)I - A)^{-1} z_0$. If $z_0 \in \mathcal{D}(A)$, then it follows that $f = \hat{y}$ with

$$y(t) = (1 - e^{-t}) C\mathbb{T}_t z_0 .$$

Using again the Paley–Wiener theorem, we obtain that for $z_0 \in \mathcal{D}(A)$,

$$\int_0^\infty |(1 - e^{-t})|^2 \|C\mathbb{T}_t z_0\|^2 \,\mathrm{d}t \leqslant \mu^2 \kappa^2 \|z_0\|^2 .$$

Since $1 - e^{-t} > \frac{1}{2}$ for $t \geqslant 1$, we obtain that

$$\frac{1}{4} \int_1^\infty \|C\mathbb{T}_t z_0\|^2 \,\mathrm{d}t \leqslant \mu^2 \kappa^2 \|z_0\|^2 \qquad \forall\, z_0 \in \mathcal{D}(A) .$$

By continuous extension to X, we obtain that

$$\|\Psi \mathbb{T}_1 z_0\|_{L^2} \leqslant 2\mu\kappa \|z_0\| \qquad \forall\, z_0 \in X .$$

Since Ran $\mathbb{T}_1 = X$, the admissibility of C follows. □

The dual counterpart of Corollary 5.2.3 is the following.

Corollary 5.2.5. *Suppose that $\mathbb{T}$ is right-invertible and let $C \in \mathcal{L}(X_1, Y)$. If for every $\mathrm{v} \in Y$ the functional $C^{\mathrm{v}} \in \mathcal{L}(X_1, \mathbb{C})$ defined by $C^{\mathrm{v}} z = \langle Cz, \mathrm{v}\rangle$ is an admissible observation operator for $\mathbb{T}$, then C is an admissible observation operator for $\mathbb{T}$.*

5.3 Admissibility for diagonal semigroups

In this section we consider only diagonal semigroups, as introduced in Example 2.6.6. Moreover, we restrict our attention to semigroups with eigenvalues in the open left half-plane, as this does not lead to a loss of generality: if a semigroup generator A is diagonal, we can always replace A by a shifted version $A - \gamma I$, with $\gamma > 0$ large enough, and the admissible observation (or control) operators for the shifted semigroup remain the same. Of course, infinite-time admissibility changes after such a shift, but infinite-time admissibility is at any rate only meaningful for diagonal semigroups that have their eigenvalues in the open left half-plane.

Diagonal semigroups may seem a very narrow class of semigroups, but they are not: many examples of semigroups that we deal with are diagonalizable, which means that they are isomorphic to diagonal semigroups, as explained in Example 2.6.6. For example, we have seen that self-adjoint or skew-adjoint generators with compact resolvents are diagonalizable; see Proposition 3.2.12. Notationally, it is more convenient to deal with diagonal semigroups than with diagonalizable ones.

We introduce the notation for this section. Our state space is $X = l^2$, and (λ_k) is a sequence in $\mathbb{C}$ such that

$$\operatorname{Re}\lambda_k < 0 \qquad \forall\, k \in \mathbb{N}.$$

The semigroup generator $A : \mathcal{D}(A) \to X$ is defined by

$$(Az)_k = \lambda_k z_k\,, \qquad \mathcal{D}(A) = \left\{ z \in l^2 \mid \sum_{k\in\mathbb{N}} (1+|\lambda_k|^2)|z_k|^2 < \infty \right\}. \tag{5.3.1}$$

As already explained in Example 2.6.6, $\sigma(A)$ is the closure in $\mathbb{C}$ of the sequence (λ_k) (this may contain points on the imaginary axis). We have

$$((sI-A)^{-1}z)_k = \frac{z_k}{s-\lambda_k} \qquad \forall\, s \in \rho(A), \tag{5.3.2}$$

and A is the generator of the diagonal contraction semigroup

$$(\mathbb{T}_t\, z)_k = e^{\lambda_k t} z_k \qquad \forall\, k \in \mathbb{N}. \tag{5.3.3}$$

We remark that $\mathbb{T}$ is strongly stable, as defined in Section 5.1 (this is easy to verify).

The space X_1 is, as usual, $\mathcal{D}(A)$ with the graph norm. This norm is equivalent to

$$\|z\|_1^2 = \sum_{k\in\mathbb{N}} |z_k|^2 (1+|\lambda_k|^2).$$

It is clear that the adjoint generator A^* is represented in the same way, with the sequence $(\overline{\lambda}_k)$ in place of (λ_k). Hence $\mathcal{D}(A^*) = \mathcal{D}(A)$ and the space X_1^d (the analogue of X_1 for the adjoint semigroup) is the same as X_1.

As explained in Example 2.10.9, X_{-1} is the space of all the sequences $z = (z_k)$ for which

$$\sum_{k\in\mathbb{N}} \frac{|z_k|^2}{1+|\lambda_k|^2} < \infty$$

and the square root of the above series gives an equivalent norm on X_{-1}. The space X_{-1}^d (the analogue of X_{-1} for A^*) is the same as X_{-1}.

Since X_{-1} is (by definition) the dual of $X_1^d = X_1$ with respect to the pivot space X, any sequence $c = (c_k) \in X_{-1}$ can be regarded as an operator $C \in \mathcal{L}(X_1, \mathbb{C})$, defined by

$$Cz = \langle z, \overline{c} \rangle = \sum_{k\in\mathbb{N}} c_k z_k\,. \tag{5.3.4}$$

Conversely, every $C \in \mathcal{L}(X_1, \mathbb{C})$ can be regarded as a sequence in X_{-1}.

For $h > 0$ and $\omega \in \mathbb{R}$ we denote

$$R(h,\omega) = \{\, s \in \mathbb{C} \mid 0 < \operatorname{Re} s \leqslant h, \quad |\operatorname{Im} z - \omega| \leqslant h \,\}\,.$$

Definition 5.3.1. A sequence (c_k) satisfies the *Carleson measure criterion* for the sequence (λ_k) if for every $h > 0$ and $\omega \in \mathbb{R}$,

$$\sum_{-\overline{\lambda}_k \in R(h,\omega)} |c_k|^2 \leqslant Mh, \tag{5.3.5}$$

where $M > 0$ is independent of h and ω.

The reason for the name of this criterion is that if (c_k) satisfies it, then the discrete measure on $\mathbb{C}_0$ with weights $|c_k|^2$ in the points $-\overline{\lambda}_k$ is a Carleson measure, as defined in Section 12.4. (It does not matter if we write λ_k in place of $\overline{\lambda}_k$ in (5.3.5), it would look simpler, but for the proofs it is more natural to write it as above.)

Using the above concept, we give a characterization of admissible observation operators for $\mathbb{T}$ with scalar output (i.e., the output space is $Y = \mathbb{C}$).

Theorem 5.3.2. *Suppose that c is a sequence that satisfies the Carleson measure criterion for (λ_k). Then $c \in X_{-1}$ and, when regarded as an operator $C \in \mathcal{L}(X_1, \mathbb{C})$, it is an infinite-time admissible observation operator for $\mathbb{T}$.*

Conversely, if $c \in X_{-1}$ determines an infinite-time admissible observation operator for $\mathbb{T}$, then c satisfies the Carleson measure criterion for (λ_k).

Proof. We have, denoting

$$\Delta_n = R(2^{n+1}, 0) \setminus R(2^n, 0)\ ,$$

that the union of the sets Δ_n for $n \in \mathbb{Z}$ is $\mathbb{C}_0$. Hence, for any complex sequence c,

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{|c_k|^2}{1 + |\lambda_k|^2} &= \sum_{n\in\mathbb{Z}} \sum_{-\lambda_k \in \Delta_n} \frac{|c_k|^2}{1 + |\lambda_k|^2} \\ &\leqslant \sum_{n\in\mathbb{Z}} \frac{1}{1 + 2^{2n}} \sum_{-\lambda_k \in R(2^{n+1},0)} |c_k|^2 .\end{aligned}$$

We get that if c satisfies (5.3.5), then

$$\sum_{k=1}^{\infty} \frac{|c_k|^2}{1 + |\lambda_k|^2} \leqslant \sum_{n\in\mathbb{Z}} \frac{1}{1 + 2^{2n}} \cdot M \cdot 2^{n+1} < \infty,$$

so that $c \in X_{-1}$, as claimed.

We show that if the sequence c satisfies the Carleson measure criterion, then the corresponding operator $C \in \mathcal{L}(X_1, \mathbb{C})$ is infinite-time admissible. The operator $C^* \in \mathcal{L}(\mathbb{C}, X_{-1}) = X_{-1}$ is represented by the sequence $(\overline{c}_k)$. According to Remark 4.6.5 is it enough to prove that C^* is an infinite-time admissible control operator for the diagonal semigroup $\mathbb{T}^*$ corresponding to the conjugate eigenvalues $(\overline{\lambda}_k)$. We denote by Φ_T^d the input map corresponding to the semigroup $\mathbb{T}^*$ with the control operator C^* and the time T (see Section 4.2). For every $u \in L^2[0,\infty)$, $k \in \mathbb{N}$ and $T \geqslant 0$ we can express the kth component of $\Phi_T^d \boldsymbol{\mathcal{R}}_T u$ by

$$\left(\Phi_T^d \boldsymbol{\mathcal{R}}_T u\right)_k = \left(\int_0^T \mathbb{T}_t^* C^* u(t)\,\mathrm{d}t\right)_k = \int_0^T e^{\overline{\lambda}_k t}\overline{c}_k u(t)\mathrm{d}t = \overline{c}_k(\widehat{\mathbf{P}_T u})(-\overline{\lambda}_k),$$

where $\boldsymbol{\mathcal{R}}_T$ is the time-reflection operator from Section 1.4. It follows that

$$\|\Phi_T^d \boldsymbol{\mathcal{R}}_T u\|^2 = \sum_{k\in\mathbb{N}} |c_k|^2 \cdot |(\widehat{\mathbf{P}_T u})(-\overline{\lambda}_k)|^2. \tag{5.3.6}$$

Define a positive measure μ on the Borel subsets E of $\mathbb{C}_0$ by

$$\mu(E) = \sum_{-\overline{\lambda}_k\in E} |c_k|^2.$$

It is easy to see that this is a Carleson measure and the right-hand side of (5.3.6) can be regarded as an integral with respect to μ. Therefore, by the Carleson measure theorem (see Section 12.4 in Appendix II) there exists $m_c \geqslant 0$ (independent of u and T) such that

$$\|\Phi_T^d \boldsymbol{\mathcal{R}}_T u\|^2 = \int_{\mathbb{C}_0} |\widehat{\mathbf{P}_T u}|^2 \mathrm{d}\mu \leqslant m_c^2 \|\widehat{\mathbf{P}_T u}\|^2_{\mathcal{H}^2}.$$

By the Paley–Wiener theorem (see again Section 12.4) we have

$$\|\widehat{\mathbf{P}_T u}\|_{\mathcal{H}^2} = \|\mathbf{P}_T u\|_{L^2} \leqslant \|u\|_{L^2}.$$

Thus, we obtain that

$$\|\Phi_T^d \boldsymbol{\mathcal{R}}_T u\| \leqslant m_c \|u\|_{L^2}.$$

Since $\boldsymbol{\mathcal{R}}_T$ is a unitary operator on $L^2[0,T]$, it follows that $\|\Phi_T^d\| \leqslant m_c$, so that indeed C^* (and hence also C) is infinite-time admissible.

Now assume that the sequence $c \in X_{-1}$ determines an infinite-time admissible observation operator $C \in \mathcal{L}(X_1, \mathbb{C})$ via (5.3.4). Hence its adjoint C^* (represented by the sequence $\overline{c} = (\overline{c}_k)$) is an infinite-time admissible control operator for $\mathbb{T}^*$. As explained at the beginning of Section 4.6, for any $u \in L^2[0,\infty)$ with $\|u\| = 1$ we have

$$\left\|\lim_{T\to\infty} \int_0^T \mathbb{T}_t^* C^* u(t)\mathrm{d}t\right\|_{l^2} \leqslant K,$$

with $K > 0$ independent of u. Using the fact that the extension of $\mathbb{T}^*$ to X_{-1} is still given by (5.3.3) (with $\overline{\lambda}_k$ in place of λ_k), we can rewrite the last estimate as

$$\sum_{k=1}^{\infty} |c_k|^2 \cdot \left|\int_0^\infty e^{\overline{\lambda}_k t} u(t)\,\mathrm{d}t\right|^2 \leqslant K^2. \tag{5.3.7}$$

(These integrals exist, there is no longer a need to write them as limits.)

Let $h > 0$ and $\omega \in \mathbb{R}$. We have to prove that (5.3.5) holds with M independent of h and ω. For $h > 0$ we define

$$u(t) = \begin{cases} \sqrt{h} \cdot e^{i\omega t} & \text{for} \quad t \in [0, \frac{1}{h}], \\ 0 & \text{for} \quad t > \frac{1}{h}. \end{cases}$$

We have $\|u\| = 1$ and hence (5.3.7) holds. This means that

$$K^2 \geqslant h \sum_{k=1}^{\infty} |c_k|^2 \cdot \left| \int_0^{\frac{1}{h}} e^{(\overline{\lambda}_k + i\omega)t} \mathrm{d}t \right|^2 \geqslant \frac{1}{h} \sum_{-\overline{\lambda}_k \in R(h,\omega)} |c_k|^2 \cdot \left| \frac{e^{\frac{\overline{\lambda}_k + i\omega}{h}} - 1}{\frac{\overline{\lambda}_k + i\omega}{h}} \right|^2 .$$

Let us denote

$$m = \min_{-z \in R(1,0)} \left| \frac{e^z - 1}{z} \right| \tag{5.3.8}$$

(for $z = 0$, we consider the extension by continuity). Since $m > 0$, the previous inequality implies

$$\sum_{-\overline{\lambda}_k \in R(h,\omega)} |c_k|^2 \leqslant \frac{K^2}{m^2} \cdot h,$$

which is equivalent to (5.3.5). □

Remark 5.3.3. Let (λ_k) be a sequence in $\mathbb{C}$ with $\operatorname{Re} \lambda_k < 0$. On the vector space of all the sequences (c_k) which satisfy the Carleson measure criterion for (λ_k) we define a norm by

$$|\!|\!| c |\!|\!|^2 = \sup_{h>0,\, \omega \in \mathbb{R}} \frac{1}{h} \sum_{-\lambda_k \in R(h,\omega)} |c_k|^2 .$$

If c is such a sequence, we denote by Ψ^c the extended output map corresponding to the diagonal semigroup $\mathbb{T}$ from (5.3.3) with the observation operator $C \in \mathcal{L}(X_1, \mathbb{C})$ corresponding to the sequence c. Then we have

$$0.6 |\!|\!| c |\!|\!| < \|\Psi^c\| < 20 |\!|\!| c |\!|\!| .$$

The proof of this fact is contained between the lines of the last proof. Indeed, the left inequality follows from the last part of the proof, after we verify that the constant m from (5.3.8) satisfies $m > 0.6$. The right inequality follows from the estimate $\|\Phi_T^d\| \leqslant m_c$ that has been derived towards the middle of the proof of Theorem 5.3.2. Indeed, if we combine this with $\|\Psi^c\| = \lim_{T\to\infty} \|\Psi_T^c\| = \lim_{T\to\infty} \|\Phi_T^d\|$ and with the estimate $m_c < 20\sqrt{M}$, given as part of the Carleson measure theorem, we obtain $\|\Psi^c\| < 20\sqrt{M}$. Here, M is the constant from (5.3.5). If M is chosen optimally (i.e., the smallest possible value for our c), then $M = |\!|\!| c |\!|\!|^2$.

Remark 5.3.4. Let $\mathbb{T}$ be a diagonal semigroup generated by A, and let (λ_k) be the sequence of the eigenvalues of A. In this remark, we make no stability assumption on $\mathbb{T}$. Often we want to check the admissibility of an observation operator C for $\mathbb{T}$, not its infinite-time admissibility. To accomplish this, we may replace $\mathbb{T}$ by the semigroup $\tilde{\mathbb{T}}$ generated by $A - \alpha I$, where $\alpha \geqslant 0$ is large enough to make $\tilde{\mathbb{T}}$ exponentially stable. Clearly C is admissible for $\mathbb{T}$ iff it is admissible for $\tilde{\mathbb{T}}$. As already mentioned in Section 4.6, C is admissible for $\tilde{\mathbb{T}}$ iff it is infinite-time admissible for $\tilde{\mathbb{T}}$. Thus, according to the last theorem, C is admissible iff the sequence (c_k) satisfies the Carleson measure criterion for the sequence $(\lambda_k - \alpha)$.

The following proposition shows that for diagonal groups (i.e., diagonal semigroups with $\operatorname{Re}\lambda_k$ bounded from below, see Remark 2.7.9), admissibility can be tested by a simpler condition than the Carleson measure criterion.

Proposition 5.3.5. *Let $\mathbb{T}$ be a diagonal group on $X = l^2$, with generator A, as in* (5.3.1) *and* (5.3.3), *and* (*as usual in this section*) *we assume that $\operatorname{Re}\lambda_k < 0$ for all $k \in \mathbb{N}$. Let $C \in \mathcal{L}(X_1, \mathbb{C})$ be represented by the sequence (c_k), as in* (5.3.4).

Then C is an admissible observation operator for $\mathbb{T}$ if and only if there exists $m \geqslant 0$ such that

$$\sum_{\operatorname{Im}\lambda_k \in [n,n+1)} |c_k|^2 \leqslant m \qquad \forall\, n \in \mathbb{Z}. \tag{5.3.9}$$

Moreover, we have the following numerical estimates: If (5.3.9) *holds then*

$$\|\Psi_1\| < 20e\sqrt{3m}, \tag{5.3.10}$$

where Ψ_1 is the output map of (A, C) for unit time.

Conversely, if C is admissible for $\mathbb{T}$, then (5.3.9) *holds for*

$$m = \frac{25a}{9(1-e^{-2})}\, \|\Psi_1\|^2,$$

where $a \geqslant 1$ is such that $1 - \operatorname{Re}\lambda_k \leqslant a$ for all $k \in \mathbb{N}$.

Proof. We shall use Remark 5.3.4 (which is based on Theorem 5.3.2). As in Remark 5.3.4 we replace A with $A - I$, which generates the exponentially stable semigroup $\tilde{\mathbb{T}}$. Admissibility for $\mathbb{T}$ is equivalent to admissibility for $\tilde{\mathbb{T}}$.

First we prove that (5.3.9) is sufficient for admissibility. It is easy to see that condition (5.3.9) implies that for all $h > 0$ and $\omega \in \mathbb{R}$ we have the estimate

$$\sum_{1-\overline{\lambda}_k \in R(h,\omega)} |c_k|^2 \leqslant (h+2)m \tag{5.3.11}$$

(for $h \in (0, 1)$ the inequality is trivial). Since $h + 2 \leqslant 3h$, we obtain from (5.3.11) that the Carleson measure criterion (5.3.5) holds with $M = 3m$, and with $\lambda_k - 1$ in place of λ_k. According to Remark 5.3.4, C is admissible.

Conversely, suppose that C is admissible, so that (c_k) satisfies (5.3.5) with $\lambda_k - 1$ in place of λ_k. Let $a \geqslant 1$ be such that $1 - \operatorname{Re}\lambda_k \in [1, a]$ for all $k \in \mathbb{N}$. Since, for every $n \in \mathbb{Z}$, the set $\{s \in \mathbb{C} \mid \operatorname{Re} s \in [1,a],\ \operatorname{Im} s \in [n,n+1)\}$ is contained in $R(a,n)$, it follows that condition (5.3.9) holds with $m = Ma$.

To prove the "moreover" part of the proposition, assume that (5.3.9) holds, so that (as proved earlier) (5.3.5) holds with $M = 3m$, and with $\lambda_k - 1$ in place of λ_k. We define the Carleson norm $|||c|||$ as in Remark 5.3.3, with $\lambda_k - 1$ in place of λ_k, then clearly $|||c||| \leqslant \sqrt{3m}$. We denote by Ψ^c the extended output map corresponding to $\tilde{\mathbb{T}}$ and C. According to Remark 5.3.3 we have $\|\Psi^c\| < 20|||c||| \leqslant 20\sqrt{3m}$. From here it follows that for every $z \in \mathcal{D}(A)$ we have

$$\|\Psi_1 z\|^2 = \int_0^1 e^{2t}\|e^{-t}C\mathbb{T}_t z\|^2 \mathrm{d}t \leqslant e^2 \int_0^1 \|C\tilde{\mathbb{T}}_t z\|^2 \leqslant e^2\|\Psi^c z\|^2 \leqslant e^2 20^2 3m\|z\|^2.$$

Clearly this implies the estimate (5.3.10).

To prove the converse numerical estimate, assume that C is admissible, then it is admissible also for $\tilde{\mathbb{T}}$. According to Remark 5.3.3 we have

$$0.6|||c||| < \|\Psi^c\|. \tag{5.3.12}$$

Let us denote by Ψ^c_τ the output maps corresponding to $\tilde{\mathbb{T}}$ and C, so that $\|\Psi^c\| = \lim_{\tau\to\infty}\|\Psi^c_\tau\|$. Since $\|\tilde{\mathbb{T}}_t\| \leqslant e^{-t}$, it follows from (4.3.5) that

$$\|\Psi^c_n\| \leqslant \|\Psi^c_1\|\left(1 + e^{-2} + \cdots + e^{-2(n-1)}\right)^{\frac{1}{2}} \leqslant \|\Psi^c_1\|\frac{1}{\sqrt{1-e^{-2}}}.$$

Taking the limit as $n \to \infty$ and then combining the result with (5.3.12), we obtain

$$\frac{3}{5}|||c||| < \frac{1}{\sqrt{1-e^{-2}}}\ \|\Psi^c_1\|.$$

Since, by elementary considerations, $\|\Psi^c_1\| \leqslant \|\Psi_1\|$, we obtain

$$|||c|||^2 < \frac{25}{9}\cdot\frac{1}{1-e^{-2}}\ \|\Psi_1\|^2.$$

Let again $a \geqslant 1$ be such that $1 - \operatorname{Re}\lambda_k \in [1,a]$ for all $k \in \mathbb{N}$. Since, for every $n \in \mathbb{Z}$, the set $\{s \in \mathbb{C} \mid \operatorname{Re} s \in [1,a],\ \operatorname{Im} s \in [n,n+1)\}$ (which contains all the eigenvalues of $A - I$ with $\operatorname{Im}\lambda_k \in [n,n+1)$) is contained in $R(a,n)$, it follows that

$$\frac{1}{a}\sum_{\operatorname{Im}\lambda_k\in[n,n+1)}|c_k|^2 < \frac{25}{9(1-e^{-2})}\|\Psi_1\|^2.$$

This shows that (5.3.9) holds with m as given at the end of the proposition. □

Remark 5.3.6. The first part of the above proposition remains valid also without the assumption that $\operatorname{Re}\lambda_k < 0$ for all $k \in \mathbb{N}$ (which is a standing assumption in this section). Indeed, both condition (5.3.9) and the admissibility of C are invariant properties with respect to a shift of A (i.e., replacing A with $A - \alpha I$ for some $\alpha \in \mathbb{R}$). In the "moreover" part of the proposition, the condition $\operatorname{Re}\lambda_k < 0$ can be relaxed to $\operatorname{Re}\lambda_k \leqslant 0$ (with the same proof). We mention that (5.3.9) is sufficient for the admissibility of C also for non-invertible diagonal semigroups.

Proposition 5.3.7. *Let $\mathbb{T}$ be a diagonal semigroup on $X = l^2$, as in* (5.3.3)*, let Y be a Hilbert space and let (c_k) be a sequence in Y such that the sequence $(\|c_k\|)$ represents an* (*infinite-time*) *admissible observation operator for $\mathbb{T}$. Then the operator*

$$Cz = \sum_{k\in\mathbb{N}} c_k z_k\,, \tag{5.3.13}$$

first defined for sequences $z = (z_k)$ in $\mathbb{C}$ with finitely many non-zero terms, is in $\mathcal{L}(X_1, Y)$ and it is an (*infinite-time*) *admissible observation operator for $\mathbb{T}$.*

Proof. Suppose that $(\|c_k\|)$ represents an admissible observation operator $\tilde{C}$ for $\mathbb{T}$, so that in particular $\tilde{C} \in \mathcal{L}(X_1, \mathbb{C})$. As explained around (5.3.4), the fact that $\tilde{C} \in \mathcal{L}(X_1, \mathbb{C})$ implies that the sequence $(\|c_k\|)$ is in X_{-1}, which means that

$$L = \sum_{k\in\mathbb{N}} \frac{\|c_k\|^2}{1+|\lambda_k|^2} < \infty.$$

Let $z = (z_k)$ be a sequence with finitely many non-zero terms. We have

$$\|Cz\|_Y \leqslant \sum_{k\in\mathbb{N}} \|c_k\|\cdot|z_k| \leqslant \left(\sum_{k\in\mathbb{N}} \frac{\|c_k\|^2}{1+|\lambda_k|^2}\right)\left(\sum_{k\in\mathbb{N}} (1+|\lambda_k|^2)|z_k|^2\right) \leqslant L\|z\|_1^2.$$

Hence, C can be extended such that $C \in \mathcal{L}(X_1, Y)$.

We have to show that $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$. According to Remark 4.6.5 is it enough to prove that C^* is an admissible control operator for $\mathbb{T}^*$. We denote by Φ_T^d the input map corresponding to $\mathbb{T}^*$ with the control operator C^* and the time T. We know that if $u \in L^2([0,\infty); Y)$ is a step function on $[0,T]$, then $\Phi_T^d u \in X = l^2$ (see Remark 4.2.3). For every $T \geqslant 0$, every $u \in L^2([0,\infty); Y)$ and every $k \in \mathbb{N}$ we can express the kth component of $\Phi_T^d u$ by

$$\left(\Phi_T^d u\right)_k = \left(\int_0^T \mathbb{T}_t^* C^* u(T-t)\,\mathrm{d}t\right)_k = \int_0^T e^{\overline{\lambda}_k t}\langle u(T-t), c_k\rangle\,\mathrm{d}t.$$

It follows that if $u \in L^2([0,\infty); Y)$ is a step function on $[0,T]$, then

$$\|\Phi_T^d u\|^2 = \sum_{k\in\mathbb{N}} \left|\left\langle \int_0^T e^{\overline{\lambda}_k t} u(T-t)\,\mathrm{d}t, c_k\right\rangle\right|^2 \leqslant \sum_{k\in\mathbb{N}} \left\|\int_0^T e^{\overline{\lambda}_k t} u(T-t)\,\mathrm{d}t\right\|^2 \|c_k\|^2.$$

Let $(e_j)_{j\in\mathcal{J}}$ be an orthonormal basis in Y and define $u_j \in L^2[0,\infty)$ by $u_j(t) = \langle u, e_j\rangle$. Clearly

$$\left\|\int_0^T e^{\overline{\lambda}_k t} u(T-t)\,\mathrm{d}t\right\|^2 = \sum_{j\in\mathcal{J}} \left|\int_0^T e^{\overline{\lambda}_k t} u_j(T-t)\,\mathrm{d}t\right|^2.$$

Interchanging the order of summation, we obtain

$$\|\Phi_T^d u\|^2 \leqslant \sum_{j\in\mathcal{J}} \left(\sum_{k\in\mathbb{N}} \left| \int_0^T e^{\overline{\lambda}_k t} u_j(T-t)\,\mathrm{d}t \right|^2 \|c_k\|^2 \right).$$

Let $\tilde{\Phi}_T^d$ be the input map corresponding to $\mathbb{T}^*$ with the control operator $\tilde{C}^*$ and the time T ($\tilde{C}$ has been defined at the beginning of this proof). Then the last formula can be rewritten as

$$\|\Phi_T^d u\|^2 \leqslant \sum_{j\in\mathcal{J}} \|\tilde{\Phi}_T^d u_j\|^2 .$$

Since $\tilde{C}^*$ is an admissible control operator for $\mathbb{T}^*$, we have $\tilde{\Phi}_T^d \in \mathcal{L}(L^2[0,\infty), l^2)$. Hence,

$$\|\Phi_T^d u\|^2 \leqslant \sum_{j\in\mathcal{J}} \|\tilde{\Phi}_T^d\| \cdot \|u_j\|_{L^2}^2 = \|\tilde{\Phi}_T^d\| \cdot \|u\|_{L^2}^2 .$$

Since the functions u as above (which are step functions on $[0,T]$) are dense in $L^2([0,\infty);U)$, it follows that C^* is admissible, hence C is admissible.

Note that in the above argument we have also proved that $\|\Phi_T^d\| \leqslant \|\tilde{\Phi}_T^d\|$. If $\tilde{C}$ is infinite-time admissible, then so is $\tilde{C}^*$ (see Remark 4.6.5), so that the operators $\tilde{\Phi}_T^d$ (with $T \geqslant 0$) are uniformly bounded. It follows that also the operators Φ_T^d are uniformly bounded, so that C^* is infinite-time admissible, hence so is C. □

Remark 5.3.8. If we combine Propositions 5.3.5 and 5.3.7, we obtain the following: Let $\mathbb{T}$ be a diagonal and invertible semigroup on $X = l^2$, let (c_k) be a sequence in a Hilbert space Y and assume that there exists $m \geqslant 0$ such that

$$\sum_{\operatorname{Im}\lambda_k \in [n,n+1)} \|c_k\|^2 \leqslant m \qquad \forall\, n \in \mathbb{Z}.$$

Then C defined by (5.3.13) (first for sequences with finitely many non-zero terms) is in $\mathcal{L}(X_1, Y)$ and it is an admissible observation operator for $\mathbb{T}$.

Theorem 5.3.9. *Let (λ_k), $\mathbb{T}$, A, X_1 be as at the beginning of this section and let $C \in \mathcal{L}(X_1, \mathbb{C})$. Then C is an infinite-time admissible observation operator for $\mathbb{T}$ if and only if there is a $K \geqslant 0$ such that*

$$\|C(sI-A)^{-1}\| \leqslant \frac{K}{\sqrt{2\operatorname{Re} s}} \qquad \forall\, s \in \mathbb{C}_0 . \tag{5.3.14}$$

Proof. The "only if" part has been proved in Proposition 4.6.6. We remark that K is now the same constant as in the infinite-time admissibility estimate (4.6.6).

To prove the "if" part, recall that C is represented by a sequence $c \in X_{-1}$. We show that c satisfies the Carleson measure criterion for (λ_k). According to

(5.3.2) the estimate (5.3.14) implies that

$$\sum_{k\in\mathbb{N}} \left| \frac{c_k}{s-\lambda_k} \right|^2 \leqslant \frac{K^2}{2\mathrm{Re}\, s} \qquad \forall\, s \in \mathbb{C}_0\,.$$

Take $h>0$ and $\omega \in \mathbb{R}$. Restricting the above summation only to those k for which $-\lambda_k \in R(h,\omega)$, and then taking $s = h - i\omega$, we get

$$\sum_{-\lambda_k \in R(h,\omega)} \frac{h}{|h - i\omega - \lambda_k|^2} |c_k|^2 \leqslant \frac{K^2}{2}\,.$$

Since

$$\min_{-\lambda_k \in R(h,\omega)} \frac{h}{|h - i\omega - \lambda_k|^2} = \frac{1}{5h}\,,$$

the previous inequality implies that (5.3.5) holds with $M = \frac{5K^2}{2}$. According to Theorem 5.3.2, C is an infinite-time admissible observation operator for $\mathbb{T}$. □

We mention that in the last theorem we could replace the condition $\mathrm{Re}\,\lambda_k < 0$ with the weaker one $\sup \mathrm{Re}\,\lambda_k < \infty$, using the same proof. However, this is an insignificant generalization, the components c_k corresponding to $\lambda_k \geqslant 0$ would have to be zero, so that these components would play no role.

The following corollary is a partial converse of Theorem 4.3.7.

Corollary 5.3.10. *Let $\mathbb{T}$ be a diagonal semigroup on $X = l^2$ with generator A and let $C \in \mathcal{L}(X_1, \mathbb{C})$. If there exists $\alpha \in \mathbb{R}$ and $K_\alpha \geqslant 0$ such that $\mathrm{Re}\,\lambda_k < \alpha$ and*

$$\|C(sI - A)^{-1}\| \leqslant \frac{K_\alpha}{\sqrt{\mathrm{Re}\, s - \alpha}} \qquad \forall\, s \in \mathbb{C}_\alpha\,,$$

then C is an admissible observation operator for $\mathbb{T}$.

Proof. Introduce the semigroup $\tilde{\mathbb{T}}$ generated by $A - \alpha I$. Since $A - \alpha I$ with C satisfy the estimate (5.3.14), according to Theorem 5.3.9, C is infinite-time admissible (hence, admissible) for $\tilde{\mathbb{T}}$. It follows that C is admissible also for $\mathbb{T}$. □

Example 5.3.11. As promised in Section 5.1, we show that for $A \leqslant 0$ the sufficient condition $C \in \mathcal{L}(X_{\frac{1}{2}}, Y)$ is not necessary for C to be admissible.

Take $X = l^2$ and let $\mathbb{T}$ be the diagonal semigroup corresponding to the sequence $\lambda_k = -2^k$ as in (5.3.3). Let $C \in \mathcal{L}(X_1, \mathbb{C}) = X_{-1}$ be defined by the sequence $c_k = 2^{\frac{k}{2}}$. It is easy to verify that (c_k) satisfies the Carleson measure criterion for (λ_k). According to Theorem 5.3.2, C is infinite-time admissible for $\mathbb{T}$. If C were bounded on $X_{\frac{1}{2}}$, then $C(-A)^{-\frac{1}{2}}$ would be bounded on X, so that it would be a sequence in l^2. However, $C(-A)^{-\frac{1}{2}} = (1, 1, 1, \ldots)$, so that C is not bounded on $X_{\frac{1}{2}}$.

5.4 Some unbounded perturbations of generators

In Section 2.11 we have seen that by adding a bounded perturbation to a semigroup generator we get another semigroup generator. This property is actually true also for many classes of unbounded perturbations, of which we present here a simple one: perturbations that are admissible observation operators, multiplied with a bounded operator. Moreover, we show that the admissible observation operators for the perturbed semigroup are the same as for the original semigroup.

We continue to use the standard notation of this chapter, such as U, X, Y, $\mathbb{T}$, A, X_1, X_{-1}, $\mathbf{P}_\tau$ and $\mathbf{S}_\tau$ (the latter is the unilateral right shift operator on $L^2_{\rm loc}([0,\infty);U)$). In addition, we will need $\mathbf{S}^*_\tau$, the *unilateral left shift* operator by $\tau \geqslant 0$ on the space $L^2_{\rm loc}([0,\infty);U)$, which means that $(\mathbf{S}^*_\tau u)(t) = u(t+\tau)$. Note that

$$\mathbf{S}^*_\tau \mathbf{S}_\tau = I, \qquad \mathbf{S}_\tau \mathbf{S}^*_\tau = I - \mathbf{P}_\tau. \tag{5.4.1}$$

For the proof of the first lemma, the reader needs to recall the version of the Paley–Wiener theorem for Hilbert space-valued functions; see Proposition 12.5.4 in Appendix I. As usual, if u is a function defined on $[0,\infty)$ that has a Laplace transform, then we denote this Laplace transform by $\hat{u}$.

Lemma 5.4.1. *For every $\omega \in \mathbb{R}$ and every Hilbert space U we define the space*

$$L^2_\omega([0,\infty);U) = e_\omega L^2([0,\infty);U), \quad \textit{where} \quad e_\omega(t) = e^{\omega t},$$

with the norm

$$\|u\|^2_\omega = \int_0^\infty e^{-2\omega t}\|u(t)\|^2\,{\rm d}t.$$

Assume that $\omega_0 \in \mathbb{R}$ and $\mathbf{G} : \mathbb{C}_{\omega_0} \to \mathcal{L}(U,Y)$ is analytic and bounded. Then for every $\omega \geqslant \omega_0$ there exists a unique operator

$$\mathbb{F}_\omega \in \mathcal{L}(L^2_\omega([0,\infty);U), L^2_\omega([0,\infty);Y))$$

such that $y = \mathbb{F}_\omega u$ if and only if $\hat{y} = \mathbf{G}\hat{u}$. Moreover,

$$\|\mathbb{F}_\omega\|_{\mathcal{L}(L^2_\omega)} \leqslant \sup_{s\in\mathbb{C}_\omega} \|\mathbf{G}(s)\| \qquad \forall\, \omega \geqslant \omega_0 \tag{5.4.2}$$

and

$$\mathbf{P}_\tau \mathbb{F}_\omega (I - \mathbf{P}_\tau) = 0 \qquad \forall\, \tau \geqslant 0. \tag{5.4.3}$$

We mention that in fact we have equality in (5.4.2). This would require some extra effort to prove but we do not need it, so we only state the inequality. The identity (5.4.3) is called *causality* and $\mathbf{G}$ is called the *transfer function* of $\mathbb{F}_\omega$.

Proof. We shall regard the function e_ω also as a pointwise multiplication operator. Then clearly e_ω is a unitary operator from $L^2([0,\infty);U)$ to $L^2_\omega([0,\infty);U)$, whose inverse is $e_{-\omega}$. It is easy to see that

$$(\widehat{e_{-\omega}u})(s) = \hat{u}(s+\omega). \tag{5.4.4}$$

Define a shifted transfer function $\mathbf{G}_\omega(s) = \mathbf{G}(s+\omega)$, so that $\mathbf{G}_\omega$ is a bounded analytic function on $\mathbb{C}_0$. Hence, when we regard $\mathbf{G}_\omega$ as a pointwise multiplication operator acting from $\mathcal{H}^2(\mathbb{C}_0;U)$ to $\mathcal{H}^2(\mathbb{C}_0;Y)$, then

$$\|\mathbf{G}_\omega\|_{\mathcal{L}(\mathcal{H}^2)} \leqslant \sup_{s\in\mathbb{C}_0} \|\mathbf{G}_\omega(s)\|_{\mathcal{L}(U,Y)} = \sup_{s\in\mathbb{C}_\omega} \|\mathbf{G}(s)\|_{\mathcal{L}(U,Y)}.$$

Indeed, this follows from the definition of the norm on $\mathcal{H}^2(\mathbb{C}_0;Y)$. We define $\mathbb{F}_\omega$ by

$$\mathbb{F}_\omega = e_\omega \mathcal{L}^{-1}\mathbf{G}_\omega \mathcal{L} e_{-\omega},$$

where $\mathcal{L}$ denotes the Laplace transformation, a unitary operator from $L^2([0,\infty);U)$ to $\mathcal{H}(\mathbb{C}_0;U)$ (see Proposition 12.5.4). It is now clear that $\mathbb{F}_\omega \in \mathcal{L}(L^2_\omega([0,\infty);Y))$ and $\|\mathbb{F}_\omega\| = \|\mathbf{G}_\omega\|$, so that $\|\mathbb{F}_\omega\|$ satisfies (5.4.2). It is now easy to see from (5.4.4) that $\widehat{\mathbb{F}_\omega u} = \mathbf{G}\hat{u}$, for all $u \in L^2_\omega([0,\infty);U)$.

It is easy to see that $\mathbb{F}_\omega$ satisfies the shift-invariance identity

$$\mathbf{S}_\tau \mathbb{F}_\omega = \mathbb{F}_\omega \mathbf{S}_\tau \qquad \forall\, \tau \geqslant 0.$$

Indeed, this follows from $\widehat{\mathbf{S}_\tau u}(s) = e^{-s\tau}\hat{u}(s)$. Multiplying with $\mathbf{S}^*_\tau$ from the right and using (5.4.1), we obtain

$$\mathbf{S}_\tau \mathbb{F}_\omega \mathbf{S}^*_\tau = \mathbb{F}_\omega (I - \mathbf{P}_\tau).$$

Applying $\mathbf{P}_\tau$ to both sides, we obtain (5.4.3). $\square$

Theorem 5.4.2. *Assume that $B \in \mathcal{L}(Y,X)$ and $C : \mathcal{D}(A) \to Y$ is an admissible observation operator for $\mathbb{T}$. Then the operator $A+BC : \mathcal{D}(A) \to X$ is the generator of a strongly continuous semigroup $\mathbb{T}^{cl}$ on X. This semigroup satisfies the integral equation*

$$\mathbb{T}^{cl}_t z_0 = \mathbb{T}_t z_0 + \int_0^t \mathbb{T}_{t-\sigma} BC \mathbb{T}^{cl}_\sigma z_0 \,\mathrm{d}\sigma \qquad \forall\, z_0 \in \mathcal{D}(A),\ t \geqslant 0.$$

Moreover, for any Hilbert space Y_1, the space of all admissible observation operators for $\mathbb{T}$ that map into Y_1 is equal to the corresponding space for $\mathbb{T}^{cl}$.

In the above context, BC is called a *perturbation of the generator* A and $\mathbb{T}^{cl}$ is called the *perturbed semigroup* (in the system theory community, $\mathbb{T}^{cl}$ would also be called the *closed-loop semigroup*).

Proof. The first step is to define an input-output map $\mathbb{F}$ associated with the operators A, B, C. We denote by $\mathcal{H}^1_{\mathrm{comp}}((0,\infty);Y)$ the subspace of those functions in $\mathcal{H}^1((0,\infty);Y)$ that have compact support (contained in $[0,\infty)$). We define the operator

$$\mathbb{F} : \mathcal{H}^1_{\mathrm{comp}}((0,\infty);Y) \to C([0,\infty);Y)$$

by

$$(\mathbb{F}u)(t) = C\int_0^t \mathbb{T}_{t-\sigma} Bu(\sigma)\,\mathrm{d}\sigma \qquad \forall\, t \geqslant 0.$$

First we show that this operator makes sense (i.e., the integral is in $\mathcal{D}(A)$ and the resulting function is continuous in Y). We apply Theorem 4.1.6 with the state space X_1 in place of X (hence with X in place of X_{-1}) and with $f = Bu$, and obtain that the function z defined by $z(t) = \int_0^t \mathbb{T}_{t-\sigma} Bu(\sigma)$ is in $C([0,\infty); X_1)$. Since $(\mathbb{F}u)(t) = Cz(t)$, we obtain that the definition of $\mathbb{F}$ is correct.

In terms of Laplace transforms, if $y = \mathbb{F}u$, then it follows from Remark 4.1.9 that for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \omega_0(\mathbb{T})$ we have $\hat{y}(s) = \mathbf{G}(s)\hat{u}(s)$, where

$$\mathbf{G}(s) \,=\, C(sI - A)^{-1}B \qquad \text{for} \;\; \operatorname{Re} s > \omega_0(\mathbb{T}) .$$

Take $\alpha > \omega_0(\mathbb{T})$. It follows from Theorem 4.3.7 that there exists $K_\alpha \geqslant 0$ such that

$$\|C(sI - A)^{-1}\| \,\leqslant\, \frac{K_\alpha}{\sqrt{\operatorname{Re} s - \alpha}} \qquad\qquad \forall\, s \in \mathbb{C}_\alpha .$$

It follows that for every $\omega > \alpha$,

$$\sup_{s \in \mathbb{C}_\omega} \|\mathbf{G}(s)\| \,\leqslant\, \frac{K_\alpha \|B\|}{\sqrt{\omega - \alpha}} . \tag{5.4.5}$$

According to Lemma 5.4.1, for every $\omega > \alpha$, $\mathbb{F}$ has a continuous extension to a bounded linear operator $\mathbb{F}_\omega$ acting on $L^2_\omega([0,\infty); Y)$. This extension is unique, because $\mathcal{H}^1_{\mathrm{comp}}((0,\infty); Y)$ is dense in $L^2_\omega([0,\infty); Y)$. The norm of $\mathbb{F}_\omega$ can be estimated by (5.4.2) together with (5.4.5).

The second step is to consider the system described by $\dot{z} = Az + Bu$, $y = Cz$ with the unity feedback $u = y$. First we express the resulting function y and then the operators $\mathbb{T}^{cl}_t$ that give the evolution of z. Let Ψ be the extended output map of (A, C). We claim that for large $\omega > \alpha$ and for any $z_0 \in X$ the equation

$$y \,=\, \Psi z_0 + \mathbb{F}_\omega y \tag{5.4.6}$$

has a unique solution $y \in L^2_\omega([0,\infty); Y)$. According to (5.4.5) we can choose ω sufficiently large such that

$$\sup_{s \in \mathbb{C}_\omega} \|\mathbf{G}(s)\| \,<\, 1 , \tag{5.4.7}$$

hence $\|\mathbb{F}_\omega\| < 1$. For the remainder of this proof, ω will be a fixed real number with the property (5.4.7). Notice that $\Psi \in \mathcal{L}(X, L^2_\omega([0,\infty); Y))$ according to Proposition 4.3.6. Hence (5.4.6) has a unique solution given by

$$y \,=\, (I - \mathbb{F}_\omega)^{-1} \Psi z_0 . \tag{5.4.8}$$

It will be convenient to introduce the operator $\Psi^{cl} \in \mathcal{L}(X, L^2_\omega([0,\infty); Y))$ by

$$\Psi^{cl} \,=\, (I - \mathbb{F}_\omega)^{-1} \Psi .$$

We shall see later that this is the extended output map of the closed-loop semigroup with the observation operator C. We define the operators $\mathbb{T}_t^{cl}$ (for $t \geqslant 0$) by taking y from (5.4.8) as the input function of the system $\dot{z} = Az + Bu$:

$$\mathbb{T}_t^{cl} = \mathbb{T}_t + \Phi_t \Psi^{cl} . \tag{5.4.9}$$

Here, Φ_t is defined by (4.2.1), and due to its causality (see the comments before (4.2.2)) Φ_t has a continuous extension to $L^2_\omega([0,\infty);Y)$, so that $\mathbb{T}_t^{cl} \in \mathcal{L}(X)$.

The third step is to show that the family $\mathbb{T}^{cl} = (\mathbb{T}_t^{cl})_{t\geqslant 0}$ is a strongly continuous semigroup on X. As a preparation for this, first we check that

$$\mathbf{S}_\tau^* \mathbb{F}_\omega = \mathbb{F}_\omega \mathbf{S}_\tau^* + \Psi \Phi_\tau . \tag{5.4.10}$$

To prove (5.4.10), apply both sides to $u \in \mathcal{H}^1((0,\infty);Y)$. We have seen at the beginning of this proof that $\Phi_t u$ is a continuous X_1-valued function and $(\mathbb{F}_\omega u)(t) = (\mathbb{F}u)(t) = C\Phi_t u$. With this, (5.4.10) (applied to u) can be recognized as being C applied to both sides of the composition property (4.2.2). Since $\mathcal{H}^1((0,\infty);Y)$ is dense in $L^2_\omega([0,\infty);Y)$, it follows that (5.4.10) holds in general.

We rewrite (5.4.10) in the equivalent form

$$(I - \mathbb{F}_\omega)\mathbf{S}_\tau^* - \mathbf{S}_\tau^*(I - \mathbb{F}_\omega) = \Psi \Phi_\tau ,$$

which in turn is equivalent to

$$\mathbf{S}_\tau^*(I - \mathbb{F}_\omega)^{-1} - (I - \mathbb{F}_\omega)^{-1}\mathbf{S}_\tau^* = \Psi^{cl}\Phi_\tau (I - \mathbb{F}_\omega)^{-1} .$$

We shall also need the following easily verifiable identities:

$$\Psi \mathbb{T}_\tau = \mathbf{S}_\tau^* \Psi , \qquad \Phi_{t+\tau} = \mathbb{T}_t \Phi_\tau + \Phi_t \mathbf{S}_\tau^* . \tag{5.4.11}$$

which hold for all $t, \tau \geqslant 0$ (they are just alternative ways to write (4.3.7) and (4.2.2)).

Now we have all the necessary tools to verify the semigroup property for $\mathbb{T}^{cl}$:

$$\begin{aligned}
\mathbb{T}_t^{cl}\mathbb{T}_\tau^{cl} &= \mathbb{T}_t\mathbb{T}_\tau + \Phi_t\Psi^{cl}\mathbb{T}_\tau + \mathbb{T}_t\Phi_\tau\Psi^{cl} + \Phi_t\Psi^{cl}\Phi_\tau(I - \mathbb{F}_\omega)^{-1}\Psi \\
&= \mathbb{T}_{t+\tau} + \Phi_t\Psi^{cl}\mathbb{T}_\tau + \mathbb{T}_t\Phi_\tau\Psi^{cl} + \Phi_t\left[\mathbf{S}_\tau^*(I - \mathbb{F}_\omega)^{-1} - (I - \mathbb{F}_\omega)^{-1}\mathbf{S}_\tau^*\right]\Psi \\
&= \mathbb{T}_{t+\tau} + \Phi_t(I - \mathbb{F}_\omega)^{-1}\left[\Psi\mathbb{T}_\tau - \mathbf{S}_\tau^*\Psi\right] + \left[\mathbb{T}_t\Phi_\tau + \Phi_t\mathbf{S}_\tau^*\right](I - \mathbb{F}_\omega)^{-1}\Psi \\
&= \mathbb{T}_{t+\tau} + \Phi_{t+\tau}\Psi^{cl} = \mathbb{T}_{t+\tau}^{cl} .
\end{aligned}$$

Obviously $\mathbb{T}_0^{cl} = I$. The strong continuity of the family $\mathbb{T}^{cl}$ is clear from (5.4.9), as both families $\mathbb{T}$ and Φ are strongly continuous (see Proposition 4.2.4).

The fourth step is to show that the generator of $\mathbb{T}^{cl}$, denoted by A^{cl}, is the restriction of $A + BC$ to $\mathcal{D}(A^{cl})$, which is a subspace of $\mathcal{D}(A)$. (Later we shall see that these spaces are actually equal.) We also show that C^{cl}, which is the restriction of C to $\mathcal{D}(A^{cl})$, is admissible for $\mathbb{T}^{cl}$. We apply the Laplace transformation to

$y = \Psi^{cl} z_0$, where $z_0 \in X$. We have seen in the second step of this proof that y satisfies (5.4.6), whence (using Theorem 4.3.7) we get

$$\hat{y}(s) = C(sI - A)^{-1} z_0 + \mathbf{G}(s)\hat{y}(s) \qquad \forall\, s \in \mathbb{C}_\omega\,.$$

From here we see (using also (5.4.7)) that

$$\hat{y}(s) = (I - \mathbf{G}(s))^{-1} C(sI - A)^{-1} z_0 \qquad \forall\, s \in \mathbb{C}_\omega\,.$$

From (5.4.9) and the definition of y we see that $\mathbb{T}^{cl}_t z_0 = \mathbb{T}_t z_0 + \Phi_t y$. Applying here the Laplace transformation, we obtain (using Proposition 2.3.1 for both semigroups) that for $\operatorname{Re} s$ sufficiently large and every $z_0 \in X$,

$$(sI - A^{cl})^{-1} z_0 = (sI - A)^{-1} z_0 + (sI - A)^{-1} B(I - \mathbf{G}(s))^{-1} C(sI - A)^{-1} z_0\,. \quad (5.4.12)$$

Since $\mathcal{D}(A^{cl}) = \operatorname{Ran}\,(sI - A^{cl})^{-1}$, we see from the above that $\mathcal{D}(A^{cl}) \subset \mathcal{D}(A)$. We apply C to both sides of (5.4.12) and obtain that for $\operatorname{Re} s$ sufficiently large,

$$\begin{aligned} C(sI - A^{cl})^{-1} z_0 &= C(sI - A)^{-1} z_0 + \mathbf{G}(s)(I - \mathbf{G}(s))^{-1} C(sI - A)^{-1} z_0 \\ &= (I - \mathbf{G}(s))^{-1} C(sI - A)^{-1} z_0 = \hat{y}(s)\,. \end{aligned}$$

We see from the last formula that if $z_0 \in \mathcal{D}(A^{cl})$, then $y(t) = C\mathbb{T}^{cl}_t z_0$. Since y is given by (5.4.8), it depends continuously (as an element of $L^2_\omega([0,\infty); Y)$) on z_0 (as an element of X). This shows that C^{cl}, the restriction of C to $\mathcal{D}(A^{cl})$, is an admissible observation operator for $\mathbb{T}^{cl}$, and the corresponding extended output map is Ψ^{cl}:

$$(\Psi^{cl} z_0)(t) = C\mathbb{T}^{cl}_t z_0 \qquad \forall\, t \geqslant 0,\ z_0 \in \mathcal{D}(A^{cl}) \subset \mathcal{D}(A)\,. \quad (5.4.13)$$

If $z_0 \in \mathcal{D}(A^{cl})$, then according to Proposition 4.3.4 we have that $y = \Psi^{cl} z_0$ belongs to $\mathcal{H}^1_{\mathrm{loc}}((0,\infty); Y)$. We see from (5.4.9) that $\mathbb{T}^{cl}_t z_0 = \mathbb{T}_t z_0 + \Phi_t y$. According to Theorem 4.1.6 (with X_1 in place of X and X in place of X_{-1}) we have $z \in C^1([0,\infty); X)$ and $\dot{z}(t) = Az(t) + By(t)$ holds for all $t \geqslant 0$. In particular, for $t = 0$ we obtain $A^{cl} z_0 = Az_0 + By(0) = Az_0 + BCz_0$. Thus,

$$A^{cl} z_0 = (A + BC) z_0 \qquad \forall\, z_0 \in \mathcal{D}(A^{cl})\,.$$

(This conclusion could be obtained also by a computation starting from (5.4.12).)

The fifth step is to show that in fact $\mathcal{D}(A^{cl}) = \mathcal{D}(A)$ and $\mathbb{T}^{cl}$ satisfies the integral equation stated in the theorem. We start from the operators A^{cl}, $-B$ and C^{cl} and we redo with them the first four steps of this proof. We obtain a closed-loop semigroup $\mathbb{T}^{cl,cl}$ with a generator $A^{cl,cl}$ defined on a domain $\mathcal{D}(A^{cl,cl}) \subset \mathcal{D}(A^{cl})$. According to the last conclusion in step four, we have

$$A^{cl,cl} z_0 = (A^{cl} - BC^{cl}) z_0 = Az_0 \qquad \forall\, z_0 \in \mathcal{D}(A^{cl,cl})\,.$$

Since a restriction of the generator A to a strictly smaller subspace cannot be a generator (because $sI - A$ must be invertible for large $\operatorname{Re} s$), it follows that in fact $\mathcal{D}(A^{cl,cl}) = \mathcal{D}(A)$. Clearly this implies $\mathcal{D}(A^{cl}) = \mathcal{D}(A)$. Finally, the integral equation in the theorem follows easily by combining (5.4.9) with (5.4.13).

The sixth step is to show that admissibility for $\mathbb{T}$ is equivalent to admissibility for $\mathbb{T}^{cl}$. Let Y_1 be a Hilbert space and let $C_1 : \mathcal{D}(A) \to Y_1$ be an admissible observation operator for $\mathbb{T}$. We denote by Ψ^1 and by $\Psi^{1,cl}$ the extended output maps of (A, C_1) and (A^{cl}, C_1), respectively. We also introduce the input-output map $\mathbb{F}^1$ associated with the operators A, B, C_1 exactly as we did it for A, B, C in the first step of the proof, and we extend it in the same way, obtaining an operator

$$\mathbb{F}^1_\omega \in \mathcal{L}(L^2_\omega([0,\infty);Y), L^2_\omega([0,\infty);Y_1)\,.$$

By the definition of $\mathbb{F}^1$ we have

$$(\mathbb{F}^1 y)(t) = C_1 \Phi_t y \qquad \forall\, t \geqslant 0,\ y \in \mathcal{H}^1_{\rm comp}((0,\infty);Y)\,.$$

Using the causality of $\mathbb{F}^1$ (see (5.4.3)), the above formula can be extended:

$$(\mathbb{F}^1_\omega y)(t) = C_1 \Phi_t y \qquad \forall\, t \geqslant 0,\ y \in \mathcal{H}^1_{\rm loc}((0,\infty);Y) \cap L^2_\omega([0,\infty);Y)\,.$$

Applying the terms of (5.4.9) to $z_0 \in \mathcal{D}(A)$ and then applying C_1 to the resulting equation, we obtain (using that $\Psi^{cl} z_0 \in \mathcal{H}^1_{\rm loc}((0,\infty);Y)$ by Proposition 4.3.4) that

$$\Psi^{1,cl} z_0 = \Psi^1 z_0 + \mathbb{F}^1_\omega \Psi^{cl} z_0 \qquad \forall\, z_0 \in \mathcal{D}(A)\,.$$

Since the operators on the right-hand side have continuous extensions to X, the same is true for $\Psi^{1,cl}$, meaning that C_1 is an admissible observation operator for $\mathbb{T}^{cl}$.

To show that every admissible observation operator for $\mathbb{T}^{cl}$ is admissible also for $\mathbb{T}$, we repeat the same argument, but with the roles of $\mathbb{T}$ and $\mathbb{T}^{cl}$ reversed and with $-B$ in place of B (we did a similar trick in step five). □

Proposition 5.4.3. *With the assumptions and the notation of Theorem* 5.4.2*, let $C_1 \in \mathcal{L}(X_1, Y_1)$ be an admissible observation operator for $\mathbb{T}$. We denote by Ψ and Ψ^1 the extended output maps of (A, C) and (A, C_1), respectively. Similarly, let Ψ^{cl} and $\Psi^{1,cl}$ be the extended output maps of $(A+BC, C)$ and $(A+BC, C_1)$, respectively. For any $\omega > \omega_0(\mathbb{T})$ we denote by*

$$\mathbb{F}_\omega : L^2_\omega([0,\infty);Y) \to L^2_\omega([0,\infty);Y)\,, \qquad \mathbb{F}^1_\omega : L^2_\omega([0,\infty);Y) \to L^2_\omega([0,\infty);Y_1)$$

the input-output maps corresponding to the transfer functions $C(sI-A)^{-1}B$ and $C_1(sI-A)^{-1}B$, respectively. Then

$$\Psi^{cl} = (I - \mathbb{F}_\omega)^{-1}\Psi\,, \qquad \Psi^{1,cl} = \Psi^1 + \mathbb{F}^1_\omega \Psi^{cl}\,.$$

The proof of this proposition is contained in the proof of Theorem 5.4.2, in the second, fourth and sixth steps. We could have appended the above proposition to Theorem 5.4.2, but this would have made the theorem very heavy. Proposition 5.4.3 will be needed in a proof in Section 6.3, otherwise it is probably of little interest, which is why we separated it from the theorem.

Example 5.4.4. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We shall introduce the operator semigroup corresponding to the *convection-diffusion equation*

$$\frac{\partial z}{\partial t} = \Delta z + b \cdot \nabla z + cz \quad \text{in} \quad \Omega \times (0, \infty), \tag{5.4.14}$$

with the boundary condition

$$z = 0 \quad \text{on} \quad \partial\Omega. \tag{5.4.15}$$

Here we assume that $b \in L^\infty(\Omega; \mathbb{C}^n)$ and $c \in L^\infty(\Omega)$. We shall regard this as a perturbation of the heat equation, of the type discussed in this section.

We denote $H = Y = L^2(\Omega)$, A is the Dirichlet Laplacian on Ω, so that (as shown in Section 3.6) $A_0 = -A$ is a strictly positive operator and

$$\mathcal{D}(A) = \left\{ \phi \in \mathcal{H}_0^1(\Omega) \;\middle|\; \Delta\phi \in L^2(\Omega) \right\}, \qquad H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}}) = \mathcal{H}_0^1(\Omega).$$

We define $C \in \mathcal{L}(H_{\frac{1}{2}}, Y)$ by

$$Cz = b \cdot \nabla z + cz.$$

As already explained in Example 5.1.4, C is admissible for the semigroup $\mathbb{T}$ generated by A. According to Theorem 5.4.2 with $B = I$, the operator $A + C$ (with domain $\mathcal{D}(A)$) generates a semigroup $\mathbb{T}^{cl}$ on H and any admissible observation operator for $\mathbb{T}$ is admissible also for $\mathbb{T}^{cl}$ (and the other way round). Note that $\mathbb{T}^{cl}$ corresponds to solutions of the convection-diffusion equation (5.4.14) with the homogeneous boundary condition (5.4.15). Clearly, C is admissible also for $\mathbb{T}^{cl}$.

To illustrate the admissibility statement made a few lines earlier, consider $\mathcal{O}$ to be an open subset of Ω such that $\operatorname{clos} \mathcal{O} \subset \Omega$ and $\partial\mathcal{O}$ is Lipschitz. Let $Y_1 = L^2(\partial\mathcal{O})$ and define $C_1 \in \mathcal{L}(H_{\frac{1}{2}}, Y_1)$ by $C_1 z = z|_{\partial\mathcal{O}}$ (i.e., C_1 is the Dirichlet trace operator corresponding to the boundary of $\mathcal{O}$). The continuity of C_1 on $H_{\frac{1}{2}} = \mathcal{H}_0^1(\Omega)$ follows from Theorem 13.6.1. According to Proposition 5.1.3, C_1 is admissible for $\mathbb{T}$. According to the last part of Theorem 5.4.2, C_1 is admissible also for $\mathbb{T}^{cl}$.

Finally, we derive a simple additional result that holds in the context of Theorem 5.4.2. For this, we have to introduce the concept of an analytic semigroup.

Definition 5.4.5. An operator semigroup $\mathbb{T}$ with generator A is *analytic* if there exists $\lambda \geqslant 0$ and $m > 0$ such that

$$\|(sI - A)^{-1}\| \leqslant \frac{m}{|s|} \qquad \text{if} \;\; \operatorname{Re} s > \lambda. \tag{5.4.16}$$

Remark 5.4.6. We mention a few well known facts from the theory of analytic semigroups. These can be found in the books dealing with operator semigroups quoted at the beginning of Chapter 2. We do not give proofs, and we shall not use these facts. An operator semigroup $\mathbb{T}$ with generator A is analytic iff there exist numbers $\lambda \geqslant 0$, $\alpha \in (0, \frac{\pi}{2})$ and $m > 0$ such that

$$\|((s+\lambda)I - A)^{-1}\| \leqslant \frac{m}{|s|} \qquad \text{if } \ |\arg(s+\lambda)| < \frac{\pi}{2} + \alpha .$$

If $\mathbb{T}$ is analytic, then $\mathbb{T}_t$ (as a function of t) has an analytic extension into the open sector where $|\arg t| < \alpha$, which satisfies the semigroup property. Moreover, $\mathbb{T}_t z \in \mathcal{D}(A^\infty)$ holds for every $z \in X$ and every $t \neq 0$ with $|\arg t| < \alpha$.

Proposition 5.4.7. *Let $A : \mathcal{D}(A) \to X$ be such that $A < 0$. Assume that $B \in \mathcal{L}(Y, X)$ and $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for the semigroup $\mathbb{T}$ generated by A. Then the semigroup $\mathbb{T}^{cl}$ generated by $A + BC$ is analytic.*

Proof. As in the proof of Theorem 5.4.2, we denote $A^{cl} = A + BC$ and we recall that for $\operatorname{Re} s$ sufficiently large, the resolvents of A^{cl} can be obtained from the resolvents of A via (5.4.12), where $\mathbf{G}(s) = C(sI - A)^{-1}B$. According to (5.4.7) the factor $(I - \mathbf{G}(s))^{-1}$ is uniformly bounded in $\mathcal{L}(Y)$ for all s in some right half-plane. Since $(sI - A)^{-1}$ satisfies (5.4.16) and since $C(sI - A)^{-1}$ is uniformly bounded in $\mathcal{L}(X, Y)$ for all s in some right half-plane (see Theorem 4.3.7), it is clear that $(sI - A^{cl})^{-1}$ satisfies an estimate similar to (5.4.16) for some (possibly larger) $\lambda \geqslant 0$. According to Definition 5.4.5, this implies that $\mathbb{T}^{cl}$ is analytic. $\square$

5.5 Admissible control operators for perturbed semigroups

In this section we investigate admissible control operators for semigroups that have been obtained by a perturbation as in Theorem 5.4.2 or its dual. We start with the dual version of Theorem 5.4.2.

Corollary 5.5.1. *Assume that $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$ and $C \in \mathcal{L}(X, U)$. Then the operator $A + BC : \mathcal{D}(A + BC) \to X$, where*

$$\mathcal{D}(A + BC) = \{z \in X \mid (A + BC)z \in X\},$$

is the generator of a strongly continuous semigroup $\mathbb{T}^{cl}$ on X. This semigroup satisfies the integral equation

$$\mathbb{T}^{cl}_t z_0 = \mathbb{T}_t z_0 + \int_0^t \mathbb{T}_{t-\sigma} BC \mathbb{T}^{cl}_\sigma z_0 \,\mathrm{d}\sigma \qquad \forall\, z_0 \in \mathcal{D}(A + BC), \ t \geqslant 0 .$$

Moreover, for any Hilbert space U_1, the space of all admissible control operators for $\mathbb{T}$ defined on U_1 is equal to the corresponding space for $\mathbb{T}^{cl}$.

This is almost an immediate consequence of Theorem 5.4.2, except for the minor trouble that one has to verify that, if A, B, C are as in the theorem, then

$$\mathcal{D}((A+BC)^*) = \{z \in X \mid (A^* + C^*B^*)z \in X\}.$$

A direct proof seems a little more complicated than for Theorem 5.4.2.

In what follows we investigate when admissible control operators for a semigroup remain admissible for the perturbed semigroup obtained as in Theorem 5.4.2. In general, this is not true. We use the assumptions and the notation of Theorem 5.4.2. Thus, $\mathbb{T}^{cl}$ is the semigroup generated by $A+BC$, where $B \in \mathcal{L}(Y, X)$ and $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$. Strictly speaking, the question posed above makes no sense, for the following reason: If B_1 is an admissible control operator for $\mathbb{T}$, then it must be an element of $\mathcal{L}(U, X_{-1})$. Let us denote by X^{cl}_{-1} the analog of the space X_{-1} for the semigroup $\mathbb{T}^{cl}$; i.e., X^{cl}_{-1} is the completion of X with respect to the norm

$$\|z\|^{cl}_{-1} = \|(\beta I - (A+BC))^{-1}z\|.$$

Since, in general, X^{cl}_{-1} is different from X_{-1}, B_1 does not qualify to be an admissible control operator for $\mathbb{T}^{cl}$ (the integral in (4.2.1) does not make sense with $\mathbb{T}^{cl}$ in place of $\mathbb{T}$ and B_1 in place of B). In order to regard B_1 as a control operator for $\mathbb{T}^{cl}$, we must identify a part of X^{cl}_{-1} with a part of X_{-1} containing Ran B_1. In other words, we must find an operator J that maps a part of X_{-1} into X^{cl}_{-1} and which, when restricted to X, is the identity operator. If we identify z with Jz, then B_1 is identified with JB_1, which is an element of $\mathcal{L}(U, X^{cl}_{-1})$. There is no unique way to find such a J, and different identifications may lead to different control operators for $\mathbb{T}^{cl}$ (from the same B_1). We shall see that redefining B_1 as an element of $\mathcal{L}(U, X^{cl}_{-1})$ can be achieved by defining the product $C(\beta I - A)^{-1}B_1$ for some $\beta \in \rho(A)$. A priori, the product $C(sI - A)^{-1}B_1$ makes no sense, because $(sI - A)^{-1}B_1$ maps into X and C is only defined on X_1. However, the product will make sense if we use a suitable extension of C in place of C. The precise statement is as follows.

Proposition 5.5.2. *With the assumptions and the notation of Theorem* 5.4.2*, assume that there exists a Banach space $\mathcal{D}(C^e)$ such that $X_1 \subset \mathcal{D}(C^e) \subset X$, with continuous embeddings and C has an extension $C^e \in \mathcal{L}(\mathcal{D}(C^e), Y)$.*

(1) *We define an operator $J \in \mathcal{L}((\beta I - A)\mathcal{D}(C^e), X^{cl}_{-1})$ by*

$$J = (\beta I - (A+BC))(\beta I - A)^{-1} + BC^e(\beta I - A)^{-1}. \tag{5.5.1}$$

Here, $A+BC$ is the extended operator acting from X to X^{cl}_{-1}. Then J is independent of β and it is an extension of the identity operator on X. We have

$$(A+BC)z = JAz + BC^e z \qquad \forall\, z \in \mathcal{D}(C^e), \tag{5.5.2}$$

where (again) $A+BC$ is the extended operator acting from X to X^{cl}_{-1}.

(2) *Let* $B_1 \in \mathcal{L}(U, X_{-1})$ *such that for some (hence, for every)* $\beta \in \rho(A)$*, we have*

$$\operatorname{Ran} B_1 \subset (\beta I - A)\mathcal{D}(C^e).$$

Then for every $\beta \in \rho(A)$ *we have* $C^e(\beta I - A)^{-1}B_1 \in \mathcal{L}(U, Y)$*, and hence*

$$JB_1 \in \mathcal{L}(U, X_{-1}^{cl}).$$

(3) *If* B_1 *as in part (2) is an admissible control operator for* $\mathbb{T}$*, and if in addition there exist* $\alpha \in \mathbb{R}$ *and* $M \geqslant 0$ *such that* $\mathbb{C}_\alpha \subset \rho(A)$ *and*

$$\|C^e(sI - A)^{-1}B_1\|_{\mathcal{L}(U,Y)} \leqslant M \qquad \forall\, s \in \mathbb{C}_\alpha, \tag{5.5.3}$$

then JB_1 *is an admissible control operator for* $\mathbb{T}^{cl}$*.*

According to the terminology of the systems theory literature, condition (5.5.3) expresses that the transfer function $C^e(sI - A)^{-1}B_1$ is *proper*.

Proof. We prove (1). Let C^e and J be as in part (1). If $z \in X$, then $(\beta I - A)^{-1}z \in \mathcal{D}(A) = \mathcal{D}(A + BC)$ and hence we may use the non-extended versions of $A + BC$ and of C in (5.5.1). Then we immediately get that $Jz = z$.

To prove that J is independent of β, we cannot use a density argument, because X need not be dense in the domain of J. We denote for a moment by J_β the operator from (5.5.1) and by J_s the operator obtained with $s \in \rho(A)$ in place of β. Then

$$\begin{aligned} J_s - J_\beta = (sI - (A + BC))^{-1}[(sI - A)^{-1} - (\beta I - A)^{-1}] \\ + (s - \beta)(sI - A)^{-1} + BC^e[(sI - A)^{-1} - (\beta I - A)^{-1}]. \end{aligned}$$

From the resolvent identity (Remark 2.2.5) we see that $(sI - A)^{-1} - (\beta I - A)^{-1}$ maps X_{-1} into $\mathcal{D}(A)$, so that we may replace C^e with C in the above formula, and $A + BC$ is no longer the extended operator, but just the original one (from $\mathcal{D}(A)$ to X). From here we easily get that $J_s = J_\beta$.

Finally, apply both sides of (5.5.1) to $(\beta I - A)z$, where $z \in \mathcal{D}(C^e)$ (and $Az \in X_{-1}$). After some cancellation, we obtain (5.5.2).

We prove (2). The operator $(\beta I - A)^{-1}B_1$ is closed from U to $\mathcal{D}(C^e)$ (because of the continuity of the embedding $\mathcal{D}(C^e) \subset X$). It follows from the closed-graph theorem (Theorem 12.1.1) that $(\beta I - A)^{-1}B_1 \in \mathcal{L}(U, \mathcal{D}(C^e))$, and hence $C^e(\beta I - A)^{-1}B_1 \in \mathcal{L}(U, Y)$. It is now clear (using (5.5.1)) that $JB_1 \in \mathcal{L}(U, X_{-1}^{cl})$.

We prove (3). Multiplying (5.5.1) by B_1 from the right and then by the resolvent $(sI - (A + BC))^{-1}$ from the left, we obtain that for all $s \in \rho(A)$,

$$(sI - (A + BC))^{-1}JB_1 = (sI - A)^{-1}B_1 + (sI - (A + BC))^{-1}BC^e(sI - A)^{-1}B_1.$$

Take $u \in L^2([0, \infty); U)$ and define the function $y \in L^2_{\text{loc}}([0, \infty); Y)$ via its Laplace transform:

$$\hat{y}(s) = C^e(sI - A)^{-1}B_1\hat{u}(s) \qquad \forall\, s \in \mathbb{C}_\alpha,$$

where $\alpha > 0$ is such that (5.5.3) holds. According to Lemma 5.4.1 (with $\mathbf{G}(s) = C^e(sI - A)^{-1}B_1$) we have $y \in L^2_\alpha([0,\infty);Y)$, so that indeed $y \in L^2_{\rm loc}([0,\infty);Y)$.

Define the function $z : [0,\infty) \to X^{cl}_{-1}$ by

$$z(t) = \int_0^t \mathbb{T}^{cl}_{t-\sigma} J B_1 u(\sigma) \mathrm{d}\sigma .$$

Using Remark 4.1.9 (with $A+BC$ in place of A) and our earlier formula to express $(sI - (A+BC))^{-1}JB_1$, we obtain that the Laplace transform of z is given by

$$\hat{z}(s) = (sI - A)^{-1}B_1\hat{u}(s) + (sI - (A+BC))^{-1}B\hat{y}(s) \qquad \forall\, s \in \mathbb{C}_\alpha ,$$

whence

$$z(t) = \int_0^t \mathbb{T}_{t-\sigma} B_1 u(\sigma)\mathrm{d}\sigma + \int_0^t \mathbb{T}^{cl}_{t-\sigma} B y(\sigma)\mathrm{d}\sigma .$$

Since B_1 is admissible for $\mathbb{T}$ and B is bounded, it follows that $z \in C([0,\infty);X)$. Remembering the definition of z, this means that JB_1 is admissible for $\mathbb{T}^{cl}$. $\square$

The following simple example is meant to illustrate Proposition 5.5.2 and to highlight the difficulties in identifying a part of X_{-1} with a part of X^{cl}_{-1} (see the discussion before the proposition). This example has been constructed such that there is no natural way to avoid the ambiguity in choosing an extension for C, and we get infinitely many candidates for the operator JB_1. A more substantial example (a boundary controlled convection-diffusion equation) relying on Proposition 5.5.2, where there is a natural way to extend C, will be discussed in Section 10.8.

Example 5.5.3. Let $X = L^2[0,\infty)$ and let $\mathbb{T}$ be the unilateral left shift semigroup on X, as discussed in Example 2.3.7. We have seen in Example 2.10.7 that

$$X_1 = \mathcal{H}^1(0,\infty), \qquad X_{-1} = \mathcal{H}^{-1}(0,\infty), \qquad X_1^d = \mathcal{H}^1_0(0,\infty).$$

We define the admissible observation operator $C \in \mathcal{L}(X_1, \mathbb{C})$ by

$$Cz = z(1).$$

(This is a slight modification of the observation operator from Example 4.4.4.) We define $B \in \mathcal{L}(\mathbb{C}, X)$ by $(B\mathrm{v})(x) = b(x)\mathrm{v}$, where $b \in L^2[0,\infty)$, $b \neq 0$. According to Theorem 5.4.2, $A + BC$ generates a strongly continuous semigroup $\mathbb{T}^{cl}$ on X.

Consider $B_1 \in \mathcal{L}(\mathbb{C}, X_{-1})$ defined by $B_1 = \delta_1$, where

$$\langle \varphi, \delta_1 \rangle_{X_1^d, X_{-1}} = \varphi(1) \qquad \forall\, \varphi \in X_1^d .$$

It is easy to see that B_1 is an admissible control operator for $\mathbb{T}$. To regard B_1 as a control operator for $\mathbb{T}^{cl}$, according to Proposition 5.5.2, we have to find an extension of C, denoted C^e, such that $C^e(sI-A)^{-1}B_1$ makes sense (it should be a bounded operator from $\mathbb{C}$ to $\mathbb{C}$, i.e., a number). We have, for $\operatorname{Re} s > 0$,

$$(sI-A)^{-1}B_1 = \begin{cases} -e^{s(x-1)} & \text{for } x \leqslant 1\,, \\ 0 & \text{for } x > 1\,. \end{cases}$$

A possible way of extending C is by choosing $\mathcal{D}(C^e)$ to be the piecewise $\mathcal{H}^1$ functions, with a possible jump at $x=1$,

$$\mathcal{D}(C^e) = \mathcal{H}^1(0,1) \times \mathcal{H}(1,\infty)\,,$$

and by defining C^e as a combination of the left and right limits at $x=1$,

$$C^e z = \gamma \lim_{x\to 1,\ x<1} z(x) + (1-\gamma) \lim_{x\to 1,\ x>1} z(x)\,,$$

where $\gamma \in \mathbb{R}$. We have

$$C^e(sI-A)^{-1}B_1 = -\gamma \qquad \forall\, s \in \mathbb{C}_0\,,$$

so that all the conditions in Proposition 5.5.2 are satisfied. Thus, JB_1 is an admissible control operator for $\mathbb{T}^{cl}$. Note that each choice of the parameter γ leads to a different operator J in (5.5.1), and hence to a different control operator JB_1 for $\mathbb{T}^{cl}$. If we choose $\gamma = 0$, then the input signal u that enters the system through B_1 never enters the feedback loop, and hence it has no influence on $z(x)$ for $x > 1$.

5.6 Remarks and bibliographical notes on Chapter 5

For papers covering much of the material of this chapter we refer the reader again to Jacob and Partington [112] and Staffans [209, Chapter 10] (see also the bibliographical notes on the previous chapter).

Section 5.1. Theorem 5.1.1 appeared in Hansen and Weiss [89] (in dual form), but important parts of this theorem were present already in Grabowski [73]. Even earlier, some related results for bounded observation operators were contained in Datko [41]. The connection between the Gramian and strong stability has been known long before the papers cited above, usually considering bounded observation or control operators (we cannot trace the first references on this).

Section 5.2. Theorem 5.2.2 is a generalization of Proposition 3.6 in Hansen and Weiss [88], where $\mathbb{T}$ was assumed to be exponentially stable and invertible, and $(sI-A)^{-1}B$ was assumed to be bounded on a right half-plane. The proof in [88] was based in part on a result in Weiss [228]. The alternative proof for Corollary 5.2.4 is due to Zwart [247]. In the latter paper, other admissibility results were given in terms of estimates on $\|C(sI-A)^{-1}\|$, of which we mention the following:

(1) If A and C satisfy (4.3.9), then for every $r \in [1, 2)$ there is a $K_r \geqslant 0$ such that

$$\int_0^1 \|C\mathbb{T}_t z_0\|^r \, \mathrm{d}t \leqslant K_r \|z_0\|^r \qquad \forall\, z_o \in \mathcal{D}(A).$$

(2) A sufficient condition for the admissibility of C is that for some $\alpha > 0$,

$$\|C(sI - A)^{-1}\| \leqslant \frac{K}{\log(\operatorname{Re} s)\sqrt{\operatorname{Re} s}} \qquad \forall\, s \in \mathbb{C}_\alpha .$$

Section 5.3. The admissibility statement in the first part of Theorem 5.3.2 (which is the main part of the theorem) is due to Ho and Russell [100], and the remaining more minor parts appeared in Weiss [226]. Actually, both of these references considered admissibility, not infinite-time admissibility, which is not a big difference. The version of Theorem 5.3.2 for infinite-time admissibility has appeared in Grabowski [74], and this reference provided additional insights, including the following strengthening of Theorem 5.3.9: $C \in \mathcal{L}(X_1, \mathbb{C})$ is an infinite-time admissible observation operator for the diagonal semigroup $\mathbb{T}$ iff there is a $K \geqslant 0$ such that

$$\|C(-\overline{s}I - A)^{-1}\| \leqslant \frac{K}{\sqrt{2\operatorname{Re} s}} \qquad \forall\, s \in \sigma(A).$$

For the “only if” part, the above K is again the constant from (4.6.6).

Theorem 5.3.9 has been generalized to normal semigroups in Weiss [233] (the necessary and sufficient condition for infinite-time admissibility remains the same). This generalization needed a slight generalization of the Carleson measure theorem, in which the Carleson measure μ is defined on the Borel subsets of the *closed* right half-plane. (This is not the generalization that the title of [233] refers to.)

The part of Theorem 5.3.2 which states that (5.3.5) implies $c \in X_{-1}$ can be replaced with a stronger statement: (5.3.5) implies $c \in X_{-\mu}$ for all $\mu > \frac{1}{2}$, where X_μ is defined for all $\mu > 0$ as the completion of X with respect to the norm

$$\|z\|_{-\mu} = \sum_{k\in\mathbb{N}} \frac{|z_k|^2}{(1 + |\lambda_k|^2)^\mu}. \tag{5.6.1}$$

This can be shown by the same elementary method that was employed in the proof of Theorem 5.3.2 for $\mu = 1$. A more general statement (not restricted to diagonal semigroups) appeared in Weiss [230, Remark 3.3] (see also Rebarber and Weiss [188, Theorem 1.4]). The infimum of those $\mu > 0$ for which $C \in X_{-\mu}$ is the degree of unboundedness of C – this follows from Triebel [220, Chapter 1].

The second (converse) part of Theorem 5.3.2 is easy to generalize in the following way. We work in the dual framework, i.e., we talk about admissible control operators. First introduce p-admissibility as the natural generalization of admissibility for the case when the inputs are of class L^p $(1 \leqslant p \leqslant \infty)$ rather than of class L^2. Let $\mathbb{T}$ be a diagonal semigroup on the Banach space l^r, where $1 \leqslant r < \infty$. Let (λ_k) be the sequence of eigenvalues of the generator A of $\mathbb{T}$, with $\operatorname{Re}\lambda_k < 0$. Assume that the sequence $b = (b_k)$ determines an infinite-time p-admissible control operator for $\mathbb{T}$. Denote $q = \frac{p}{p-1}$ (for $p = 1$ we set $q = \infty$). Then there exists $M \geqslant 0$ such that

$$\sum_{-\lambda_k \in R(h,\omega)} |b_k|^r \leqslant M h^{r/q} \qquad \forall\, h > 0, \quad \omega \in \mathbb{R}. \tag{5.6.2}$$

The proof of this is an easy extension of the proof of the corresponding part of Theorem 5.3.2, as has been remarked in [226], with a mistake (p was written in place of q). It is much more delicate to generalize the first part of Theorem 5.3.2. The first result in this direction is in Unteregge [224]. He showed that for $p \leqslant 2$ and $q \leqslant r$, condition (5.6.2) is sufficient for the p-admissibility of b.

Haak [80] has also investigated p-admissibility for diagonal semigroups on l^r. He obtained a sufficient condition for admissibility in the case $q > r$. Using different techniques from [224] (not relying on Fourier transforms) he showed that for analytic diagonal semigroups on l^r, with $1 < p \leqslant r < \infty$, the following condition is equivalent to infinite-time p-admissibility:

$$\sum_{-\lambda_k^{-1} \in R(h,\omega)} \left| \frac{b_k}{\lambda_k} \right|^r \leqslant M h^{r/p} \qquad \forall\, h > 0, \quad \omega \in \mathbb{R}.$$

Admissible observation operators for diagonal semigroups with infinite-dimensional output space. If $C \in \mathcal{L}(X_1, \mathbb{C}^n)$, then it is clear that C is an admissible observation operator for $\mathbb{T}$ iff each of its rows C^j ($j \in \{1, \ldots, n\}$) is admissible. If C maps into an infinite-dimensional Hilbert space Y, then the admissibility question becomes more difficult. Without loss of generality (using an orthonormal basis in Y) we may assume that $Y = l^2$. In Hansen and Weiss [88, 89] Theorem 5.3.2 has been partially generalized to the case when $Y = l^2$. Condition (5.3.5) has to be replaced with

$$\left\| \sum_{-\overline{\lambda}_k \in R(h,\omega)} c_k c_k^* \right\|_{\mathcal{L}(l^2)} \leqslant M h, \tag{5.6.3}$$

where $c_k = C e_k$ is the k column of C (here (e_k) is the standard basis of l^2), so that $c_k c_k^*$ is an infinite matrix of rank one. It was shown in [88] that (5.6.3) is equivalent to the following fact: For every $v \in \mathcal{L}(Y, \mathbb{C})$, vC is an infinite-time admissible observation operator for $\mathbb{T}$. Hence, (5.6.3) is a necessary condition for the infinite-time admissibility of C. It was shown in [88] that (5.6.3) is a sufficient condition for the infinite-time admissibility of C if $\mathbb{T}$ is exponentially stable and invertible (i.e, the eigenvalues λ_k are in a closed vertical strip in the open left half-plane) or exponentially stable and analytic (i.e., there are constants $\rho < 0$ and

$\gamma \geqslant 0$ such that the eigenvalues λ_k satisfy $\operatorname{Re}\lambda_k \leqslant \rho$, $|\operatorname{Im}\lambda_k| \leqslant \gamma|\operatorname{Re}\lambda_k|$). It was shown in [89] that (5.6.3) is sufficient for infinite-time admissibility also for various other classes of diagonal semigroups, that we do not describe here. Another result from [89] is that (5.6.3) is equivalent to the estimate (5.3.14).

It was conjectured in [88] that (5.6.3) is sufficient for the admissibility of $C \in \mathcal{L}(X_1, l^2)$ for every exponentially stable diagonal semigroup. This is false, as follows from results in Nazarov, Treil and Volberg [175]. They have shown that the operator-valued version of the Carleson measure theorem is not true. The paper by Jacob, Partington and Pott [115] contains (among other things) a presentation of the result of [175] in the context of our admissibility problem. Another counterexample for a closely related admissibility conjecture can be found in Zwart, Jacob and Staffans [248], where the semigroup is analytic and compact.

Propositions 5.3.5 and 5.3.7 are new, as far as we know. Proposition 5.3.7 is related to [89, Proposition 6.2]. A generalization of Proposition 5.3.7 to diagonal semigroups on l^r has been given in Haak [80, Theorem 4.1].

The Jacob–Partington theorem. In [230] it has been conjectured that if $\mathbb{T}$ is a strongly continuous semigroup and $C \in \mathcal{L}(X_1, \mathbb{C})$, then the estimate in Corollary 5.3.10 (which is known to follow from admissibility) is also sufficient for the admissibility of C (actually, the dual conjecture was formulated in [230]). In [233] this conjecture has been slightly modified: there it was conjectured that (5.3.14) (which is known to follow from infinite-time admissibility) is also sufficient for the infinite-time admissibility of C. (The version in [233] would imply the version in [230].) In support of the conjecture from [233], it was known that it holds for normal semigroups (see our earlier comments), as well as for exponentially stable and right-invertible semigroups (this follows from Corollary 5.2.4). The conjecture turned out to be false: Jacob and Zwart [121] gave a counterexample using an analytic semigroup.

However, an important positive result in this direction has been obtained by Jacob and Partington [111]: *If $\mathbb{T}$ is a contraction semigroup and $C \in \mathcal{L}(X_1, \mathbb{C})$ is such that* (5.3.14) *holds, then C is infinite-time admissible for* $\mathbb{T}$. This is probably the most important theorem in the area of admissibility. In particular, the parallel result for normal semigroups can be derived from it easily. The proof uses functional models. An alternative proof using dilation theory has been given by Staffans [209]. We cannot reproduce any of these proofs here because it would not be compatible with the elementary nature of this book.

The paper by Jacob, Partington and Pott [116] contains a wealth of new results related to the conjecture mentioned above and to the Jacob–Partington theorem. We mention two of these. The first: *If $\mathbb{T}$ is a bounded strongly continuous semigroup on X, Y is a Hilbert space and $C \in \mathcal{L}(X_1, Y)$, then C satisfies the estimate* (5.3.14) *if and only if there exists $m \geqslant 0$ such that*

$$\frac{1}{\sqrt{\tau}}\left\|\int_0^\tau e^{i\omega t} C\mathbb{T}_t z \,\mathrm{d}t\right\| \leqslant m\|z\| \qquad \forall\, z \in \mathcal{D}(A),\ \tau > 0,\ \omega \in \mathbb{R}.$$

In the above condition, the interval of integration $[0,\tau]$ may be replaced with $[\tau, 2\tau]$. The second result that we quote from [116] is: *Suppose that* $\mathbb{T}$ *is a contraction semigroup on* X*,* Y *is a Hilbert space and* $C \in \mathcal{L}(X_1, Y)$ *satisfies, for some* $k \geqslant 0$*,*

$$\|C(sI-A)^{-1}\|_{HS} \leqslant \frac{k}{\sqrt{\operatorname{Re} s}} \qquad \forall\, s \in \mathbb{C}_0\,.$$

Then C *is an infinite-time admissible observation operator for* $\mathbb{T}$*.* Here, $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt norm.

Sections 5.4 and 5.5. There is a large literature devoted to perturbations of operator semigroups, and each of the books on operator semigroups that we have quoted at the beginning of Chapter 2 covers some results in this direction. We shall only mention references that have results related to our Theorem 5.4.2. Related classes of perturbations were considered in Desch and Schappacher [48], Morris [174], Engel and Nagel [57], Davies [44], and surely we have left out many good references here. The following references consider not only the perturbed semigroup, but also the admissibility of control and observation operators for the perturbed semigroup: Hadd [84], Hansen and Weiss [89], Staffans [209] and Weiss [232]. Actually, Theorem 5.4.2 and parts of Theorem 5.5.2 follow from the (more general) results in [232] and [89, Proposition 4.2]. Proposition 5.4.7 is inspired by Haak, Haase and Kunstmann [81], which contains much more sophisticated results in this direction.

For various generalizations of the concept of an admissible observation operator we refer to Haak and Kunstmann [82] and to Haak and Le Merdy [83].

Chapter 6

Observability

Notation. Throughout this chapter, X and Y are complex Hilbert spaces which are identified with their duals. $\mathbb{T}$ is a strongly continuous semigroup on X, with generator $A : \mathcal{D}(A) \to X$ and growth bound $\omega_0(\mathbb{T})$. Recall from Section 2.10 that X_1 is $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, where $\beta \in \rho(A)$ is fixed.

For $y \in L^2_{\text{loc}}([0,\infty); Y)$ and $\tau \geqslant 0$, the truncation of y to $[0,\tau]$ is denoted by $\mathbf{P}_\tau y$. This function is regarded as an element of $L^2([0,\infty); Y)$ which is zero for $t > \tau$. For every $\tau > 0$, $\mathbf{P}_\tau$ is an operator of norm 1 on $L^2([0,\infty); Y)$.

For any open interval J, the spaces $\mathcal{H}^1(J; Y)$ and $\mathcal{H}^2(J; Y)$ are defined as at the beginning of Chapter 2. $\mathcal{H}^1_{\text{loc}}((0,\infty); Y)$ is defined as the space of those functions on $(0,\infty)$ whose restriction to $(0,n)$ is in $\mathcal{H}^1((0,n); Y)$, for every $n \in \mathbb{N}$. The space $\mathcal{H}^2_{\text{loc}}((0,\infty); Y)$ is defined similarly.

6.1 Some observability concepts

For finite-dimensional LTI systems, we had one concept of observability, see Section 1.4, which was shown to be independent of time. For infinite-dimensional systems, the picture is much more complicated: we have at least three important observability concepts, each depending on time. In this section we introduce these concepts and explore how they are related to each other.

In what follows we assume that Y is a complex Hilbert space and that $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$. Let $\tau > 0$, and let Ψ_τ be the output operator associated with (A, C), which has been introduced in (4.3.1).

Definition 6.1.1. Let $\tau > 0$.

- The pair (A, C) is *exactly observable in time* τ if Ψ_τ is bounded from below.
- (A, C) is *approximately observable in time* τ if $\operatorname{Ker} \Psi_\tau = \{0\}$.
- The pair (A, C) is *final state observable in time* τ if there exists a $k_\tau > 0$ such that $\|\Psi_\tau z_0\| \geqslant k_\tau \|\mathbb{T}_\tau z_0\|$ for all $z_0 \in X$.

It is easy to see (using the density of $\mathcal{D}(A^\infty)$ in X) that the exact observability of (A, C) in time τ is equivalent to the fact that there exists $k_\tau > 0$ such that

$$\int_0^\tau \|C\mathbb{T}_t z_0\|^2 \mathrm{d}t \geqslant k_\tau^2 \|z_0\|^2 \qquad \forall\, z_0 \in \mathcal{D}(A^\infty). \tag{6.1.1}$$

Remark 6.1.2. The following relations among the three observability concepts introduced earlier are easy to verify: Exact observability implies the other two observability concepts. If $\mathbb{T}$ is left-invertible, then (A, C) is exactly observable in time τ iff (A, C) is final state observable in time τ. If $\operatorname{Ker} \mathbb{T}_\tau = \{0\}$ and if (A, C) is final state observable in time τ, then (A, C) is approximately observable in time τ. Note that $\operatorname{Ker} \mathbb{T}_\tau = \{0\}$ holds, in particular, for every diagonalizable semigroup.

Remark 6.1.3. The following very simple observation will be needed several times: Assume that $0 \in \rho(A)$. The pair (A, C) is exactly observable in time τ iff the pair $(A|_{\mathcal{D}(A^2)}, CA)$ (with state space X_1) is exactly observable in time τ. Thus, the exact observability of (A, C) in time τ is equivalent to the estimate

$$\|\dot{y}\|_{L^2([0,\tau];Y)} \geqslant k_\tau \|Az_0\| \qquad \forall\, z_0 \in \mathcal{D}(A^\infty),$$

where z_0 is the initial state and y is the corresponding output signal ($y = \Psi_\tau z_0$). Similar statements hold if we replace exact observability with admissibility or with approximate observability or with final state observability.

Remark 6.1.4. Recall from Section 5.1 that for every $\tau > 0$, $Q_\tau = \Psi_\tau^* \Psi_\tau$ is the observability Gramian (for time τ) of (A, C). It is easy to see that (A, C) is exactly observable in time τ iff $Q_\tau > 0$. Indeed, Ψ_τ is bounded from below iff $\Psi_\tau^* \Psi_\tau > 0$ (see Proposition 12.1.3 in Appendix I). Similarly, it is easy to see that (A, C) is approximately observable in time τ iff $\operatorname{Ker} Q_\tau = \{0\}$.

Remark 6.1.5. It is easy to see that exact observability in time τ is equivalent to the following property: Any initial state $z_0 \in X$ can be expressed from the corresponding truncated output function $y = \Psi_\tau z_0$ via a bounded operator. Indeed, suppose that (A, C) is exactly observable in time τ. By the last remark Q_τ is invertible, and this implies

$$z_0 = Q_\tau^{-1} \Psi_\tau^* y .$$

The converse implication is obvious. Some facts about observability Gramians for finite-dimensional systems were given in Section 1.5. These facts remain valid with very little change for infinite-dimensional systems. For example, Corollary 1.5.10 remains valid (with the same proof) if we insert “exactly” before “observable”.

Approximate observability in time τ is equivalent to the following: For any $z_0 \in X$, if the corresponding truncated output function y is zero, then $z_0 = 0$. The following proposition shows that final state observability in time τ is equivalent to the following: For any initial state $z_0 \in X$, the final state $\mathbb{T}_\tau z_0$ can be expressed from the corresponding truncated output function $y = \Psi_\tau z_0$ via a bounded operator $\mathbf{E}_\tau$.

Proposition 6.1.6. *Suppose that (A, C) is final state observable in time τ. Then there exist operators $\mathbf{E}_\tau \in \mathcal{L}(L^2([0,\infty); Y), X)$ such that*

$$\mathbb{T}_\tau = \mathbf{E}_\tau \Psi_\tau .$$

Any such $\mathbf{E}_\tau$ is called a *final state estimation operator* associated with (A, C).

Proof. If we take in Proposition 12.1.2 $F = (\mathbb{T}_\tau)^*$ and $G = (\Psi_\tau)^*$, we obtain that there exists a bounded operator $L \in \mathcal{L}(X, L^2([0,\infty); U))$ such that $\mathbb{T}_\tau^* = \Psi_\tau^* L$. Taking adjoints, we obtain the desired identity with $\mathbf{E}_\tau = L^*$. □

Often we need the above observability concepts without having to specify the time τ. For this reason we introduce the following.

Definition 6.1.7. (A, C) is *exactly observable* if it is exactly observable in some finite time $\tau > 0$. (A, C) is *approximately observable* if it is approximately observable in some finite time $\tau > 0$. The pair (A, C) is *final state observable* if it is final state observable in some finite time $\tau > 0$.

Remark 6.1.8. If (A, C) is approximatively observable and ϕ is an eigenvector of A, then $C\phi \neq 0$. Indeed, if we had $C\phi = 0$, then it is easy to check that we would have $\Psi\phi = 0$, which contradicts the approximate observability of (A, C).

For some systems described by PDEs, it might be useful to express the approximate observability of (A, C) in terms of $\Psi_\tau z$ for $z \in \mathcal{D}(A^\infty)$ only, as follows.

Proposition 6.1.9. *Suppose that for some $\tau > 0$,*

$$\operatorname{Ker} \Psi_\tau \cap \mathcal{D}(A^\infty) = \{0\} .$$

Then (A, C) is approximately observable in time $\tau + \varepsilon$ for any $\varepsilon > 0$.

Proof. The proof is by contradiction: we assume that the conclusion is false. Then there exists $\varepsilon > 0$ and $z_0 \in X$ such that $z_0 \neq 0$ and $\Psi_{\tau+\varepsilon} z_0 = 0$. We need the operators T_φ introduced in (2.3.6) with $\varphi \in \mathcal{D}(0, \varepsilon)$. By the arguments in the proof of Proposition 2.3.6, φ can be chosen such that $z_1 = T_\varphi z_0 \neq 0$ and we have $z_1 \in \mathcal{D}(A^\infty)$. For all $t \in [0, \tau]$ we have

$$(\Psi_\tau z_1)(t) = C\mathbb{T}_t \int_0^\varepsilon \varphi(\sigma)\mathbb{T}_\sigma z_0 \, \mathrm{d}\sigma = \int_0^\varepsilon \varphi(\sigma)(\Psi z_0)(t + \sigma) \, \mathrm{d}\sigma .$$

Indeed, the last equality is easy to prove for every $z_0 \in \mathcal{D}(A)$, and it remains valid for $z_0 \in X$ by continuous extension.

Since in the above expression, $t + \sigma \in [0, \tau + \varepsilon]$, we have $(\Psi z_0)(t + \sigma) = (\Psi_{\tau+\varepsilon} z_0)(t + \sigma) = 0$, so that $\Psi_\tau z_1 = 0$. This contradicts the assumption of the proposition. □

The conclusion of the above proposition could not be improved even if we replace $\mathcal{D}(A^\infty)$ by $\mathcal{D}(A)$, as the following example shows.

Example 6.1.10. Take $X = \mathbb{C} \times L^2[0,1]$ and consider the operator A defined by

$$A \begin{bmatrix} \varphi \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\mathrm{d}w}{\mathrm{d}x} \end{bmatrix}, \qquad \mathcal{D}(A) = \left\{ \left[\begin{smallmatrix} \varphi \\ w \end{smallmatrix} \right] \in \mathbb{C} \times \mathcal{H}^1(0,1) \;\middle|\; w(1) = \varphi \right\}.$$

We define the observation operator $C : \mathcal{D}(A) \to \mathbb{C}$ by

$$C \begin{bmatrix} \varphi \\ w \end{bmatrix} = w(0).$$

A simple reasoning shows that A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X defined as follows: if $t \geqslant 0$ and $\left[\begin{smallmatrix} \varphi(t) \\ w(t) \end{smallmatrix} \right] = \mathbb{T}_t \left[\begin{smallmatrix} \varphi_0 \\ w_0 \end{smallmatrix} \right]$, then

$$\varphi(t) = \varphi_0, \qquad w(t)(x) = \begin{cases} w_0(x+t) & \text{if } \; x + t < 1, \\ \varphi_0 & \text{else}. \end{cases}$$

The observation operator C is admissible since for almost every $t \leqslant 1$, we have

$$\left(\Psi_1 \begin{bmatrix} \varphi_0 \\ w_0 \end{bmatrix} \right)(t) = w_0(t).$$

It is now easy to see that $\operatorname{Ker} \Psi_1 \cap \mathcal{D}(A) = \{0\}$. This is stronger than the condition in Proposition 6.1.9, so that, according to this proposition, (A, C) is approximately observable in any time $\tau > 1$. In fact, this pair is exactly observable in any time $\tau > 1$. However, (A, C) is not approximately observable in time 1. Indeed, if $\varphi_0 \neq 0$ and $w_0 = 0$, then the corresponding output function is 0 for almost every $t \leqslant 1$.

We know from Proposition 4.3.4 that if $z_0 \in \mathcal{D}(A)$, then $\Psi_\tau z_0 \in \mathcal{H}^1((0,\tau); Y)$. In the proposition below we give a partial converse of this statement (the proposition will be needed in Section 6.4).

Lemma 6.1.11. *Let $y \in \mathcal{H}^1((0,\infty); Y)$ and for every $\varepsilon > 0$ define the function $y_\varepsilon \in \mathcal{H}^1((0,\infty); Y)$ by*

$$y_\varepsilon(t) = \frac{y(t+\varepsilon) - y(t)}{\varepsilon}.$$

Then $\lim_{\varepsilon \to 0} y_\varepsilon = y'$ (the derivative of y) in $L^2([0,\infty); Y)$.

Proof. Let $\mathbb{T}$ be the left shift semigroup on $L^2([0,\infty); Y)$, with a slight generalization of the unilateral left shift semigroup from Example 2.3.7. It is not difficult to verify (by the same reasoning as in Example 2.3.7) that its generator is

$$A = \frac{\mathrm{d}}{\mathrm{d}x}, \qquad \mathcal{D}(A) = \mathcal{H}^1((0,\infty); Y).$$

Therefore, y_ε from the lemma can be written as

$$y_\varepsilon = \frac{1}{\varepsilon}(\mathbb{T}_\varepsilon y - y).$$

Now the lemma follows from the definition of the infinitesimal generator. $\square$

Proposition 6.1.12. *Suppose that* (A, C) *is exactly observable in time* τ_0. *If* $z_0 \in X$ *and* $\tau > \tau_0$ *are such that* $\Psi_\tau z_0 \in \mathcal{H}^1((0,\tau); Y)$, *then* $z_0 \in \mathcal{D}(A)$. *For* $\tau = \tau_0$, *the implication is not true in general.*

Proof. Denote $y = \Psi_\tau z_0$, so that $y \in \mathcal{H}^1((0,\tau); Y)$. Extend y to a function in $\mathcal{H}^1((0,\infty); Y)$ (still denoted by y). It follows from Lemma 6.1.11 that

$$\sup_{\varepsilon\in(0,\tau-\tau_0)} \int_0^{\tau_0} \left\| \frac{y(t+\varepsilon) - y(t)}{\varepsilon} \right\|_Y^2 \mathrm{d}t < \infty .$$

Since, for almost every $t \in [0, \tau_0]$, $y(t+\varepsilon) - y(t) = (\Psi_{\tau_0}(\mathbb{T}_\varepsilon - I)z_0)(t)$, it follows that

$$\sup_{\varepsilon\in(0,\tau-\tau_0)} \left\| \Psi_{\tau_0} \frac{\mathbb{T}_\varepsilon - I}{\varepsilon} z_0 \right\|_{L^2([0,\tau_0];Y)} < \infty .$$

Because of the definition of the exact observability we get that

$$\sup_{\varepsilon\in(0,\tau-\tau_0)} \left\| \frac{\mathbb{T}_\varepsilon - I}{\varepsilon} z_0 \right\|_X < \infty .$$

By Proposition 2.10.10 it follows that $z_0 \in \mathcal{D}(A)$. To see that for $\tau = \tau_0$ the implication is false, consider the left shift semigroup $\mathbb{T}$ on $X = L^2[0,1]$ with point observation at the left end. Thus $A = \frac{d}{d\xi}$, $\mathcal{D}(A) = \left\{x \in \mathcal{H}^1(0,1) \mid x(1) = 0\right\}$ and $Cx = x(0)$. This system is exactly observable in time $T_0 = 1$. However, if $z_0(\xi) = 1$ for all $\xi \in (0,1)$, then $\Psi_1 z_0 \in \mathcal{H}^1(0,1)$ but $z_0 \notin \mathcal{D}(A)$. $\square$

Proposition 6.1.13. *Assume that* (A, C) *is final state observable and* C *is infinite-time admissible for* $\mathbb{T}$. *Then* $\mathbb{T}$ *is exponentially stable.*

Proof. As usual, we denote by Ψ the extended output map of (A, C). Infinite-time admissibility means that $\Psi \in \mathcal{L}(X, L^2([0,\infty); Y))$, so that there exists $K > 0$ with

$$\int_0^\infty \|(\Psi z_0)(t)\|^2 \mathrm{d}t \leqslant K^2 \|z_0\|^2 \qquad \forall\, z_0 \in X .$$

Final state observability means that there exist $\tau > 0$ and $k_\tau > 0$ such that

$$\|\Psi_\tau z_0\| \geqslant k_\tau \|\mathbb{T}_\tau z_0\| \qquad \forall\, z_0 \in X .$$

Notice that his implies that for every $T \geqslant 0$,

$$\int_T^{\tau+T} \|(\Psi z_0)(t)\|^2 \mathrm{d}t = \int_0^\tau \|(\Psi_\tau \mathbb{T}_T z_0)(t)\|^2 \geqslant k_\tau^2 \|\mathbb{T}_{\tau+T} z_0\|^2 \qquad \forall\, z_0 \in X .$$

Hence,

$$K^2 \|z_0\|^2 \geqslant \sum_{k\in\mathbb{N}} \int_{(k-1)\tau}^{k\tau} \|(\Psi z_0)(t)\|^2 \mathrm{d}t \geqslant k_\tau^2 \sum_{k\in\mathbb{N}} \|\mathbb{T}_{k\tau} z_0\|^2 . \tag{6.1.2}$$

In particular, we see from the above that $\|\mathbb{T}_{k\tau}\| \leqslant \frac{K}{k_\tau}$ for every $k \in \mathbb{N}$ (and this holds also for $k = 0$). Hence, for every $n \in \mathbb{N}$ and every $z_0 \in X$,

$$\|\mathbb{T}_{n\tau} z_0\|^2 = \frac{1}{n} \sum_{k=1}^{n} \|\mathbb{T}_{(n-k)\tau} \mathbb{T}_{k\tau} z_0\|^2 \leqslant \frac{K^2}{n k_\tau^2} \sum_{k=1}^{n} \|\mathbb{T}_{k\tau} z_0\|^2 .$$

By (6.1.2) we get that

$$\|\mathbb{T}_{n\tau} z_0\|^2 \leqslant \frac{K^2}{n k_\tau^2} \cdot \frac{K^2}{k_\tau^2} \|z_0\|^2 ,$$

whence $\|\mathbb{T}_{n\tau}\| < 1$ for some large n. According to the definition (2.1.3) of the growth bound, $\mathbb{T}$ is exponentially stable. $\square$

The following corollary is known as *Datko's theorem*.

Corollary 6.1.14. *The semigroup* $\mathbb{T}$ *has the property*

$$\int_0^\infty \|\mathbb{T}_t z_0\|^2 \mathrm{d}t < \infty \qquad \forall\, z_0 \in X$$

if and only if it is exponentially stable.

Proof. The "if" part is obvious. To prove the "only if" part, first notice that the condition in the corollary implies that there exists $K > 0$ such that

$$\int_0^\infty \|\mathbb{T}_t z_0\|^2 \mathrm{d}t \leqslant K^2 \|z_0\|^2 \qquad \forall\, z_0 \in X .$$

This follows from the closed-graph theorem, applied to the operator that maps z_0 into the function $t \mapsto \mathbb{T}_t z_0$. Hence, the identity I is an infinite-time admissible observation operator for $\mathbb{T}$ (with the output space X).

Take $\tau > 0$ and let $M \geqslant 1$ be such that $\|\mathbb{T}_t\| \leqslant M$ for all $t \in [0, \tau]$. Then

$$\|\mathbb{T}_\tau z_0\|^2 = \frac{1}{\tau} \int_0^\tau \|\mathbb{T}_{\tau - t} \mathbb{T}_t z_0\|^2 \mathrm{d}t \leqslant \frac{M^2}{\tau} \int_0^\tau \|\mathbb{T}_t z_0\|^2 \mathrm{d}t \qquad \forall\, z_0 \in X ,$$

which shows that (A, I) is final state observable in time τ. Now we can apply Proposition 6.1.13 to conclude that $\mathbb{T}$ is exponentially stable. $\square$

Proposition 6.1.15. *Suppose that* (A, C) *is exactly observable and that*

$$\lim_{\eta \to 0} \|\Psi_\eta\| = 0 .$$

Then $\mathbb{T}$ *is bounded from below (i.e., left-invertible).*

Proof. Let $\tau_0 > 0$ and $k > 0$ be such that $\|\Psi_{\tau_0} z\| \geqslant k\|z\|$ for all $z \in X$. We have for all $\eta \in (0, \tau_0)$ and $z \in \mathcal{D}(A)$, using the dual composition property (4.3.2), that

$$k^2\|z\|^2 \leqslant \|\Psi_{\tau_0} z\|^2 = \|\Psi_\eta z\|^2 + \|\Psi_{\tau_0-\eta}\mathbb{T}_\eta z\|^2 \leqslant \|\Psi_\eta\|^2\|z\|^2 + \|\Psi_{\tau_0}\|^2\|\mathbb{T}_\eta z\|^2$$

(we have used that $\|\Psi_{\tau_0-\eta}\| \leqslant \|\Psi_{\tau_0}\|$). Hence we have that

$$\|\mathbb{T}_\eta z\|^2 \geqslant \frac{k^2 - \|\Psi_\eta\|^2}{\|\Psi_{\tau_0}\|^2}\|z\|^2 .$$

For η sufficiently small, the above fraction becomes positive. $\square$

6.2 Some examples based on the string equation

In this section we give several simple examples of exactly observable systems based on the string equation, as discussed in Examples 2.7.13 and 2.7.15.

Denote $X = \mathcal{H}_0^1(0,\pi) \times L^2[0,\pi]$ and $A : \mathcal{D}(A) \to X$ is defined by

$$\mathcal{D}(A) = \left[\mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi)\right] \times \mathcal{H}_0^1(0,\pi), \tag{6.2.1}$$

$$A\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A). \tag{6.2.2}$$

Define $\varphi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$, for every $n \in \mathbb{Z}^*$. We recall from Example 2.7.13 that the family $(\phi_n)_{n\in\mathbb{Z}^*}$, defined by

$$\phi_n = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{in}\varphi_n \\ \varphi_n \end{bmatrix} \qquad \forall\, n \in \mathbb{Z}^*, \tag{6.2.3}$$

is an orthonormal basis in X formed by eigenvectors of A and that the corresponding eigenvalues are $\lambda_n = in$, with $n \in \mathbb{Z}^*$. We also recall from Example 2.7.13 that A generates a unitary group $\mathbb{T}$ on X, which is given by

$$\mathbb{T}_t\begin{bmatrix} f \\ g \end{bmatrix} = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}^*} e^{int}\left(\frac{i}{n}\left\langle \frac{\mathrm{d}f}{\mathrm{d}x}, \frac{\mathrm{d}\varphi_n}{\mathrm{d}x}\right\rangle_{L^2[0,\pi]} + \langle g, \varphi_n\rangle_{L^2[0,\pi]}\right)\phi_n . \tag{6.2.4}$$

Recall from Remark 2.7.14 that the interpretation in terms of PDEs of the above facts is the following: For $f \in \mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi)$ and $g \in \mathcal{H}_0^1(0,\pi)$, there exists a unique function w continuous from $[0,\infty)$ to $\mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi)$ and continuously differentiable from $[0,\infty)$ to $\mathcal{H}_0^1(0,\pi)$, satisfying (2.7.3).

Our first result concerns the string equation with Neumann boundary observation.

Proposition 6.2.1. *Let $X = \mathcal{H}_0^1(0,\pi) \times L^2[0,\pi]$ and let A be the operator defined by (6.2.1), (6.2.2). Denote $Y = \mathbb{C}$ and consider the observation operator $C \in \mathcal{L}(X_1, Y)$ defined by*

$$C \begin{bmatrix} f \\ g \end{bmatrix} = \frac{\mathrm{d}f}{\mathrm{d}x}(0) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A). \tag{6.2.5}$$

Then the pair (A, C) is exactly observable in any time $\tau \geqslant 2\pi$. For $\tau < 2\pi$, the pair (A, C) is not approximately observable in time τ.

Proof. By using formulas (6.2.3) and (6.2.4), we have that, for all $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{D}(A)$,

$$C\mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}^*} e^{int} \left(\left\langle \frac{\mathrm{d}f}{\mathrm{d}x}, \psi_n \right\rangle_{L^2[0,\pi]} - i\langle g, \varphi_n \rangle_{L^2[0,\pi]} \right), \tag{6.2.6}$$

where $\psi_n(x) = \sqrt{\frac{2}{\pi}} \cos(nx)$ for all $n \in \mathbb{Z}$. The above formula and the orthogonality of the family $(e^{int})_{n\in\mathbb{Z}^*}$ in $L^2[0, 2\pi]$ imply that

$$\int_0^{2\pi} \left| C\mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} \right|^2 \mathrm{d}t = \sum_{n\in\mathbb{Z}^*} \left| \left\langle \frac{\mathrm{d}f}{\mathrm{d}x}, \psi_n \right\rangle_{L^2[0,\pi]} - i\langle g, \varphi_n \rangle_{L^2[0,\pi]} \right|^2 . \tag{6.2.7}$$

Since $\varphi_{-n} = -\varphi_n$ and $\psi_{-n} = \psi_n$, from (6.2.7) it follows that

$$\int_0^{2\pi} \left| C\mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} \right|^2 \mathrm{d}t = 2\sum_{n\in\mathbb{N}} \left(\left| \left\langle \frac{\mathrm{d}f}{\mathrm{d}x}, \psi_n \right\rangle_{L^2[0,\pi]} \right|^2 + \left| \langle g, \varphi_n \rangle_{L^2[0,\pi]} \right|^2 \right) .$$

The above relation, together with the facts that $(\psi_n)_{n\geqslant 0}$ and $(\varphi_n)_{n\geqslant 1}$ are orthonormal bases in $L^2[0,\pi]$, implies that

$$\int_0^{2\pi} \left| C\mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} \right|^2 \mathrm{d}t = 2 \left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|^2 \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A).$$

This clearly implies that C is an admissible observation operator for $\mathbb{T}$ and that (A, C) is exactly observable in any time $\tau \geqslant 2\pi$.

In order to show that (A, C) is not approximately observable in any time $\tau < 2\pi$, we first notice that from (6.2.6) it follows, by density, that the output map Ψ_τ of (A, C) is given by the right-hand side of (6.2.6), for every $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in X$. On the other hand, for $0 < \tau < 2\pi$ we take $F \in L^2[0, 2\pi]$, $F \not\equiv 0$, satisfying $F(t) = 0$ for $t \in [0, \tau]$ and $\int_0^{2\pi} F(t)\,\mathrm{d}t = 0$. It follows that there exists a sequence $c = (c_n)_{n\in\mathbb{Z}^*} \in l^2$, $c \neq 0$, such that

$$F(t) = \sum_{n\in\mathbb{Z}^*} c_n e^{int},$$

the convergence being understood in $L^2[0, 2\pi]$. Using the fact, easy to check, that for every sequence $c \in l^2(\mathbb{Z}^*)$ different from zero there exists $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in X \setminus \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ such that

$$\left\langle \frac{\mathrm{d}f}{\mathrm{d}x}, \psi_n \right\rangle_{L^2[0,\pi]} - i\langle g, \varphi_n\rangle_{L^2[0,\pi]} = \sqrt{2\pi}\, c_n \qquad \forall\, n \in \mathbb{Z}^*,$$

it follows that there exists $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in X \setminus \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ such that $\Psi_\tau \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] = 0$ in $L^2[0, \tau]$. Thus the pair (A, C) is not approximately observable in any time $\tau < 2\pi$. □

Remark 6.2.2. In terms of PDEs, the above proposition can be restated as follows: For every $\tau \geqslant 2\pi$ there exists $k_\tau > 0$ such that the solution w of (2.7.3) satisfies

$$\int_0^\tau \left| \frac{\partial w}{\partial x}(0, t) \right|^2 \mathrm{d}t \geqslant k_\tau^2 \left(\|f\|^2_{\mathcal{H}_0^1(0,\pi)} + \|g\|^2_{L^2[0,\pi]} \right) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X_1 .$$

Moreover, the above estimate is false for every $\tau < 2\pi$ and $k_\tau > 0$.

The next example concerns the string equation with distributed observation.

Proposition 6.2.3. *Let $X = \mathcal{H}_0^1(0, \pi) \times L^2[0, \pi]$ and let A be the operator defined by* (6.2.1), (6.2.2). *Denote $Y = L^2[0, \pi]$, take $\xi, \eta \in [0, \pi]$ with $\xi < \eta$ and consider the observation operator $C \in \mathcal{L}(X_1, Y)$ defined by*

$$C \begin{bmatrix} f \\ g \end{bmatrix} = g\chi_{[\xi,\eta]} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X_1 , \tag{6.2.8}$$

where $\chi_{[\xi,\eta]}$ is the characteristic function of $[\xi, \eta] \subset [0, \pi]$.

Then the pair (A, C) is exactly observable in any time $\tau \geqslant 2\pi$.

Proof. Since C is bounded, it is an admissible observation operator for $\mathbb{T}$. Moreover, following the same steps as in the proof of Proposition 6.2.1 we obtain that

$$\int_0^{2\pi} \left\| C\mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} \right\|^2 \mathrm{d}x\, \mathrm{d}t = \frac{1}{4} \sum_{n\in\mathbb{Z}^*} \left| i \left\langle \frac{\mathrm{d}f}{\mathrm{d}x}, \psi_n \right\rangle_{L^2[0,\pi]} + \langle g, \varphi_n\rangle_{L^2[0,\pi]} \right|^2 \int_\xi^\eta |\varphi_n|^2\, \mathrm{d}x .$$

The sequence $n \mapsto \int_\xi^\eta |\varphi_n(x)|^2\, \mathrm{d}x$ converges to $\frac{1}{2}(\eta - \xi)$, hence it is bounded away from zero. From here we can deduce, using a similar reasoning as in the proof of Proposition 6.2.1, that (A, C) is exactly observable in any time $\tau \geqslant 2\pi$. □

Remark 6.2.4. If we consider again the initial and boundary value problem (2.7.3), the last proposition implies that for every $\tau \geqslant 2\pi$ there exists $k_\tau > 0$ such that

$$\int_0^\tau \int_\xi^\eta \left| \frac{\partial w}{\partial t}(x, t) \right|^2 \mathrm{d}x\, \mathrm{d}t \geqslant k_\tau^2 \left(\|f\|^2_{\mathcal{H}_0^1(0,\pi)} + \|g\|^2_{L^2[0,\pi]} \right) \tag{6.2.9}$$

holds for every $f \in \mathcal{H}^2(0, \pi) \cap \mathcal{H}_0^1(0, \pi)$ and $g \in \mathcal{H}_0^1(0, \pi)$.

In the remaining part of this section we consider a related but different semigroup, corresponding to a vibrating string of length π with a Neumann boundary condition at $x = 0$, as discussed in Example 2.7.15. We denote $X = \mathcal{H}^1_R(0,\pi) \times L^2[0,\pi]$, where

$$\mathcal{H}^1_R(0,\pi) = \{f \in \mathcal{H}^1(0,\pi) \mid f(\pi) = 0\},$$

with the inner product as in (2.7.4), and $A : \mathcal{D}(A) \to X$ is defined by

$$\mathcal{D}(A) = \left\{ f \in \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_R(0,\pi) \;\middle|\; \frac{\mathrm{d}f}{\mathrm{d}x}(0) = 0 \right\} \times \mathcal{H}^1_R(0,\pi), \tag{6.2.10}$$

$$A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A). \tag{6.2.11}$$

For $n \in \mathbb{N}$, denote $\varphi_n(x) = \sqrt{\frac{2}{\pi}} \cos\left[\left(n - \frac{1}{2}\right) x\right]$ and $\mu_n = n - \frac{1}{2}$. If $-n \in \mathbb{N}$ we set $\varphi_n = -\varphi_{-n}$ and $\mu_n = -\mu_{-n}$. We recall from Example 2.7.15 that the family $(\phi_n)_{n\in\mathbb{Z}^*}$, defined by

$$\phi_n = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{i\mu_n}\varphi_n \\ \varphi_n \end{bmatrix} \qquad \forall\, n \in \mathbb{Z}^*, \tag{6.2.12}$$

is an orthonormal basis in X formed by eigenvectors of A and the corresponding eigenvalues are $\lambda_n = i\mu_n$, with $n \in \mathbb{Z}^*$. We also recall from Example 2.7.15 that A generates a unitary group $\mathbb{T}$ on X, which is given by

$$\mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}^*} e^{i\mu_n t} \left(\frac{i}{\mu_n} \left\langle \frac{\mathrm{d}f}{\mathrm{d}x}, \frac{\mathrm{d}\varphi_n}{\mathrm{d}x} \right\rangle_{L^2[0,\pi]} + \langle g, \varphi_n \rangle_{L^2[0,\pi]} \right) \phi_n . \tag{6.2.13}$$

The interpretation of $\mathbb{T}$ in terms of PDEs has been discussed starting with (2.7.7).

Proposition 6.2.5. *Let $X = \mathcal{H}^1_R(0,\pi) \times L^2[0,\pi]$ and let A be the operator defined by* (6.2.10), (6.2.11). *Consider the observation operator $C \in \mathcal{L}(X_1, \mathbb{C})$ defined by*

$$C \begin{bmatrix} f \\ g \end{bmatrix} = g(0) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A). \tag{6.2.14}$$

Then C is an admissible observation operator for the semigroup $\mathbb{T}$ generated by A and the pair (A, C) is exactly observable in any time $\tau \geqslant 2\pi$. For $\tau < 2\pi$, the pair (A, C) is not approximatively observable in time τ.

Proof. By using formulas (6.2.12) and (6.2.13), we have that, for all $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{D}(A)$,

$$C\mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} = \frac{1}{\sqrt{\pi}} \sum_{n\in\mathbb{Z}^*} e^{i\mu_n t} \left(\frac{i}{\mu_n} \left\langle \frac{\mathrm{d}f}{\mathrm{d}x}, \frac{\mathrm{d}\varphi_n}{\mathrm{d}x} \right\rangle_{L^2[0,\pi]} + \langle g, \varphi_n \rangle_{L^2[0,\pi]} \right) . \tag{6.2.15}$$

The above formula and the orthogonality of the family $(e^{i\mu_n t})_{n\in\mathbb{Z}^*}$ in $L^2[0,2\pi]$ imply that

$$\int_0^{2\pi}\left|C\mathbb{T}_t\begin{bmatrix}f\\g\end{bmatrix}\right|^2\mathrm{d}t = \sum_{n\in\mathbb{Z}^*}\left|\frac{i}{\mu_n}\left\langle\frac{\mathrm{d}f}{\mathrm{d}x},\frac{\mathrm{d}\varphi_n}{\mathrm{d}x}\right\rangle_{L^2[0,\pi]} + \langle g,\varphi_n\rangle_{L^2[0,\pi]}\right|^2. \qquad (6.2.16)$$

Since $\varphi_{-n} = -\varphi_n$ and $\mu_{-n} = \mu_n$, from (6.2.16) it follows that

$$\int_0^{2\pi}\left|C\mathbb{T}_t\begin{bmatrix}f\\g\end{bmatrix}\right|^2\mathrm{d}t = 2\sum_{n\in\mathbb{N}}\left(\frac{1}{\mu_n^2}\left|\left\langle\frac{\mathrm{d}f}{\mathrm{d}x},\frac{\mathrm{d}\varphi_n}{\mathrm{d}x}\right\rangle_{L^2[0,\pi]}\right|^2 + \left|\langle g,\varphi_n\rangle_{L^2[0,\pi]}\right|^2\right).$$

The above relation, together with the facts that $\left(\frac{1}{\mu_n}\frac{\mathrm{d}\varphi_n}{\mathrm{d}x}\right)_{n\in\mathbb{N}}$ and $(\varphi_n)_{n\in\mathbb{N}}$ are orthonormal in $L^2[0,\pi]$, implies that

$$\int_0^{2\pi}\left|C\mathbb{T}_t\begin{bmatrix}f\\g\end{bmatrix}\right|^2\mathrm{d}t = 2\left\|\begin{bmatrix}f\\g\end{bmatrix}\right\|^2 \qquad \forall\begin{bmatrix}f\\g\end{bmatrix}\in\mathcal{D}(A).$$

This clearly implies that C is an admissible observation operator for $\mathbb{T}$ and that (A,C) is exactly observable in any time $\tau \geqslant 2\pi$.

In order to show that (A,C) is not approximately observable in any time $\tau < 2\pi$, we can follow the same steps as in the proof of the similar result in Proposition 6.2.1, so that we skip the details. □

Remark 6.2.6. In terms of PDEs, the above proposition can be restated as follows: For every $\tau \geqslant 2\pi$ there exists $k_\tau > 0$ such that the solution w of (2.7.7) satisfies

$$\int_0^{\tau}\left|\frac{\partial w}{\partial t}(0,t)\right|^2\mathrm{d}t \geqslant k_\tau^2\left(\|f\|^2_{\mathcal{H}^1(0,\pi)} + \|g\|^2_{L^2[0,\pi]}\right) \qquad \forall\begin{bmatrix}f\\g\end{bmatrix}\in X_1.$$

Moreover, the above estimate is false for every $\tau < 2\pi$ and $k_\tau > 0$.

Let us compute the space X_{-1} for the generator A defined in (6.2.10) and (6.2.11). For this, notice that A fits the framework of Proposition 3.7.6, with $H = L^2[0,\pi]$,

$$H_1 = \left\{f\in\mathcal{H}^2(0,\pi)\cap\mathcal{H}^1_R(0,\pi)\;\middle|\;\frac{\mathrm{d}f}{\mathrm{d}x}(0) = 0\right\},$$

$A_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $H_{\frac12} = \mathcal{H}^1_R(0,\pi)$. According to Proposition 3.7.6, $X_{-1} = H\times H_{-\frac12}$, where

$$H_{-\frac12} = \left(\mathcal{H}^1_R(0,\pi)\right)'$$

(the dual of $\mathcal{H}^1_R(0,\pi)$ with respect to the pivot space $L^2[0,\pi]$). We would like to have a more concrete description of the space $H_{-\frac12}$. For this, define $q:[0,\pi]\to\mathbb{C}$ by

$$q(x) = \frac{\pi - x}{\pi}$$

and notice that every $f \in \mathcal{H}^1_R(0,\pi)$ has the orthogonal decomposition

$$f(x) = f_0(x) + f(0)q(x), \qquad f_0 \in \mathcal{H}^1_0(0,\pi).$$

Hence, every $v \in H_{-\frac{1}{2}}$ can be thought of as a pair $(v_0, \alpha) \in \mathcal{H}^{-1}(0,\pi) \times \mathbb{C}$ such that

$$\langle v, f\rangle_{H_{-\frac{1}{2}},H_{\frac{1}{2}}} = \langle (v_0,\alpha), f_0 + f(0)q\rangle_{H_{-\frac{1}{2}},H_{\frac{1}{2}}} = \langle v_0, f_0\rangle_{\mathcal{H}^{-1},\mathcal{H}^1_0} + \alpha\overline{f}(0).$$

In the next proposition and its proof, we deviate from our habit of denoting extensions of an operator by the same symbol as the original operator.

Corollary 6.2.7. *We denote by* $\tilde{\mathbb{T}}$ *the extension of the operator semigroup from Proposition* 6.2.5 *to the space* $X_{-1} = L^2[0,\pi] \times (\mathcal{H}^1_R(0,\pi))'$, *so that its generator is* $\tilde{A} : X \to X_{-1}$, *an extension of* A *from Proposition* 6.2.5. *We define the observation operator* $\tilde{C} \in \mathcal{L}(X,\mathbb{C})$ *by*

$$\tilde{C}\begin{bmatrix} f \\ g \end{bmatrix} = f(0) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X. \tag{6.2.17}$$

Then $\tilde{C}$ *is an admissible observation operator for* $\tilde{\mathbb{T}}$ *and the pair* $(\tilde{A},\tilde{C})$ *is exactly observable in any time* $\tau \geqslant 2\pi$. *For* $\tau < 2\pi$, *the pair* $(\tilde{A},\tilde{C})$ *is not approximatively observable in time* τ.

This corollary follows from Proposition 6.2.5 together with Remark 6.1.3 (with X_{-1} in place of X). Note that in terms of PDEs, the first part of the conclusion of the above corollary can be restated as follows: For every $\tau \geqslant 2\pi$ there exists $k_\tau > 0$ such that the solution w of (2.7.7) satisfies

$$\int_0^\tau |w(0,t)|^2 \,\mathrm{d}t \geqslant k_\tau^2 \left(\|f\|^2_{L^2[0,\pi]} + \|g\|^2_{(\mathcal{H}^1_R(0,\pi))'}\right) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X_1.$$

6.3 Robustness of exact observability with respect to admissible perturbations of the generator

In this section we show that if (A, C_1) is exactly observable in time τ, then for certain possibly unbounded perturbations P, the pair $(A+P, C_1)$ is again exactly observable in time τ. We decompose $P = BC$, with $C = DC_1 + C_2$, where B, D are bounded and C_2 is admissible (like C_1). We show that if C_2 is small in a suitable sense, then exact observability is preserved. The operator B could be omitted from this theory without loss of generality (by taking $B = I$). However, we have included it, because its presence corresponds more to the engineering intuition, where the output $y = Cz$ is in a different space from the state. We also include a version of our main result where C_2 is only small on an $(A+P)$-invariant subspace

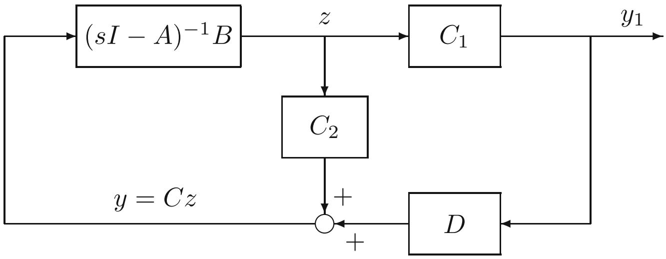

Figure 6.1: The block diagram of the system $\dot{z} = Az + By$ with the feedback $y = Cz$, where $C = DC_1 + C_2$. If (A, C_1) is exactly observable and C_2 is sufficiently small, then $(A + BC, C_1)$ is exactly observable.

of X, and we conclude that the exact observability estimate remains true on this subspace. This system is shown as a block diagram in Figure 6.1.

As usual in this chapter, $\mathbb{T}$ will denote a strongly continuous semigroup on X, with generator A, $X_1 = \mathcal{D}(A)$ with the graph norm, and Y_1, Y are other Hilbert spaces. For every $\tau > 0$, we introduce the following norm on the space of all admissible observation operators in $\mathcal{L}(X_1, Y)$:

$$|||C|||_\tau = \sup_{\|z_0\| \leqslant 1} \left(\int_0^\tau \|\Psi z_0(t)\|^2 \, \mathrm{d}t \right)^{\frac{1}{2}} = \|\Psi_\tau\|_{\mathcal{L}(X, L^2([0,\infty); Y))},$$

where Ψ and Ψ_τ are as in Section 4.3. If $C \in \mathcal{L}(X_1, Y)$ is not admissible, then we set $|||C|||_\tau = \infty$. This norm is useful for estimating the norm of an input-output operator on the interval $[0, \tau]$, as the following proposition shows.

Proposition 6.3.1. *Suppose that $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$, U is a Hilbert space and $B \in \mathcal{L}(U, Y)$. Let $\mathbb{F}_\omega$ be the input-output map associated with the transfer function $C(sI - A)^{-1}B$, as in Lemma* 5.4.1. *We regard $\mathbf{P}_\tau \mathbb{F}_\omega = \mathbf{P}_\tau \mathbb{F}_\omega \mathbf{P}_\tau$ as an operator in $\mathcal{L}(L^2([0, \tau]; U), L^2([0, \tau]; Y))$. Then*

$$\|\mathbf{P}_\tau \mathbb{F}_\omega\|_{\mathcal{L}(L^2[0,\tau])} \leqslant \sqrt{\tau}\, |||C|||_\tau \cdot \|B\|.$$

Proof. As in the first step of the proof of Theorem 5.4.2, we consider

$$u \in \mathcal{H}^1_{\mathrm{comp}}((0, \infty); U).$$

Then we can see that $\mathbb{F}_\omega u$ is independent of ω and it is a continuous Y-valued function given by

$$(\mathbb{F}_\omega u)(t) = C \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) \, \mathrm{d}\sigma \qquad \forall\, t \geqslant 0.$$

We denote by Ψ the extended output map of (A, C). Let $\varphi \in L^2([0, \tau]; Y)$. We have, using Fubini's theorem,

$$\begin{aligned}\langle \mathbb{F}_\omega u, \varphi\rangle_{L^2([0,\tau];Y)} &= \int_0^\tau \int_0^t \langle [\Psi B u(\sigma)]\,(t-\sigma), \varphi(t)\rangle_Y \,\mathrm{d}\sigma\,\mathrm{d}t \\ &= \int_0^\tau \int_0^{\tau-\sigma} \langle [\Psi B u(\sigma)]\,(\mu), \varphi(\mu+\sigma)\rangle_Y \,\mathrm{d}\mu\,\mathrm{d}\sigma\,.\end{aligned}$$

Applying the Cauchy–Schwarz inequality for the integral with respect to μ, we obtain

$$\begin{aligned}|\langle \mathbb{F}_\omega u, \varphi\rangle_{L^2([0,\tau];Y)}| &\leqslant \int_0^\tau \|\Psi B u(\sigma)\|_{L^2([0,\tau];Y)} \cdot \|\varphi\|_{L^2([0,\tau];Y)}\,\mathrm{d}\sigma \\ &\leqslant \|\!|C|\!\|_\tau \cdot \|B\| \cdot \|\varphi\|_{L^2([0,\tau];Y)} \int_0^\tau \|u(\sigma)\|_U \,\mathrm{d}\sigma\,.\end{aligned}$$

Since this is true for every $\varphi \in L^2([0,\tau];Y)$, we conclude that

$$\|\mathbf{P}_\tau \mathbb{F}_\omega u\| \leqslant \|\!|C|\!\|_\tau \cdot \|B\| \cdot \|u\|_{L^1([0,\tau];U)} \leqslant \|\!|C|\!\|_\tau \cdot \|B\| \cdot \sqrt{\tau} \cdot \|u\|_{L^2([0,\tau];U)}\,.$$

Since $\mathcal{H}^1([0,\tau];U)$ is dense in $L^2([0,\tau];U)$, our claim follows. □

Theorem 6.3.2. *Suppose that $C_1 \in \mathcal{L}(X_1, Y_1)$ is an admissible observation operator for $\mathbb{T}$ and (A, C_1) is exactly observable in time $\tau > 0$; i.e., there exists $k_\tau > 0$ such that*

$$\int_0^\tau \|C_1 \mathbb{T}_t z_0\|^2 \,\mathrm{d}t \geqslant k_\tau^2 \|z_0\|^2 \qquad \forall\, z_0 \in \mathcal{D}(A)\,.$$

Let $B \in \mathcal{L}(Y, X)$ and $D \in \mathcal{L}(Y_1, Y)$. If $C_2 \in \mathcal{L}(X_1, Y)$ satisfies

$$\|\!|C_2|\!\|_\tau \leqslant \frac{k_\tau}{\sqrt{\tau}\,\|B\|\,(\|\!|C_1|\!\|_\tau + k_\tau)}\,, \tag{6.3.1}$$

then denoting

$$C = DC_1 + C_2\,,$$

we have that $(A + BC, C_1)$ is exactly observable in time τ.

Proof. We know from Theorem 5.4.2 that $A + BC$, with $\mathcal{D}(A + BC) = \mathcal{D}(A)$, generates a strongly continuous semigroup $\mathbb{T}^{cl}$ on X. From the same theorem we also know that C_1 and C_2 (and hence also C) are admissible for $\mathbb{T}^{cl}$ (both of these statements are true regardless if the estimate (6.3.1) holds).

Our plan is to consider first the case $D = 0$, which means that $C = C_2$, and to determine a sufficient condition for $(A + BC_2, C_1)$ to be exactly observable in time τ. Afterwards, we show that the additional feedback through D has no influence on the exact observability. We shall use the notation C in place of C_2.

As in Theorem 5.4.2 and Proposition 5.4.3, we use the following notation: Ψ and Ψ^1 are the extended output maps of (A, C) and (A, C_1), respectively. Similarly,

Ψ^{cl} and $\Psi^{1,cl}$ are the extended output maps of $(A+BC, C)$ and $(A+BC, C_1)$, respectively. All these operators can be truncated to the interval $[0,\tau]$, and then they get a subscript τ, as in Section 4.3. Thus, for example, $\Psi_\tau^{1,cl} = \mathbf{P}_\tau \Psi^{1,cl}$, where $\mathbf{P}_\tau$ is as in Chapter 4. The operators $\mathbb{F}_\omega$ and $\mathbb{F}_\omega^1$ are the input-output maps associated with the transfer functions $C(sI-A)^{-1}B$ and $C_1(sI-A)^{-1}B$, respectively, and they are defined on $L^2_\omega([0,\infty);Y)$, where $\omega > \omega_0(\mathbb{T})$.

We know from Proposition 5.4.3 that

$$\Psi^{cl} = (I - \mathbb{F}_\omega)^{-1}\Psi, \qquad \Psi^{1,cl} = \Psi^1 + \mathbb{F}_\omega^1 \Psi^{cl}. \tag{6.3.2}$$

From the causality of $\mathbb{F}_\omega^1$ (see (5.4.3)) we know that $\mathbf{P}_\tau \mathbb{F}_\omega^1 = \mathbf{P}_\tau \mathbb{F}_\omega^1 \mathbf{P}_\tau$. Using this, we apply $\mathbf{P}_\tau$ to both sides of the second equation in (6.3.2) to obtain

$$\Psi_\tau^{1,cl} = \Psi_\tau^1 + \mathbf{P}_\tau \mathbb{F}_\omega^1 \Psi_\tau^{cl}. \tag{6.3.3}$$

If we regard $\mathbf{P}_\tau \mathbb{F}_\omega^1$ as an operator from $L^2([0,\tau];Y)$ to $L^2([0,\tau];Y_1)$, then according to Proposition 6.3.1 it satisfies $\|\mathbf{P}_\tau \mathbb{F}_\omega^1\| \leqslant \sqrt{\tau}\, |||C_1|||_\tau \|B\|$. Hence, for every $z_0 \in X$,

$$\begin{aligned} \|\Psi_\tau^{1,cl} z_0\| &\geqslant \|\Psi_\tau^1 z_0\| - \|\mathbf{P}_\tau \mathbb{F}_\omega^1\| \cdot \|\Psi_\tau^{cl} z_0\| \\ &\geqslant k_\tau \|z_0\| - \sqrt{\tau}\, |||C_1|||_\tau \cdot \|B\| \cdot \|\Psi_\tau^{cl} z_0\|. \end{aligned} \tag{6.3.4}$$

We rewrite the first formula in (6.3.2) in the form $(I - \mathbb{F}_\omega)\Psi^{cl} = \Psi$, and we apply $\mathbf{P}_\tau$ to both sides. The causality of $\mathbb{F}_\omega$ (see (5.4.3)) implies that $\mathbf{P}_\tau \mathbb{F}_\omega = \mathbf{P}_\tau \mathbb{F}_\omega \mathbf{P}_\tau$, so that we get the equation

$$(I - \mathbf{P}_\tau \mathbb{F}_\omega)\Psi_\tau^{cl} = \Psi_\tau, \tag{6.3.5}$$

with both sides in $\mathcal{L}(L^2([0.\tau];Y))$. According to Proposition 6.3.1 we have

$$\|\mathbf{P}_\tau \mathbb{F}_\omega\| \leqslant \sqrt{\tau}\, |||C|||_\tau \cdot \|B\|.$$

This, with (6.3.5), shows that if

$$|||C|||_\tau < \frac{1}{\sqrt{\tau}\,\|B\|}, \tag{6.3.6}$$

then

$$\|\Psi_\tau^{cl}\| \leqslant \frac{|||C|||_\tau}{1 - \sqrt{\tau}\, |||C|||_\tau \cdot \|B\|}.$$

Substituting this into (6.3.4), we obtain that if (6.3.6) holds, then

$$\|\Psi_\tau^{1,cl} z_0\| \geqslant k_\tau \|z_0\| - \frac{\sqrt{\tau}\, |||C_1|||_\tau \cdot \|B\| \cdot |||C|||_\tau}{1 - \sqrt{\tau}\, |||C|||_\tau \cdot \|B\|} \cdot \|z_0\|$$

for all $z_0 \in X$. Thus, if (6.3.6) holds and

$$\frac{\sqrt{\tau}\, |||C_1|||_\tau \cdot \|B\| \cdot |||C|||_\tau}{1 - \sqrt{\tau}\, |||C|||_\tau \cdot \|B\|} < k_\tau,$$

then $\Psi_\tau^{1,cl}$ is bounded from below; i.e., $(A+BC, C_1)$ is exactly observable in time τ. The last inequality is equivalent to

$$|||C|||_\tau \leqslant \frac{k_\tau}{\sqrt{\tau}\,\|B\|\,(|||C_1|||_\tau + k_\tau)}\,.$$

This condition implies (6.3.6), so we do not have to impose also (6.3.6). Thus, we got the condition in the theorem, for the particular case when $D = 0$.

Now assume that the closed-loop system corresponding to $D = 0$, i.e., the pair $(A + BC_2, C_1)$, is exactly observable in time τ. The extended output map of this system is $\Psi^{1,cl}$ from the earlier part of this proof. We denote by Ψ^D the extended output map of the closed-loop system with an arbitrary $D \in \mathcal{L}(Y_1, Y)$, i.e., the extended output map of the pair $(A + BC_2 + BDC_1, C_1)$. This pair is obtained from $(A + BC_2, C_1)$ through the perturbation BDC_1 of the generator. We denote by $\mathbb{F}_\omega^D$ the input-output maps corresponding to the transfer function

$$\mathbf{G}^D(s) = C_1(sI - (A + BC_2))^{-1}B\,.$$

In this new situation (having now $A+BC_2$ in place of A and DC_1 in place of C_2) the second formula in (6.3.2) becomes

$$\Psi^D = \Psi^{1,cl} + \mathbb{F}_\omega^D D\Psi^D\,.$$

Indeed, $D\Psi^D$ corresponds to what used to be Ψ^{cl} in the earlier part of the proof. Applying $\mathbf{P}_\tau$ to both sides and using the causality of $\mathbb{F}^D$, we obtain

$$\Psi_\tau^D = \Psi_\tau^{1,cl} + \mathbf{P}_\tau\mathbb{F}_\omega^D D\Psi_\tau^D\,. \tag{6.3.7}$$

We claim that $I - \mathbf{P}_\tau\mathbb{F}_\omega^D D$ is invertible. We know from (5.4.2) and (5.4.5) (with $A+BC_2$ in place of A and C_1 in place of C) that for ω large enough, we have $\|\mathbb{F}_\omega^D D\| < 1$, hence $I - \mathbb{F}_\omega^D D$ is invertible as an operator on $L^2_\omega([0,\infty); Y_1)$. Both this operator and its inverse are causal. It follows that the part of $I - \mathbb{F}_\omega^D D$ acting on $[0,\tau]$, namely $I - \mathbf{P}_\tau\mathbb{F}_\omega^D D$, is invertible as an operator on $L^2([0,\tau]; Y_1)$. (This can be checked by verifying that its inverse is the part of $(I - \mathbb{F}_\omega^D D)^{-1}$ acting on $[0,\tau]$.)

From (6.3.7) we now see that

$$\Psi_\tau^D = (I - \mathbf{P}_\tau\mathbb{F}_\omega^D D)^{-1}\Psi_\tau^{1,cl}\,.$$

Since $\Psi_\tau^{1,cl}$ is bounded from below, as shown earlier, so is Ψ_τ^D. □

In certain arguments, we need a version of the last theorem in which the perturbation is small only on a closed invariant subspace of the closed-loop semigroup, and we conclude exact observability only on this subspace. To simplify matters, we assume that the perturbation is bounded and we do not assume a decomposition of the perturbation as in Theorem 6.3.2.

Proposition 6.3.3. *Suppose that $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$. Assume that (A, C) is exactly observable in time $\tau > 0$; i.e., there exists $k_\tau > 0$ such that*

$$\left(\int_0^\tau \|C\mathbb{T}_t z_0\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \geqslant k_\tau \|z_0\| \qquad \forall\, z_0 \in \mathcal{D}(A).$$

Let $P \in \mathcal{L}(X)$ and let $\mathbb{T}^{cl}$ be the strongly continuous semigroup on X generated by $A + P$. Let V be a closed invariant subspace of $\mathbb{T}^{cl}$ and let $P_V \in \mathcal{L}(V, X)$ be the restriction of P to V. Denote

$$M_V = \sup\left\{ \|\mathbb{T}_t^{cl} z_0\| \;\middle|\; t \in [0,\tau],\ z_0 \in V,\ \|z_0\| \leqslant 1 \right\}.$$

If

$$\|P_V\| \leqslant \frac{k_\tau}{\tau M_V \,|\!|\!| C |\!|\!|_\tau}, \tag{6.3.8}$$

then $(A+P, C)$ is exactly observable in time τ on V; i.e., there exists $k_\tau^V > 0$ such that

$$\left(\int_0^\tau \|C\mathbb{T}_t^{cl} z_0\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \geqslant k_\tau^V \|z_0\| \qquad \forall\, z_0 \in V \cap \mathcal{D}(A).$$

Proof. This proof resembles the first part of the proof of Theorem 6.3.2. We know from Theorem 5.4.2 that $A + P$ generates a strongly continuous semigroup $\mathbb{T}^{cl}$ on X. From the same theorem we also know that C is admissible for $\mathbb{T}^{cl}$.

We use the following notation: Ψ and Ψ^P are the extended output maps of (A, C) and (A, P), respectively. Similarly, Ψ^{cl} and $\Psi^{P,cl}$ are the extended output maps of $(A + P, C)$ and $(A + P, P)$, respectively. All these operators can be truncated to the interval $[0, \tau]$, and then they get a subscript τ. The operators $\mathbb{F}_\omega$ and $\mathbb{F}_\omega^P$ are the input-output maps associated with the transfer functions $C(sI - A)^{-1}$ and $P(sI - A)^{-1}$, respectively, and they are defined on $L^2_\omega([0,\infty); X)$, where $\omega > \omega_0(\mathbb{T})$.

We know from Proposition 5.4.3 that

$$\Psi^{P,cl} = (I - \mathbb{F}_\omega^P)^{-1}\Psi^P, \qquad \Psi^{cl} = \Psi + \mathbb{F}_\omega \Psi^{P,cl}. \tag{6.3.9}$$

From the causality of $\mathbb{F}_\omega$ (see (5.4.3)) we know that $\mathbf{P}_\tau \mathbb{F}_\omega = \mathbf{P}_\tau \mathbb{F}_\omega \mathbf{P}_\tau$. Using this, we apply $\mathbf{P}_\tau$ to both sides of the second equation in (6.3.9) to obtain

$$\Psi_\tau^{cl} = \Psi_\tau + \mathbf{P}_\tau \mathbb{F}_\omega \Psi_\tau^{P,cl}. \tag{6.3.10}$$

If we regard $\mathbf{P}_\tau \mathbb{F}_\omega$ as an operator from $L^2([0,\tau]; X)$ to $L^2([0,\tau]; Y)$, then according to Proposition 6.3.1 it satisfies $\|\mathbf{P}_\tau \mathbb{F}_\omega\| \leqslant \sqrt{\tau}\,|\!|\!| C |\!|\!|_\tau$. Hence, for every $z_0 \in X$,

$$\begin{aligned}\|\Psi_\tau^{cl} z_0\| &\geqslant \|\Psi_\tau z_0\| - \|\mathbf{P}_\tau \mathbb{F}_\omega\| \cdot \|\Psi_\tau^{P,cl} z_0\| \\ &\geqslant k_\tau \|z_0\| - \sqrt{\tau}\,|\!|\!| C |\!|\!|_\tau \cdot \|\Psi_\tau^{P,cl} z_0\|.\end{aligned} \tag{6.3.11}$$

It is easy to see that for every $z_0 \in V$,

$$\|\Psi_\tau^{P,cl} z_0\|^2 = \int_0^\tau \|P\mathbb{T}_t^{cl} z_0\|^2 \,\mathrm{d}t \leqslant \|P_V\|^2 \tau M_V^2 \|z_0\|^2 .$$

Substituting this into (6.3.11) we obtain

$$\|\Psi_\tau^{cl} z_0\| \geqslant k_\tau \|z_0\| - \tau \|\!|C|\!\|_\tau \cdot \|P_V\| \cdot M_V \|z_0\| .$$

Thus, if

$$\tau \|\!|C|\!\|_\tau \cdot \|P_V\| \cdot M_V < k_\tau ,$$

then Ψ_τ^{cl} is bounded from below; i.e., $(A + P, C)$ is exactly observable in time τ. The last inequality is equivalent to condition (6.3.8). □

6.4 Simultaneous exact observability

In this section we investigate the simultaneous (exact or approximate) observability of two systems. This concept means that by observing the sum of their outputs, we can recover the initial states of both systems.

Definition 6.4.1. For $j \in \{1, 2\}$, let A_j be the generator of a strongly continuous semigroup $\mathbb{T}^j$ acting on the Hilbert space X^j. Let Y be a Hilbert space and let $C_j \in \mathcal{L}(X_1^j, Y)$ be an admissible observation operator for $\mathbb{T}^j$. For $\tau > 0$ we denote by Ψ^j the output map associated with (A_j, C_j), as defined in Section 4.3.

The pairs (A_j, C_j) are called *simultaneously exactly observable* in time $\tau > 0$ if there exists $k_\tau > 0$ such that for all $(z_0^1, z_0^2) \in \mathcal{D}(A_1) \times \mathcal{D}(A_2)$, we have

$$\|\Psi_\tau^1 z_0^1 + \Psi_\tau^2 z_0^2\|_{L^2([0,\tau];Y)} \geqslant k_\tau \left(\|z_0^1\|_{X^1} + \|z_0^2\|_{X^2}\right) . \tag{6.4.1}$$

The same pairs are called *simultaneously approximately observable* in time $\tau > 0$ if the fact that $(z_0^1, z_0^2) \in X^1 \times X^2$ satisfies

$$\Psi_\tau^1 z_0^1 + \Psi_\tau^2 z_0^2 = 0 \quad \text{for almost every } t \in [0, \tau], \tag{6.4.2}$$

implies that $(z_0^1, z_0^2) = (0, 0)$.

The main result of this section is the following.

Theorem 6.4.2. *Let A be the generator of the strongly continuous semigroup $\mathbb{T}$ on X. Let Y be another Hilbert space, let $C \in \mathcal{L}(X_1, Y)$ be an admissible observation operator for $\mathbb{T}$ and assume that (A, C) is exactly observable in time $\tau_0 > 0$. Let $a \in \mathcal{L}(\mathbb{C}^n)$ and $c \in \mathcal{L}(\mathbb{C}^n, Y)$ be such that (a, c) is observable. Assume that A and a have no common eigenvalues. Then the pairs (A, C) and (a, c) are simultaneously exactly observable in any time $\tau > \tau_0$.*

First we prove the following approximate observability result.

Lemma 6.4.3. *Suppose that* (A, C), (a, c) *and* τ_0 *satisfy the assumptions of Theorem* 6.4.2. *Then these two pairs are simultaneously approximately observable in time* τ, *for every* $\tau > \tau_0$.

Proof. Let $\tau > \tau_0$ be fixed and let Ψ_τ be the output map associated with (A, C). Denote by V the set of all $v_0 \in \mathbb{C}^n$ such that there exists a $z_0 \in X$ with

$$(\Psi_\tau z_0)(t) + c e^{at} v_0 = 0 \quad \text{for almost every } t \in [0, \tau]. \tag{6.4.3}$$

The approximate observability of (A, C) in time τ_0 implies that for every $z_0 \in X$, the function $\Psi_\tau z_0$ determines z_0. By (6.4.3), this function is determined by v_0. Thus, if $v_0 \in V$, then z_0 satisfying (6.4.3) is unique and depends linearly on v_0: $z_0 = T v_0$. Since the function $t \to c e^{at} v_0$ is smooth, by Proposition 6.1.12 we have that

$$T v_0 \in \mathcal{D}(A) \qquad\qquad \forall\, v_0 \in V.$$

Now we show that for all $v_0 \in V$, we have

$$T a v_0 = A T v_0. \tag{6.4.4}$$

Indeed, by differentiating (6.4.3) with respect to time and using Proposition 4.3.4, we obtain that

$$(\Psi_\tau A T v_0)(t) + c e^{at} a v_0 = 0, \tag{6.4.5}$$

for almost every $t \in [0, \tau]$, which shows that $a v_0 \in V$ and (6.4.4) holds.

Let $\widetilde{a}$ denote the restriction of a to its invariant subspace V. If $V \neq \{0\}$, then $\widetilde{a}$ must have an eigenvalue $\lambda \in \sigma(a)$ and a corresponding eigenvector $\widetilde{v}$. Formula (6.4.4) implies that $A T \widetilde{v} = \lambda T \widetilde{v}$. Since T is one-to-one, we have that $T \widetilde{v} \neq 0$, so that λ is an eigenvalue of A. This is in contradiction to the assumption in Theorem 6.4.2 that A and a have no common eigenvalues. Hence we must have $V = \{0\}$. Thus, (6.4.3) implies that $(z_0, v_0) = (0, 0)$, so that (A, C) and (a, c) are simultaneously approximately observable in time τ. □

Proof of Theorem 6.4.2. Let $\tau > \tau_0$ be fixed. We need to show that the pair

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & a \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} C & c \end{bmatrix} \tag{6.4.6}$$

is exactly observable in time τ. We already know from Lemma 6.4.3 that $(\mathcal{A}, \mathcal{C})$ is approximately observable in time τ. Let $\mathcal{Q}_\tau$ denote the observability Gramian for time τ of $(\mathcal{A}, \mathcal{C})$, so that Ker $\mathcal{Q}_\tau = \{0\}$ (see Remark 6.1.4). We partition $\mathcal{Q}_\tau$ in a natural way, according to the product space $X \times \mathbb{C}^n$:

$$\mathcal{Q}_\tau = \begin{bmatrix} Q_\tau & L \\ L^* & q_\tau \end{bmatrix}.$$

We want to show that $\mathcal{Q}_\tau > 0$. It is not difficult to see that Q_τ is the observability Gramian for time τ of (A, C) and q_τ is the observability Gramian for time τ of

(a, c). As (A, C) and (a, c) are exactly observable in time τ, by Remark 6.1.4, $Q_\tau > 0$ and $q_\tau > 0$. We bring in the Schur-type factorization

$$\begin{bmatrix} Q_\tau & L \\ L^* & q_\tau \end{bmatrix} = \begin{bmatrix} Q_\tau & 0 \\ L^* & I \end{bmatrix} \begin{bmatrix} Q_\tau^{-1} & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} Q_\tau & L \\ 0 & I \end{bmatrix},$$

where $\Delta = q_\tau - L^* Q_\tau^{-1} L$ (this is checked by multiplying out). Notice that the first factor is the adjoint of the last, and they are invertible. Therefore, $\Delta \geqslant 0$ and we have $\mathcal{Q}_\tau > 0$ if (and only if) the middle factor is strictly positive (i.e., > 0). Since obviously $Q_\tau^{-1} > 0$, we see that $\mathcal{Q}_\tau > 0$ if (and only if) $\Delta > 0$. Since $\operatorname{Ker} \mathcal{Q}_\tau = \{0\}$, from the factorization we see that $\operatorname{Ker} \Delta = \{0\}$. But Δ is a matrix, so that $\operatorname{Ker} \Delta = \{0\}$ and $\Delta \geqslant 0$ implies that $\Delta > 0$. Thus we have proved that $\mathcal{Q}_\tau > 0$. By Remark 6.1.4, $(\mathcal{A}, \mathcal{C})$ is exactly observable in time τ. □

The simultaneous observability result that we have just proved enables us to tackle exact observability problems for diagonalizable semigroups by separating the high frequencies from the low frequencies, as the following proposition shows. For this, we have to recall the concept of the part of A in V, as introduced in Section 2.3.

Proposition 6.4.4. *Assume that there exists an orthonormal basis $(\phi_k)_{k\in\mathbb{N}}$ formed of eigenvectors of A and the corresponding eigenvalues λ_k satisfy $\lim |\lambda_k| = \infty$. Let $C \in \mathcal{L}(X_1, Y)$ be an admissible observation operator for $\mathbb{T}$. For some bounded set $J \subset \mathbb{C}$ denote*

$$V = \operatorname{span}\{\phi_k \mid \lambda_k \in J\}^\perp$$

and let A_V be the part of A in V. Let C_V be the restriction of C to $\mathcal{D}(A_V)$. Assume that (A_V, C_V) is exactly observable in time $\tau_0 > 0$ and that $C\phi \neq 0$ for every eigenvector ϕ of A. Then (A, C) is exactly observable in any time $\tau > \tau_0$.

Proof. Denote by a the part of A in $V^\perp$ (which is finite dimensional) and let c be the restriction of C to $V^\perp$. Since $C\phi \neq 0$ for every eigenvector ϕ, according to the finite-dimensional Hautus test (a, c) is observable (see Remark 1.5.2). Since A_V and a have no common eigenvalues, we can apply Theorem 6.4.2 to get that the pairs (A_V, C_V) and (a, c) are simultaneously exactly observable in any time $\tau > \tau_0$. Thus (A, C) is exactly observable in any time $\tau > \tau_0$. □

Finally, we give a result on simultaneous approximate observability. For this we need a notation. Suppose that A is the generator of a strongly continuous semigroup on X. We denote by $\rho_\infty(A)$ the connected component of $\rho(A)$ which contains some right half-plane (obviously, there is only one such component). In particular, if $\sigma(A)$ is countable, as is often the case in applications, then $\rho_\infty(A) = \rho(A)$. (We have already encountered this set in Proposition 2.4.3.)

Proposition 6.4.5. *Let A be the generator of the strongly continuous semigroup $\mathbb{T}$ acting on X. Let $C \in \mathcal{L}(X_1, \mathbb{C}^m)$ be an admissible observation operator for $\mathbb{T}$*

and assume that (A, C) is approximately observable in time τ_0. Let $a \in \mathbb{C}^{n\times n}$ and $c \in \mathbb{C}^{m\times n}$ be matrices such that (a, c) is observable. Further, assume that

$$\sigma(a) \subset \rho_\infty(A). \tag{6.4.7}$$

Then there exists $\tau > 0$ such that the pairs (A, C) and (a, c) are simultaneously approximately observable in time τ.

Proof. To arrive at a contradiction, we assume that the opposite holds: $(\mathcal{A}, \mathcal{C})$ from (6.4.6) is not approximately observable in any time. Thus, for every $k \in \mathbb{N}$ there exist a $z_k \in X$ and a $v_k \in \mathbb{C}^n$ such that $(z_k, v_k) \neq (0, 0)$ and

$$(\Psi_k z_k)(t) + ce^{at}v_k = 0 \quad \forall\, t \in [0, k], \tag{6.4.8}$$

where Ψ_k is the output map of (A, C) on the interval $[0, k]$. It follows from the approximate observability in time τ_0 of (A, C) that for all $k > \tau_0$ we must have $v_k \neq 0$. Hence we may assume without loss of generality that $\|v_k\|_{\mathbb{C}^n} = 1$. By the compactness of the unit ball in $\mathbb{C}^n$, we may assume further that the sequence (v_k) is convergent: $\lim v_k = v_0$. Then it follows that if we define the functions $y_k \in L^2_{\text{loc}}([0, \infty); \mathbb{C}^m)$ by

$$y_k(t) = ce^{at}v_k \qquad \text{for} \quad k \in \{0, 1, 2, \dots\},$$

then $\lim y_k = y_0$ (in L^2_{loc}). Clearly (6.4.8) implies that

$$\Psi_{\tau_0} z_k + \mathbf{P}_{\tau_0} y_k = 0 \qquad \forall\, k \geqslant \tau_0 .$$

Since Ker $\Psi_{\tau_0} = \{0\}$, the above equation shows that z_k is uniquely determined by y_k, which in turn is obtained from v_k. All these dependencies are linear, so that there is an operator $R : \mathbb{C}^n \to X$ (possibly non-unique, depending on the span of all v_k) such that $z_k = Rv_k$, for all $k \in \mathbb{N}$. Hence, the sequence (z_k) is convergent and we put $z_0 = \lim z_k = Rv_0$. Now it is easy to conclude from (6.4.8) that

$$(\Psi z_0)(t) + ce^{at}v_0 = 0 \ \text{ for almost every } \ t \geqslant 0.$$

Taking Laplace transforms, we obtain from the last formula that for some $\alpha \in \mathbb{R}$ and for every $s \in \mathbb{C}_\alpha$,

$$C(sI - A)^{-1}z_0 + c(sI - a)^{-1}v_0 = 0. \tag{6.4.9}$$

By analytic continuation, this formula remains valid on $\rho_\infty(A) \setminus \sigma(a)$. (On the other connected components of $\rho(A)$ we have no such information.) Since $v_0 \neq 0$ (actually, its norm is 1) and (a, c) is observable, the rational function $c(sI-a)^{-1}v_0$ is not zero. Therefore it has poles at a nonempty subset of $\sigma(a)$, which by (6.4.7) is contained in $\rho_\infty(A)$. Since the first term in (6.4.9) is analytic around $\sigma(a)$, it follows that the left-hand side of (6.4.9) has poles, which is absurd. Thus we have proved that $(\mathcal{A}, \mathcal{C})$ must be approximately observable in some time τ. □

Note that the last proposition says nothing about the time τ in which $(\mathcal{A}, \mathcal{C})$ is approximately observable. If τ_0 is minimal for (A, C), then of course $\tau \geqslant \tau_0$.

6.5 A Hautus-type necessary condition for exact observability

We give a necessary condition for exact observability which may be regarded as a generalization of the Hautus test for finite-dimensional systems (see Section 1.5).

Notation. In this section, X and Y are Hilbert spaces, $\mathbb{T}$ is an exponentially stable semigroup on X, with generator A, and $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$. Ψ is the extended output map of (A, C), which is a bounded operator from X to $L^2([0,\infty); Y)$ (see Remark 4.3.5). We denote

$$\mathbb{C}_- = \{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}.$$

The exponential stability is assumed because it simplifies the presentation, but it is not a real restriction. Indeed, for any strongly continuous semigroup $\mathbb{T}$ with generator A, we may replace A with $A - \lambda I$, where $\lambda > \omega_0(\mathbb{T})$, obtaining a *shifted semigroup* that is exponentially stable. The admissibility of C for the original or for the shifted semigroup are equivalent. Similarly, the exact (or approximate) observability of (A, C) in time τ is equivalent to the exact (or approximate) observability of $(A - \lambda I, C)$ in time τ, as it is easy to verify.

Definition 6.5.1. The pair (A, C) is *exactly observable in infinite time* if Ψ is bounded from below. Equivalently, there is a $k > 0$ such that

$$\int_0^\infty \|C\mathbb{T}_t z\|^2 \mathrm{d}t \geqslant k^2 \|z\|^2 \qquad \forall\, z \in \mathcal{D}(A). \tag{6.5.1}$$

The pair (A, C) is *approximately observable in infinite time* if $\operatorname{Ker} \Psi = \{0\}$.

Note that the above property is equivalent to $Q > 0$, where Q is the infinite-time observability Gramian of (A, C), as defined in Section 5.1.

Proposition 6.5.2. *If (A, C) is exactly observable in infinite time, then this system is exactly observable.*

Proof. For any $z \in D(A)$ and any $\tau > 0$, we have

$$\int_0^\tau \|C\mathbb{T}_t z\|^2 \mathrm{d}t = \int_0^\infty \|C\mathbb{T}_t z\|^2 \mathrm{d}t - \int_0^\infty \|C\mathbb{T}_t \mathbb{T}_\tau x\|^2 \mathrm{d}t.$$

Note that by Remark 4.3.5 there exists $K > 0$ such that

$$\int_0^\infty \|C\mathbb{T}_t z\|^2 \mathrm{d}t \leqslant K^2 \|z\|^2 \qquad \forall\, z \in \mathcal{D}(A).$$

Combining the last two formulas with (6.5.1) we obtain

$$\int_0^\tau \|C\mathbb{T}_t z\|^2 \mathrm{d}t \geqslant k^2 \cdot \|z\|^2 - K^2 \cdot \|\mathbb{T}_\tau z\|^2 \geqslant \left(k^2 - K^2 \cdot \|\mathbb{T}_\tau\|^2\right) \cdot \|z\|^2.$$

Since $\mathbb{T}$ is exponentially stable, the parenthesis above becomes positive for τ sufficiently large. For such τ, (A, C) is exactly observable. $\square$

Theorem 6.5.3. *If* (A, C) *is exactly observable in infinite time, then there is an* $m > 0$ *such that for every* $s \in \mathbb{C}_-$ *and every* $z \in \mathcal{D}(A)$,

$$\frac{1}{|\mathrm{Re}\, s|^2} \|(sI - A)z\|^2 + \frac{1}{|\mathrm{Re}\, s|} \|Cz\|^2 \geqslant m \cdot \|z\|^2 . \tag{6.5.2}$$

We shall refer to (6.5.2) as the (infinite-dimensional) *Hautus test.*

Proof. We shall prove the following estimate: For all $s \in \mathbb{C}_-$ and $z \in \mathcal{D}(A)$,

$$\frac{1}{|\mathrm{Re}\, s|^2} \|(sI - A)z\|^2 + \frac{1}{|\mathrm{Re}\, s|} \|Cz\|^2 \geqslant \mu \cdot \|\Psi z\|^2_{L^2} , \tag{6.5.3}$$

where

$$\frac{1}{\mu} = \frac{1}{2} + \|\Psi\|^2 .$$

Clearly, this implies the theorem. We choose $s \in \mathbb{C}_-$, $z \in \mathcal{D}(A)$, we denote

$$q = (A - sI)z ,$$

and we define $\xi : [0, \infty) \to X$ by $\xi(t) = \mathbb{T}_t z$. Then

$$\dot{\xi}(t) = \mathbb{T}_t Az = \mathbb{T}_t (sz + q) = s\xi(t) + \mathbb{T}_t q ,$$

whence

$$\xi(t) = e^{st} z + \int_0^t e^{s(t-\sigma)} \mathbb{T}_\sigma q \mathrm{d}\sigma .$$

Without loss of generality, we may assume that $z \in \mathcal{D}(A^2)$ (by density in X_1) so that $q \in \mathcal{D}(A)$. Then

$$(\Psi z)(t) = C\xi(t) = e^{st} Cz + \int_0^t e^{s(t-\sigma)} C\, \mathbb{T}_\sigma q \mathrm{d}\sigma = e^{st} Cz + (e_s * \Psi q)(t) ,$$

where $*$ denotes the convolution product and e_s denotes the function $e_s(t) = e^{st}$. We use the following well-known property of convolutions:

$$\|u * v\|_{L^2} \leqslant \|u\|_{L^1} \cdot \|v\|_{L^2}$$

to obtain that

$$\begin{aligned} \|\Psi z\|_{L^2} &\leqslant \|e_s\|_{L^2} \cdot \|Cz\| + \|e_s\|_{L^1} \cdot \|\Psi q\|_{L^2} \\ &\leqslant \frac{1}{\sqrt{2|\mathrm{Re}\, s|}} \|Cz\| + \frac{1}{|\mathrm{Re}\, s|} \|\Psi\| \cdot \|q\| . \end{aligned}$$

Using that $(\alpha a + \beta b)^2 \leqslant (\alpha^2 + \beta^2)(a^2 + b^2)$, we get

$$\|\Psi z\|^2_{L^2} \leqslant \left(\frac{1}{2} + \|\Psi\|^2 \right) \left[\frac{1}{|\mathrm{Re}\, s|^2} \|q\|^2 + \frac{1}{|\mathrm{Re}\, s|} \|Cz\|^2 \right] ,$$

which is the same as (6.5.3). $\square$

Remark 6.5.4. The above theorem remains valid (with the same proof) if we replace the exponential stability assumption on $\mathbb{T}$ (from the beginning of the section) with the requirement that C is infinite-time admissible for $\mathbb{T}$.

Lemma 6.5.5. *Let $\tilde{A}$ be the generator of an operator semigroup on a Hilbert space Z. If $\|(sI-\tilde{A})^{-1}\|$ is bounded on $\rho(\tilde{A})$, then $Z=\{0\}$ (the trivial space).*

Proof. According to Remark 2.2.8, for every $s\in\rho(\tilde{A})$ we have

$$\|(sI-\tilde{A})^{-1}\| \geqslant \frac{1}{\min\limits_{\lambda\in\sigma(\tilde{A})} |s-\lambda|}\,.$$

If $\|(sI-\tilde{A})^{-1}\|$ is bounded, then it follows that $\sigma(\tilde{A})=\emptyset$, so that $(sI-\tilde{A})^{-1}$ is a bounded entire function. By Liouville's theorem, $(sI-\tilde{A})^{-1}$ is constant. We know from Corollary 2.3.3 that $\|(\lambda I-\tilde{A})^{-1}\|$ decays like $1/\lambda$ for large positive λ, so that we must have $(sI-\tilde{A})^{-1}=0$, for all $s\in\mathbb{C}$. Since the range of $(sI-\tilde{A})^{-1}$ is dense in Z, it follows that $Z=\{0\}$. $\square$

Proposition 6.5.6. *If the estimate* (6.5.2) *holds, then the system* (A,C) *is approximately observable in infinite time.*

Proof. It follows from (4.3.7) that

$$\|\Psi\mathbb{T}_\tau z\| \leqslant \|\Psi z\| \qquad \forall\,\tau\geqslant 0\,. \tag{6.5.4}$$

If we denote $Z=\operatorname{Ker}\Psi$, then (6.5.4) implies that Z is invariant under $\mathbb{T}$. Let $\tilde{\mathbb{T}}$ be the restriction of $\mathbb{T}$ to Z, so $\tilde{\mathbb{T}}$ is a strongly continuous semigroup on Z, and let $\tilde{A}$ be the generator of $\tilde{\mathbb{T}}$. It is easy to see that

$$\mathcal{D}(\tilde{A}) = \mathcal{D}(A)\cap Z\,, \qquad \mathcal{D}(\tilde{A})\subset \operatorname{Ker} C\,,$$

and $\tilde{A}$ is the restriction of A to $\mathcal{D}(\tilde{A})$.

Now suppose that (6.5.2) holds. Then for every $s\in\mathbb{C}_-$ and every $z\in\mathcal{D}(\tilde{A})$,

$$\frac{1}{|\operatorname{Re}s|^2}\,\|(sI-\tilde{A})z\|^2 \geqslant m\cdot\|z\|^2\,,$$

or equivalently, for any $s\in\rho(\tilde{A})\cap\mathbb{C}_-$,

$$\|(sI-\tilde{A})^{-1}\| \leqslant \frac{1}{\sqrt{m}\,|\operatorname{Re}s|}\,. \tag{6.5.5}$$

Since $\tilde{\mathbb{T}}$ is exponentially stable, $\|(sI-\tilde{A})^{-1}\|$ is defined and bounded on some half-plane $\mathbb{C}_\alpha$, where $\alpha<0$ (see Corollary 2.3.3). Together with (6.5.5) we obtain that $\|(sI-\tilde{A})^{-1}\|$ is bounded on all of $\rho(\tilde{A})$. By Lemma 6.5.5, $Z=\{0\}$. By definition, this means that (A,C) is approximately observable in infinite time. $\square$

Proposition 6.5.7. *If there exists $\alpha \leqslant 0$ such that the estimate* (6.5.2) *holds for all $s \in (-\infty, \alpha)$ with $m \geqslant 1$, then the system (A, C) is exactly observable.*

Proof. For $s \in (-\infty, \alpha)$, (6.5.2) with $m \geqslant 1$ implies that

$$\|(sI - A)z\|^2 - s\|Cz\|^2 \geqslant s^2\|z\|^2 \qquad \forall\, z \in \mathcal{D}(A)$$

which is clearly equivalent to

$$2\mathrm{Re}\,\langle Az, z\rangle + \frac{1}{|s|}\|Az\|^2 + \|Cz\|^2 \geqslant 0 \qquad \forall\, z \in \mathcal{D}(A).$$

Taking limits, the term containing s disappears. Now replacing z with $\mathbb{T}_t z$ and integrating from 0 to ∞, we obtain (as at (5.1.5) with $\Pi = I$) that

$$\int_0^\infty \|C\mathbb{T}_t z\|^2 \mathrm{d}t \geqslant \|z\|^2 \qquad \forall\, z \in \mathcal{D}(A),$$

so that (A, C) is exactly observable in infinite time. Now the conclusion follows from Proposition 6.5.2. $\square$

6.6 Hautus-type tests for exact observability with a skew-adjoint generator

In this section, $A : \mathcal{D}(A) \to X$ is a skew-adjoint operator, so that (by Stone's theorem) A generates a unitary group $\mathbb{T}$. Y is a Hilbert space and $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for the group $\mathbb{T}$. For such operators, the following infinite-dimensional version of the Hautus test (Proposition 1.5.1) holds.

Theorem 6.6.1. *The pair (A, C) is exactly observable if and only if there exist constants M, $m > 0$ such that*

$$M^2\|(i\omega I - A)z_0\|^2 + m^2\|Cz_0\|^2 \geqslant \|z_0\|^2 \qquad \forall\, \omega \in \mathbb{R},\ z_0 \in \mathcal{D}(A). \tag{6.6.1}$$

If (6.6.1) *holds, then (A, C) is exactly observable in time τ for any $\tau > M\pi$.*

Proof. Suppose that (A, C) is exactly observable. It is easy to see that the operator $A - I$ generates an exponentially stable semigroup on X and $(A - I, C)$ is exactly observable. According to Theorem 6.5.3, taking only s with $\mathrm{Re}\, s = -1$ in (6.5.2), we obtain that there exists $m_0 > 0$ such that

$$\|(i\omega I - A)\, z_0\|^2 + \|Cz_0\|^2 \geqslant m_0 \cdot \|z_0\|^2$$

for all $z_0 \in \mathcal{D}(A)$ and for all $\omega \in \mathbb{R}$. This clearly implies (6.6.1).

Now we prove that (6.6.1) implies that the pair (A, C) is exactly observable. We first show that, for all $\chi \in \mathcal{H}^1(\mathbb{R})$ and for all $z_0 \in \mathcal{D}(A)$, we have

$$\int_{\mathbb{R}} \|\mathbb{T}_t z_0\|^2 \left(\chi^2(t) - M^2 \dot{\chi}^2(t)\right) \mathrm{d}t \leqslant m^2 \int_{\mathbb{R}} \|C\mathbb{T}_t z_0\|^2 \chi^2(t) \, \mathrm{d}t. \tag{6.6.2}$$

Indeed, let us denote $z(t) = \mathbb{T}_t z_0$, $w(t) = \chi(t) z(t)$ and $f(t) = \dot{w}(t) - Aw(t)$. If we take the Fourier transform of the last equality, we get that $\widehat{f}(\omega) = (i\omega I - A)\widehat{w}(\omega)$ for all $\omega \in \mathbb{R}$. By applying (6.6.1) with $z_0 = \widehat{w}(\omega)$ and integrating with respect to $\omega \in \mathbb{R}$ we obtain that

$$\int_{\mathbb{R}} \|\widehat{w}(\omega)\|^2 \mathrm{d}\omega \leqslant M^2 \int_{\mathbb{R}} \|\widehat{f}(\omega)\|^2 \mathrm{d}\omega + m^2 \int_{\mathbb{R}} \|C\widehat{w}(\omega)\|^2 \mathrm{d}\omega.$$

The above inequality and Plancherel's theorem imply that

$$\int_{\mathbb{R}} \|w(t)\|^2 \, \mathrm{d}t \leqslant M^2 \int_{\mathbb{R}} \|f(t)\|^2 \, \mathrm{d}t + m^2 \int_{\mathbb{R}} \|Cw(t)\|^2 \, \mathrm{d}t.$$

The above relation and the fact that $f(t) = \dot{\chi}(t) z(t)$ for all $t \in \mathbb{R}$ imply (6.6.2).

We choose $\chi(t) = \varphi\left(\frac{t}{\tau}\right)$ with $\operatorname{supp}(\varphi) \subset [0, 1]$ and $\tau > 0$. The integral on the right-hand side of (6.6.2) satisfies

$$\int_{\mathbb{R}} \|C\mathbb{T}_t z_0\|^2 \chi^2(t) \, \mathrm{d}t \leqslant \|\varphi\|^2_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \|C\mathbb{T}_t z_0\|^2 \, \mathrm{d}t. \tag{6.6.3}$$

A lower bound for the left-hand side of (6.6.2) can be derived as follows: Since $\mathbb{T}$ is unitary, we have

$$\int_{\mathbb{R}} \|\mathbb{T}_t z_0\|^2 \left(\chi^2(t) - M^2 \dot{\chi}^2(t)\right) \mathrm{d}t = \|z_0\|^2 I_\tau(\varphi), \tag{6.6.4}$$

where

$$I_\tau(\varphi) = \int_0^\tau \left(\varphi^2\left(\frac{t}{\tau}\right) - \frac{M^2}{\tau^2}\dot{\varphi}^2\left(\frac{t}{\tau}\right)\right) \mathrm{d}t = \tau \int_0^1 \varphi^2(t) \, \mathrm{d}t - \frac{M^2}{\tau} \int_0^1 \dot{\varphi}^2(t) \, \mathrm{d}t.$$

For $\varphi \neq 0$ and τ large enough we have that $I_\tau(\varphi) > 0$. Consequently, the relations (6.6.2), (6.6.3) and (6.6.4) imply the exact observability estimate

$$\int_0^\tau \|C\mathbb{T}_t z_0\|^2 \, \mathrm{d}t \geqslant \frac{I_\tau(\varphi)}{\|\varphi\|^2_{L^\infty(\mathbb{R})}} \|z_0\|^2 \qquad \forall\, z_0 \in \mathcal{D}(A). \tag{6.6.5}$$

If we choose $\varphi(t) = \sin(\pi t)$ for $t \in [0, 1]$ (and zero else), then a short computation shows that $I_\tau(\varphi) > 0$ for every $\tau > M\pi$. According to the comment after Definition 6.1.1, (A, C) is exactly observable for every $\tau > M\pi$. $\square$

Remark 6.6.2. It is not difficult to show that the choice of φ at the end of the last proof is optimal, in the sense that it minimizes the ratio $\|\dot{\varphi}\|_{L^2}/\|\varphi\|_{L^2}$ over all non-zero functions in $\mathcal{H}_0^1(0,1)$.

Remark 6.6.3. The first part of the last proof (the necessity of condition (6.6.1) for exact observability) can be proved also directly, without going through Theorem 6.5.3. The direct proof in Miller [170] is along the following lines.

For $z_0 \in \mathcal{D}(A)$ and $\omega \in \mathbb{R}$, we denote $z(t) = \mathbb{T}_t z_0$, $v(t) = z(t) - e^{it\omega} z_0$ and $f = (A - i\omega) z_0$. By using Proposition 2.1.5 we obtain that

$$\dot{z}(t) = \mathbb{T}_t A z_0 = \mathbb{T}_t(i\omega z_0 + f) = i\omega z(t) + \mathbb{T}_t f .$$

From the above relation we obtain that

$$\dot{v}(t) = i\omega v(t) + \mathbb{T}_t f ,$$

which implies that $v(t) = \int_0^t e^{i\omega(t-s)} \mathbb{T}_s f \, \mathrm{d}s$. The last formula, combined with the fact that $z(t) = e^{it\omega} z_0 + v(t)$, yields the estimate

$$\int_0^\tau \|Cz(t)\|^2 \mathrm{d}t \leqslant 2\tau \|Cz_0\|^2 + 2 \int_0^\tau t \int_0^t \|C\mathbb{T}_s f\|^2 \mathrm{d}s \mathrm{d}t .$$

The above relation, together with the inequality

$$\int_0^\tau t \int_0^t \|C\mathbb{T}_s f\|^2 \mathrm{d}s \mathrm{d}t \leqslant \frac{\tau^2}{2} \int_0^\tau \|C\mathbb{T}_s f\|^2 \mathrm{d}s ,$$

implies that

$$\int_0^\tau \|Cz(t)\|^2 \mathrm{d}t \leqslant 2\tau \|Cz_0\|^2 + \tau^2 \int_0^\tau \|C\mathbb{T}_s (A - i\omega) z_0\|^2 \mathrm{d}s .$$

By using the fact that the pair (A, C) is admissible (see (4.3.3)) and exactly observable in time τ (see Definition 6.1.1 and the comment after it) we conclude that (6.6.1) holds with $M = \frac{\tau K_\tau}{k_\tau}$ and $m = \frac{\sqrt{2\tau}}{k_\tau}$.

The range of exact observability times given in Theorem 6.6.1 is not sharp, in general. In some cases, a smaller exact observability time can be found based on the following proposition, which amounts to looking only at "high frequencies". For this proposition, the reader should recall the representation of self-adjoint operators with compact resolvents (Proposition 3.2.12), since obviously a similar representation holds for skew-adjoint operators with compact resolvents.

Proposition 6.6.4. *Assume that A has compact resolvents. Let $(\phi_k)_{k \in \mathcal{I}}$ (where $\mathcal{I} \subset \mathbb{Z}$) be an orthonormal basis of eigenvectors of A and denote by $i\mu_k$ the eigenvalue corresponding to ϕ_k. For any $\lambda > 0$ we denote by E_λ the closure in X of* $\operatorname{span}\{\phi_k \mid |\mu_k| > \lambda\}$. *Assume that*

1. *there exist* M, m, $\alpha > 0$ *such that for all* $\omega \in \mathbb{R}$ *with* $|\omega| > \alpha$,

$$M^2\|(i\omega I - A)z_0\|^2 + m^2\|Cz_0\|^2 \geqslant \|z_0\|^2 \qquad \forall\, z_0 \in E_\alpha \cap \mathcal{D}(A),$$

2. $C\phi \neq 0$ *for every eigenvector* ϕ *of* A.

Then (A, C) *is exactly observable in any time* $\tau > M\pi$.

Proof. Denote by A_α the part of A in $E_{\alpha+M^{-1}}$ and by C_α the restriction of C to $\mathcal{D}(A_\alpha)$. Let a_α be the part of A in $E^\perp_{\alpha+M^{-1}}$ and let c_α be the restriction of C to $E^\perp_{\alpha+M^{-1}}$. It is easy to see that if $z_0 \in E_{\alpha+M^{-1}}$ and $|\omega| \leqslant \alpha$, then

$$M^2\|(i\omega I - A)z_0\|^2 \geqslant \|z_0\|^2 .$$

The above inequality and the first assumption in the proposition imply that

$$M^2\|(i\omega I - A)z_0\|^2 + m^2\|Cz_0\|^2 \geqslant \|z_0\|^2 \qquad \forall\, z_0 \in E_{\alpha+M^{-1}},\ \omega \in \mathbb{R}.$$

By Theorem 6.6.1 the pair (A_α, C_α) is exactly observable in any time $\tau > M\pi$. The second assumption in the proposition implies, by using Proposition 6.4.4 with $V = E_{\alpha+M^{-1}}$, that (A, C) is exactly observable in any time $\tau > M\pi$. $\square$

6.7 From $\ddot{w} = -A_0 w$ to $\dot{z} = iA_0 z$

In this section we show that if a system is described by the second-order equation $\ddot{w} = -A_0 w$ and either $y = C_1 w$ or $y = C_0 \dot{w}$ (y being the output signal) and if this system is exactly observable, then this property is inherited by the system described by the first-order equation $\dot{z} = iA_0 z$, with either $y = C_1 z$ or $y = C_0 z$. Thus, we can prove the exact observability of systems governed by the Schrödinger equation, using results available for systems governed by the wave equation.

Throughout this section, H stands for a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. The operator $A_0 : \mathcal{D}(A_0) \to H$ is assumed to be strictly positive. As in Section 3.4 we denote by H_1 the space $\mathcal{D}(A_0)$ endowed with the norm $\|z\|_1 = \|A_0 z\|$ and by $H_{\frac{1}{2}}$ the completion of $\mathcal{D}(A_0)$ with respect to the norm

$$\|w\|_{\frac{1}{2}} = \sqrt{\langle A_0 w, w \rangle},$$

which coincides with $\mathcal{D}(A_0^{\frac{1}{2}})$ with the norm $\|w\|_{\frac{1}{2}} = \|A_0^{\frac{1}{2}} w\|$. We also need the space $\mathcal{D}(A_0^{\frac{3}{2}}) = A_0^{-1}\mathcal{D}(A_0^{\frac{1}{2}})$. If we restrict A_0 to a densely defined positive operator on $H_{\frac{1}{2}}$, then its domain is $\mathcal{D}(A_0^{\frac{3}{2}})$.

Define $X = H_{\frac{1}{2}} \times H$, which is a Hilbert space with the scalar product

$$\left\langle \begin{bmatrix} w_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ v_2 \end{bmatrix} \right\rangle_X = \langle A_0^{\frac{1}{2}} w_1, A_0^{\frac{1}{2}} w_2 \rangle + \langle v_1, v_2 \rangle .$$

We define a dense subspace of X by $\mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}$ and the linear operator $A : \mathcal{D}(A) \to X$ by

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad \text{i.e.,} \quad A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -A_0 f \end{bmatrix}. \tag{6.7.1}$$

Recall from Proposition 3.7.6 that A is skew-adjoint, so that it generates a unitary group $\mathbb{T}$ on X. As usual, X_1 stands for $\mathcal{D}(A)$ endowed with the graph norm.

We assume that A_0^{-1} is compact so that, according to Proposition 3.2.12, there exists an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ in H consisting of eigenvectors of A_0. We denote by $(\lambda_k)_{k\in\mathbb{N}}$ the corresponding sequence of strictly positive eigenvalues of A_0.

Proposition 6.7.1. *Let Y be a Hilbert space, let $C_1 \in \mathcal{L}(H_1, Y)$ and define $C \in \mathcal{L}(X_1, Y)$ by*

$$C = \begin{bmatrix} C_1 & 0 \end{bmatrix}. \tag{6.7.2}$$

Assume that C is an admissible observation for the unitary group $\mathbb{T}$ generated by A. Let $\mathbb{S}$ be the unitary group generated by iA_0 on $H_{\frac{1}{2}}$. Then C_1 is an admissible observation operator for $\mathbb{S}$.

Proof. It is easy to verify that for every $s \in \mathbb{C}$ for which $s^2 \in \rho(A_0)$,

$$(sI - A)^{-1} = \begin{bmatrix} (s^2 I + A_0)^{-1} & 0 \\ 0 & (s^2 I + A_0)^{-1} \end{bmatrix} \cdot \begin{bmatrix} sI & I \\ -A_0 & sI \end{bmatrix},$$

so that

$$C(sI - A)^{-1} = \begin{bmatrix} sC_1(s^2 I + A_0)^{-1} & C_1(s^2 I + A_0)^{-1} \end{bmatrix}.$$

We know from Theorem 4.3.8 that for any $\alpha > 0$, the norm of the above operator in $\mathcal{L}(X, Y)$ is bounded on $\mathbb{C}_0$ by $K(1 + \frac{1}{\operatorname{Re} s})$ (where $K \geqslant 0$). Looking at the left term of $C(sI - A)^{-1}$, which is in $\mathcal{L}(H_{\frac{1}{2}}, Y)$, we obtain that

$$|s| \cdot \|C_1(s^2 I + A_0)^{-1}\|_{\mathcal{L}(H_{\frac{1}{2}}, Y)} \leqslant K\left(1 + \frac{1}{\operatorname{Re} s}\right) \qquad \forall\, s \in \mathbb{C}_0. \tag{6.7.3}$$

We consider only the points $s = a + ib$ with $a, b > 0$ for which $s^2 = -\omega + i$, where $\omega \in \mathbb{R}$. Hence, $a, b > 0$ satisfy $a^2 - b^2 = -\omega$, $2ab = 1$. By elementary algebra,

$$a^2 = \frac{1}{2}\left[-\omega + \sqrt{\omega^2 + 1}\right], \qquad b = \frac{1}{2a}.$$

Now (6.7.3) shows that for every $\omega \in \mathbb{R}$,

$$(\omega^2 + 1)^{\frac{1}{4}} \cdot \|C_1((1 + i\omega)I - iA_0)^{-1}\|_{\mathcal{L}(H_{\frac{1}{2}}, Y)} \leqslant K\left(1 + \frac{1}{a}\right).$$

To show that the function $\omega \mapsto C_1((1+i\omega)I - iA_0)^{-1}$ is bounded in $\mathcal{L}(H_{\frac{1}{2}}, Y)$, we only have to examine its limit behavior as $\omega \to \infty$ and as $\omega \to -\infty$. For large positive ω we have $a^2 \approx \frac{1}{4\omega}$, so that

$$K\left(1 + \frac{1}{a}\right) / (\omega^2 + 1)^{\frac{1}{4}} \approx K \frac{1 + 2\sqrt{\omega}}{(\omega^2 + 1)^{\frac{1}{4}}} \to 2K.$$

For large negative ω we have $a^2 \approx -\omega - \frac{1}{4\omega}$, so that $a \approx \sqrt{|\omega|}$ and

$$K\left(1 + \frac{1}{a}\right) / (\omega^2 + 1)^{\frac{1}{4}} \approx K \frac{1 + \frac{1}{\sqrt{|\omega|}}}{(\omega^2 + 1)^{\frac{1}{4}}} \to 0.$$

It follows that we have

$$\sup_{\omega \in \mathbb{R}} \|C_1((1+i\omega)I - iA_0)^{-1}\|_{\mathcal{L}(H_{\frac{1}{2}}, Y)} < \infty.$$

According to Corollary 5.2.4, C_1 is admissible for the semigroup $\mathbb{S}$. $\square$

Theorem 6.7.2. *With the assumptions in Proposition* 6.7.1*, assume that the pair* (A, C) *is exactly observable. Then the pair* (iA_0, C_1)*, with the state space* $H_{\frac{1}{2}}$*, is exactly observable in any time* $\tau > 0$*.*

Proof. The exact observability of (A, C) implies, according to Theorem 6.6.1, that there exist M, $m > 0$ such that

$$M^2\|(i\sqrt{\omega}I - A)\widetilde{z}\|^2 + m^2\|C\widetilde{z}\|^2 \geqslant \|\widetilde{z}\|^2 \qquad \forall\, \omega > 0,\ \widetilde{z} \in \mathcal{D}(A).$$

Taking $\widetilde{z} = \begin{bmatrix} z \\ iA_0^{\frac{1}{2}} z \end{bmatrix}$, with $z \in \mathcal{D}(A_0)$, it is easy to verify that

$$\|C\widetilde{z}\|_Y = \|C_1 z\|_Y, \qquad \|\widetilde{z}\|_X = \sqrt{2}\|z\|_{\frac{1}{2}},$$
$$\|(i\sqrt{\omega}I - A)\widetilde{z}\|_X = \sqrt{2}\|(\sqrt{\omega}I - A_0^{\frac{1}{2}})z\|_{\frac{1}{2}}.$$

The last four displayed formulas imply that

$$M^2\|(\sqrt{\omega}I - A_0^{\frac{1}{2}})z\|_{\frac{1}{2}}^2 + \frac{m^2}{2}\|C_1 z\|^2 \geqslant \|z\|_{\frac{1}{2}}^2 \qquad \forall\, z \in \mathcal{D}(A_0). \tag{6.7.4}$$

Since, for all $\omega > 0$ and for all $z \in \mathcal{D}(A_0^{\frac{3}{2}})$, we have

$$\|(\omega I - A_0)z\|_{\frac{1}{2}} = \|(\sqrt{\omega}I + A_0^{\frac{1}{2}})A_0^{\frac{1}{2}}(\sqrt{\omega}I - A_0^{\frac{1}{2}})z\| \geqslant \sqrt{\omega}\|(\sqrt{\omega}I - A_0^{\frac{1}{2}})z\|_{\frac{1}{2}},$$

it follows from (6.7.4) that

$$\frac{M^2}{\omega}\|(\omega I - A_0)z\|_{\frac{1}{2}}^2 + \frac{m^2}{2}\|C_1 z\|^2 \geqslant \|z\|_{\frac{1}{2}}^2 \qquad \forall\, \omega > 0,\ z \in \mathcal{D}(A_0^{\frac{3}{2}}).$$

The above estimate implies that for every $T > 0$ and every $\omega > \frac{\pi^2 M^2}{T^2}$ we have

$$\frac{T^2}{\pi^2}\|(\omega I - A_0)z\|^2_{\frac{1}{2}} + \frac{m^2}{2}\|C_1 z\|^2 \geqslant \|z\|^2_{\frac{1}{2}} \qquad \forall\, z \in \mathcal{D}(A_0^{\frac{3}{2}})\,. \tag{6.7.5}$$

On the other hand, for every $T > 0$ and every $\omega < -\frac{\pi}{T}$, we have (using the fact that A_0 is a positive operator on $H_{\frac{1}{2}}$)

$$\frac{T^2}{\pi^2}\|(\omega I - A_0)z\|^2_{\frac{1}{2}} \geqslant \|z\|^2_{\frac{1}{2}} \qquad \forall\, z \in \mathcal{D}(A_0^{\frac{3}{2}})\,.$$

This fact, together with (6.7.5), implies, denoting $\alpha = \max\left\{\frac{\pi^2 M^2}{T^2}, \frac{\pi}{T}\right\}$, that for every $|\omega| > \alpha$, (6.7.5) holds.

In addition, the exact observability of (A, C) implies, by using Remark 6.1.8, that $C\phi \neq 0$ for every eigenvector ϕ of A. According to Proposition 3.7.7, this implies that $C_1\varphi \neq 0$ for every eigenvector φ of A_0. Applying Proposition 6.6.4, it follows that (iA_0, C) is exactly observable in any time $\tau > T$. Since $T > 0$ was arbitrary, it follows that (iA_0, C) is exactly observable in any time $\tau > 0$. □

Example 6.7.3. Let $H = L^2[0,\pi]$ and $A_0 : \mathcal{D}(A_0) \to H$ be defined by

$$\mathcal{D}(A_0) = \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_0(0,\pi)\,,$$
$$A_0 f = -\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \qquad \forall\, f \in \mathcal{D}(A_0)\,.$$

With the above choice of H and A_0, the space $X = H_{\frac{1}{2}} \times H$ and the operator A from (6.7.1) coincide with X and A considered in Section 6.2. Let $Y = \mathbb{C}$ and consider the observation operator $C \in \mathcal{L}(X_1, Y)$ defined by (6.2.5). We know from Proposition 6.2.1 that the pair (A, C) is exactly observable in any time $\tau \geqslant 2\pi$. On the other hand, C is of the form (6.7.2), with $C_1\varphi = \frac{\mathrm{d}\varphi}{\mathrm{d}x}(0)$ for all $\varphi \in \mathcal{D}(A_0)$. According to Theorem 6.7.2, the pair (iA_0, C_1), with the state space $\mathcal{H}^1_0(0,\pi)$, *is exactly observable in any time* $\tau > 0$. In PDEs terms, this means that for every $\tau > 0$ there exists $k_\tau > 0$ such that the solution z of the Schrödinger equation

$$\frac{\partial z}{\partial t}(x,t) = -i\frac{\partial^2 z}{\partial x^2}(x,t) \qquad \forall\, (x,t) \in (0,\pi) \times [0,\infty)\,,$$

with

$$z(0,t) = z(\pi,t) = 0\,, \quad t \geqslant 0\,,$$

and $z(\cdot,0) = z_0 \in \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_0(0,\pi)$, satisfies

$$\int_0^\tau \left|\frac{\partial z}{\partial x}(0,t)\right|^2 \mathrm{d}t \geqslant k_\tau^2 \|z_0\|^2_{\mathcal{H}^1_0(0,\pi)} \qquad \forall\, z_0 \in \mathcal{D}(A_0^{\frac{3}{2}})\,.$$

In this case,

$$\mathcal{D}(A_0^{\frac{3}{2}}) = \left\{ f \in \mathcal{H}^3(0,\pi) \cap \mathcal{H}^1_0(0,\pi) \;\middle|\; \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(0) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(\pi) = 0 \right\}.$$

Proposition 6.7.4. *Let Y be a Hilbert space, let $C_0 \in \mathcal{L}(H_{\frac{1}{2}}, Y)$ and define $C \in \mathcal{L}(X_1, Y)$ by*

$$C = \begin{bmatrix} 0 & C_0 \end{bmatrix} . \tag{6.7.6}$$

Assume that C is an admissible observation for the unitary group $\mathbb{T}$ generated by A. Let $\mathbb{S}$ be the unitary group generated by iA_0 on H. Then C_0 is an admissible observation operator for $\mathbb{S}$.

The proof of the above proposition can be obtained by obvious adaptations of the proof of Proposition 6.7.1, so we do not give it here.

Theorem 6.7.5. *With the assumptions in Proposition* 6.7.4*, assume that the pair (A, C) is exactly observable. Then the pair (iA_0, C_0), with state space H, is exactly observable in any time $\tau > 0$.*

Proof. The exact observability of (A, C) implies, according to Theorem 6.6.1, that there exist M, $m > 0$ such that

$$M^2 \|(i\sqrt{\omega} I - A)\widetilde{z}\|^2 + m^2 \|C\widetilde{z}\|^2 \geqslant \|\widetilde{z}\|^2 \qquad \forall\, \omega > 0,\ \widetilde{z} \in \mathcal{D}(A).$$

Taking here $\widetilde{z} = \begin{bmatrix} A_0^{-\frac{1}{2}} z \\ iz \end{bmatrix}$, with $z \in H_{\frac{1}{2}}$ and using the fact that, with the above choice of $\widetilde{z}$, we have

$$\|C\widetilde{z}\|_Y = \|C_0 z\|_Y , \qquad \|\widetilde{z}\|_X = \sqrt{2}\|z\| ,$$
$$\|(i\sqrt{\omega} I - A)\widetilde{z}\|_X = \sqrt{2}\|(\sqrt{\omega} I - A_0^{\frac{1}{2}})z\| ,$$

we obtain

$$M^2 \|(\sqrt{\omega} I - A_0^{\frac{1}{2}})z\|^2 + \frac{m^2}{2}\|C_0 z\|^2 \geqslant \|z\|^2 \qquad \forall\, z \in \mathcal{D}(A_0).$$

The proof can now be completed following line by line the corresponding part of the proof of Theorem 6.7.2, and this is left to the reader. □

Example 6.7.6. Let H, A_0, X and A be as in Example 6.7.3. Let $Y = L^2[0, \pi]$ and $C \in \mathcal{L}(X, Y)$ be the observation operator defined in (6.2.8). We know from Proposition 6.2.3 that the pair (A, C) is exactly observable in any time $\tau \geqslant 2\pi$. Since C is of the form (6.7.6), with

$$C_0 \varphi = \varphi \chi_{[\xi, \eta]} \qquad \forall\, \varphi \in L^2[0, \pi] ,$$

we can apply Theorem 6.7.5 to get that the pair (iA_0, C_0), with the state space $L^2[0, \pi]$, *is exactly observable in any time* $\tau > 0$. In PDEs terms, this means that if $\tau > 0$, then there exists $k_\tau > 0$ such that the solution z of the Schrödinger equation

$$\frac{\partial z}{\partial t}(x, t) = -i \frac{\partial z}{\partial x^2}(x, t), \quad (x, t) \in (0, \pi) \times [0, \infty) ,$$

with

$$z(0,t) = z(\pi,t) = 0, \quad t \geqslant 0,$$

and with $z(\cdot,0) = z_0 \in \mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi)$, satisfies

$$\int_0^\tau \int_\xi^\eta |z(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \geqslant k_\tau^2 \|z_0\|^2_{L^2[0,\pi]} \qquad \forall\, z_0 \in \mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi).$$

6.8 From first- to second-order equations

In this section we show how the exact observability for systems described by certain Schrödinger-type equations implies the exact observability for systems described by certain Euler–Bernoulli-type equations.

Notation and preliminaries. We use the same notation as in Section 6.7. In particular, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are the inner product and the norm on H, $A_0 > 0$ and $H_1 = \mathcal{D}(A_0)$ with the norm $\|z\|_1 = \|A_0 z\|$. We denote $\mathcal{X} = H_1 \times H$, which is a Hilbert space with the inner product

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_{\mathcal{X}} = \langle A_0 f_1, A_0 f_2\rangle + \langle g_1, g_2\rangle .$$

We define $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ by $\mathcal{D}(\mathcal{A}) = \mathcal{D}(A_0^2) \times \mathcal{D}(A_0)$ and

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0^2 & 0 \end{bmatrix}, \quad \text{i.e.,} \quad \mathcal{A}\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -A_0^2 f \end{bmatrix}. \tag{6.8.1}$$

Since $A_0^2 > 0$ (see Remark 3.3.7), according to Proposition 3.7.6, $\mathcal{A}$ is skew-adjoint and $0 \in \rho(\mathcal{A})$. By the theorem of Stone, $\mathcal{A}$ generates a unitary group $\mathbb{T}$ on $\mathcal{X}$. As usual, we denote by $\mathcal{X}_1$ the space $\mathcal{D}(\mathcal{A})$ endowed with the graph norm.

Proposition 6.8.1. *Let Y be a Hilbert space and let $C_0 \in \mathcal{L}(H_1, Y)$ be an admissible observation operator for the unitary group generated by iA_0. Define $C \in \mathcal{L}(X_1, Y)$ by $C = \begin{bmatrix} 0 & C_0 \end{bmatrix}$. Then C is an admissible observation operator for $\mathbb{T}$.*

Proof. An easy computation similar to the one in the proof of Proposition 6.7.1 shows that for all $s \in \rho(\mathcal{A})$,

$$C(sI - \mathcal{A})^{-1} = \begin{bmatrix} -C_0 A_0^2 (s^2 I + A_0^2)^{-1} & C_0 s (s^2 I + A_0^2)^{-1} \end{bmatrix}.$$

The admissibility assumption in the proposition implies that there exists $K \geqslant 0$ such that

$$\left\| C_0 (sI - iA_0)^{-1} \right\|_{\mathcal{L}(H,Y)} \leqslant K \qquad \text{for } \operatorname{Re} s = 1; \tag{6.8.2}$$

see Theorem 4.3.7. On the other hand, for all $\omega \geqslant 0$ we have

$$\left\| ((-\omega + i)I - A_0)^{-1} \right\| = \frac{1}{\min\limits_{\lambda\in\sigma(A_0)} |-\omega + i - \lambda|} < \frac{1}{|-\omega + i|},$$

because of (3.2.3) and the fact that $\sigma(A_0) \subset (0, \infty)$. Hence, for all $\omega \geqslant 0$,

$$\left\|((1+i\omega)I + iA_0)^{-1}\right\| < \frac{1}{|1+i\omega|}. \tag{6.8.3}$$

Combining this with (6.8.2), we obtain that for $\operatorname{Re} s = 1$ and $\operatorname{Im} s \geqslant 0$,

$$\left\|C_0(s^2 I + A_0^2)^{-1}\right\|_{\mathcal{L}(H,Y)} < \frac{K}{|s|}. \tag{6.8.4}$$

Now we redo the argument with every number replaced with its complex conjugate: we obtain that (6.8.2) holds with the minus sign replaced with plus, and (6.8.3) holds with $-i$ in place of i everywhere. Combining these two modified formulas, we obtain that (6.8.4) holds for $\operatorname{Re} s = 1$ and $\operatorname{Im} s \leqslant 0$. Together with the first version of (6.8.4), this implies that (6.8.4) holds for all $s \in \mathbb{C}$ with $\operatorname{Re} s = 1$.

For $s \in \mathbb{C}_0$ we have

$$\begin{aligned}
&\|C_0 A_0^2 (s^2 I + A_0^2)^{-1}\|_{\mathcal{L}(H_1,Y)} \\
&\quad = \|C_0(sI - iA_0)^{-1} A_0 (sI + iA_0)^{-1}\|_{\mathcal{L}(H,Y)} \\
&\quad \leqslant \|C_0(sI - iA_0)^{-1}\|_{\mathcal{L}(H,Y)} \cdot \|I - s(sI + iA_0)^{-1}\|_{\mathcal{L}(H)}.
\end{aligned}$$

Using (6.8.2) to estimate the first factor and (6.8.3) to estimate the second factor, we obtain that for all $s \in \mathbb{C}$ with $\operatorname{Re} s = 1$ and $\operatorname{Im} s \geqslant 0$,

$$\|C_0 A_0^2 (s^2 I + A_0^2)^{-1}\|_{\mathcal{L}(H_1,Y)} \leqslant 2K. \tag{6.8.5}$$

If we redo the computations leading to (6.8.5) but with every number replaced with its complex conjugate, we obtain that (6.8.5) also holds for $\operatorname{Re} s = 1$ and $\operatorname{Im} s \leqslant 0$. Thus, it holds for all $s \in \mathbb{C}$ with $\operatorname{Re} s = 1$.

The estimate (6.8.5), together with (6.8.4), shows that the $\mathcal{L}(\mathcal{X}, Y)$-valued function $C(sI - \mathcal{A})^{-1}$ (as decomposed into components at the beginning of this proof) is bounded on the vertical line where $\operatorname{Re} s = 1$. According to Corollary 5.2.4, C is an admissible observation operator for $\mathbb{T}$. $\square$

In what follows sequel we assume that A_0^{-1} is compact, which implies that there exists an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ in H such that $A_0\varphi_k = \lambda_k \varphi_k$, with $\lambda_k > 0$. We set $\varphi_{-n} = -\varphi_n$, for all $n \in \mathbb{N}$. According to Proposition 3.7.7, the eigenvalues of $\mathcal{A}$ are $(i\mu_n)_{n\in\mathbb{Z}^*}$ with $\mu_n = \lambda_n$ if $n > 0$ and $\mu_n = -\lambda_n$ if $n < 0$. There is in $\mathcal{X}$ an orthonormal basis formed of eigenvectors of $\mathcal{A}$, given by

$$\phi_n = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{i\mu_n}\varphi_n \\ \varphi_n \end{bmatrix} \qquad \forall\, n \in \mathbb{Z}^*. \tag{6.8.6}$$

The above facts imply that the group $\mathbb{T}$ is diagonalizable and

$$\mathbb{T}_t z = \sum_{n\in\mathbb{Z}^*} e^{i\mu_n t} \langle z, \phi_n\rangle \phi_n \qquad \forall\, (t, z) \in \mathbb{R} \times \mathcal{X}. \tag{6.8.7}$$

Proposition 6.8.2. *Assume that A_0^{-1} is compact. Let Y be a Hilbert space and let $C_0 \in \mathcal{L}(H_1, Y)$ be such that the pair (iA_0, C_0) is exactly observable in some time τ_0. Moreover, assume that there exists $d \in \mathbb{N}$ such that*

$$\sum_{j\in\mathbb{N}} \lambda_j^{-d} < \infty, \tag{6.8.8}$$

and define $C \in \mathcal{L}(X_1, Y)$ by $C = \begin{bmatrix} 0 & C_0 \end{bmatrix}$.

Then the pair $(\mathcal{A}, C)$ is exactly observable in any time $\tau > \tau_0$.

Proof. For $N \in \mathbb{N}$ which will be made precise later, let $z = \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{D}(\mathcal{A})$ be such that

$$\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \phi_k \right\rangle_{\mathcal{X}} = 0 \qquad \text{if} \quad |k| \leqslant N. \tag{6.8.9}$$

From (6.8.6) and (6.8.7) it follows that

$$\begin{aligned} \sqrt{2} C \mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} &= \sum_{n>N} e^{i\lambda_n t} \left(i\lambda_n \langle f, \varphi_n \rangle + \langle g, \varphi_n \rangle \right) C_0 \varphi_n \\ &\quad + \sum_{n>N} e^{-i\lambda_n t} \left(-i\lambda_n \langle f, \varphi_n \rangle + \langle g, \varphi_n \rangle \right) C_0 \varphi_n . \end{aligned}$$

The above relation implies that

$$\sqrt{2} C \mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} = C_0 \mathbb{T}_t^+ z^+ + C_0 \mathbb{T}_t^- z^- , \tag{6.8.10}$$

where $\mathbb{T}^+$ (respectively, $\mathbb{T}^-$) is the group of isometries on H generated by iA_0 (respectively, by $-iA_0$) and

$$\begin{aligned} z^+ &= \sum_{n>N} z_n^+ \varphi_n, \quad \text{where} \quad z_+^n = i\lambda_n \langle f, \varphi_n \rangle + \langle g, \varphi_n \rangle , \\ z^- &= \sum_{n>N} z_n^- \varphi_n, \quad \text{where} \quad z_n^- = -i\lambda_n \langle f, \varphi_n \rangle + \langle g, \varphi_n \rangle . \end{aligned}$$

Let ε be such that $\varepsilon, \varepsilon + \tau_0 \in (0, \tau)$ and let $\kappa \in \mathcal{D}(\mathbb{R})$ such that $\kappa(t) = 1$ for $t \in (\varepsilon, \varepsilon + \tau_0)$, $0 \leqslant \kappa(t) \leqslant 1$ for all $t \in \mathbb{R}$ and $\kappa(t) = 0$ if $t \notin [0, \tau]$. Then

$$\int_0^\tau \left\| C_0 \mathbb{T}_t^+ z^+ + C_0 \mathbb{T}_t^- z^- \right\|^2 \mathrm{d}t \geqslant \int_{\mathbb{R}} \kappa(t) \left\| C_0 \mathbb{T}_t^+ z^+ + C_0 \mathbb{T}_t^- z^- \right\|^2 \mathrm{d}t .$$

By using the properties of κ and by denoting by k_0 a common observability constant of (iA_0, C_0) and $(-iA_0, C_0)$, it follows that

$$\begin{aligned} &\int_0^\tau \left\| C_0 \mathbb{T}_t^+ z^+ + C_0 \mathbb{T}_t^- z^- \right\|^2 \mathrm{d}t \\ &\qquad \geqslant k_0^2 (\|z^+\|^2 + \|z^-\|^2) + 2 \int_{\mathbb{R}} \kappa(t) \mathrm{Re} \left\langle C_0 \mathbb{T}_t^+ z^+, C_0 \mathbb{T}_t^- z^- \right\rangle \mathrm{d}t . \end{aligned} \tag{6.8.11}$$

We compute the last integral term as follows:

$$\int_{\mathbb{R}} \kappa(t) \mathrm{Re} \left\langle C_0 \mathbb{T}_t^+ z^+, C_0 \mathbb{T}_t^- z^- \right\rangle \mathrm{d}t = \sum_{m,n \geqslant N} \int_{\mathbb{R}} e^{i(\lambda_m+\lambda_n)t} \kappa(t) \langle C_0 z_m^+, C_0 z_n^- \rangle \, \mathrm{d}t \, .$$

Since C_0 is admissible for iA_0, there exists a $K_0 \geqslant 0$ such that $\|C_0 \varphi_n\| \leqslant K_0$ for all $n \in \mathbb{N}$. Hence

$$\left| \int_{\mathbb{R}} \kappa(t) \mathrm{Re} \left\langle C_0 \mathbb{T}_t^+ z^+, C_0 \mathbb{T}_t^- z^- \right\rangle \mathrm{d}t \right| \leqslant K_0^2 \sqrt{2\pi} \sum_{m,n \geqslant N} \left| \widehat{\kappa}(-\lambda_m - \lambda_n) z_m^+ z_n^- \right| ,$$

where $\widehat{\kappa}$ is the Fourier transform of κ, as defined in (12.4.1). The above estimate, together with (6.8.11), implies that

$$\int_0^\tau \left\| C_0 \mathbb{T}_t^+ z^+ + C_0 \mathbb{T}_t^- z^- \right\|^2 \mathrm{d}t \geqslant k_0^2 (\|z^+\|^2 + \|z^-\|^2)$$
$$- 2K_0^2 \sqrt{2\pi} \sum_{m,n \geqslant N} \left| \widehat{\kappa}(-\lambda_m - \lambda_n) z_m^+ z_n^- \right| .$$

Since $\kappa^{(d)} \in \mathcal{D}(\mathbb{R})$ and since the Fourier transformation maps $\mathcal{D}(\mathbb{R})$ into $C_0(\mathbb{R})$ (see Section 12.4 in Appendix I), it follows that $\omega \mapsto \omega^d \widehat{\kappa}(\omega)$ is a bounded function on $\mathbb{R}$. Therefore there exists $C_1 > 0$ such that

$$|\widehat{\kappa}(-\lambda_m - \lambda_n))| \leqslant C_1 (\lambda_m + \lambda_n)^{-d} \qquad \forall \, m, n \in \mathbb{N},$$

which implies (using $2|z_m^+ z_n^-| \leqslant |z_n^+|^2 + |z_m^-|^2$) that

$$\int_0^\tau \left\| C_0 \mathbb{T}_t^+ z^+ + C_0 \mathbb{T}_t^- z^- \right\|^2 \mathrm{d}t \geqslant k_0^2 \left(\|z^+\|^2 + \|z^-\|^2 \right)$$
$$- C_1 K_0^2 \sum_{m \geqslant N} |z_m^+|^2 \sum_{n \geqslant N} (\lambda_m + \lambda_n)^{-d} - C_1 K_0^2 \sum_{n \geqslant N} |z_n^-|^2 \sum_{m \geqslant N} (\lambda_m + \lambda_n)^{-d} .$$

By choosing N from the beginning of this proof large enough, the above relation and assumption (6.8.8) imply the existence of a constant $c_\tau > 0$ such that

$$\int_0^\tau \left\| C_0 \mathbb{T}_t^+ z^+ + C_0 \mathbb{T}_t^- z^- \right\|^2 \mathrm{d}t \geqslant c_\tau^2 \left(\|z^+\|^2 + \|z^-\|^2 \right) .$$

This estimate, combined with (6.8.10), implies that the inequality

$$\int_0^\tau \left\| C \mathbb{T}_t \begin{bmatrix} f \\ g \end{bmatrix} \right\|_Y^2 \mathrm{d}t \geqslant \frac{c_\tau^2}{2} \left(\|f\|_{\frac{1}{2}}^2 + \|g\|^2 \right)$$

holds for every $\left[\begin{smallmatrix} f \\ g \end{smallmatrix} \right] \in \mathcal{D}(A)$ satisfying (6.8.9). Thus, the "high-frequency part" of $(\mathcal{A}, C)$ is exactly observable in time τ.

To apply Proposition 6.4.4, we notice that, according to Proposition 3.7.7, if ϕ is an eigenvector of $\mathcal{A}$, corresponding to the eigenvalue $i\mu$ (where $\mu \in \mathbb{R}$), then

$$\phi = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{i\mu}\varphi \\ \varphi \end{bmatrix},$$

where φ is an eigenvector of A_0. It follows that

$$C\phi = \frac{1}{\sqrt{2}} C_0 \varphi,$$

so that $C\phi \neq 0$, since (iA_0, C_0) is exactly observable. Thus, $(\mathcal{A}, C)$ is exactly observable in time τ. □

Example 6.8.3. Let $H = L^2[0,\pi]$ and let $-A_0$ be the Dirichlet Laplacian on $[0,\pi]$ as in Examples 6.7.3 and 6.7.6. Then A_0^2 is the fourth-order derivative operator defined on the space of all $f \in \mathcal{H}^4(0,\pi)$ with f and $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ vanishing at $x = 0$ and at $x = \pi$. The space $X = \mathcal{D}(A_0) \times H$ is, in this case, given by $X = \mathcal{H}_0^2(0,\pi) \times L^2[0,\pi]$. Let $Y = L^2[0,\pi]$, ξ, $\eta \in [0,\pi]$ with $\xi < \pi$ and let $C_0 \in \mathcal{L}(H,Y)$ be the observation operator defined by

$$C_0 f = f\chi_{[\xi,\eta]} \qquad \forall\, f \in L^2[0,\pi].$$

We have seen in Example 6.7.6 that the pair (iA_0, C_0), with the state space $L^2[0,\pi]$, is exactly observable in any time $\tau > 0$. Moreover, the eigenvalues of A_0 clearly satisfy (6.8.8) for $d = 1$. By applying Theorem 6.8.2 we get that the pair (A, C), with A given by (6.8.1) and $C = \begin{bmatrix} 0 & C_0 \end{bmatrix}$, is exactly observable in any time $\tau > 0$. In PDEs terms, this means that if $\tau > 0$, then there exists $k_\tau > 0$ such that the solution w of the Euler–Bernoulli beam equation

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^4 w}{\partial x^4}(x,t) = 0, \quad (x,t) \in (0,\pi) \times [0,\infty),$$

with

$$w(0,t) = w(\pi,t) = 0, \quad t \geqslant 0,$$
$$\frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2} w(\pi,t) = 0, \quad t \geqslant 0,$$

and $w(\cdot,0) = w_0 \in \mathcal{H}^2(0,\pi) \times \mathcal{H}_0^1(0,\pi)$, $\frac{\partial w}{\partial t}(\cdot,0) = w_1 \in L^2[0,\pi]$, satisfies

$$\int_0^\tau \int_\xi^\eta \left| \frac{\partial w}{\partial t}(x,t) \right|^2 \mathrm{d}x\,\mathrm{d}t \geqslant k_\tau^2 \left(\|w_0\|^2_{\mathcal{H}^2(0,\pi)} + \|w_1\|^2_{L^2[0,\pi]} \right) \qquad \forall \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in X.$$

Example 6.8.4. We show here that the admissibility and the exact observability of a boundary observed hinged Euler–Bernoulli equation can be obtained from the corresponding properties of a one-dimensional Schrödinger equation. Let $H = \mathcal{H}_0^1(0,\pi)$ and

$$\mathcal{D}(A_0) = \left\{ f \in H \;\middle|\; \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \in H \right\}.$$

Let $Y = \mathbb{C}$ and let $C_0 \in \mathcal{L}(H_1, Y)$ be the observation operator defined by

$$C_0 f = \frac{\mathrm{d}f}{\mathrm{d}x}(0) \qquad \forall\, f \in H_1 .$$

We have seen in Example 6.7.3 that the pair (iA_0, C_0) is exactly observable in any time $\tau > 0$. Moreover, the eigenvalues of A_0 clearly satisfy (6.8.8) for $d = 1$. We define $H_1 = \mathcal{D}(A_0)$ with the graph norm, which is equivalent to the norm inherited from $\mathcal{H}^3(0,\pi)$. By applying Proposition 6.8.2 we get that the pair $(\mathcal{A}, C)$, with $\mathcal{A}$ given by (6.8.1) and $C = [0 \ C_0]$ on the state space $\mathcal{X} = H_1 \times H$, is exactly observable in any time $\tau > 0$. In PDEs terms, this means that if $\tau > 0$, then there exists $k_\tau > 0$ such that the solution w of the Euler–Bernoulli beam equation with hinged ends

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^4 w}{\partial x^4}(x,t) = 0, \quad (x,t) \in (0,\pi) \times [0,\infty),$$

with

$$w(0,t) = w(\pi,t) = 0, \quad t \geqslant 0,$$
$$\frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2} w(\pi,t) = 0, \quad t \geqslant 0,$$

and $w(\cdot,0) = w_0 \in \mathcal{D}(A_0^2)$, $\frac{\partial w}{\partial t}(\cdot,0) = w_1 \in \mathcal{D}(A_0)$, satisfies

$$\int_0^\tau \left| \frac{\partial^2 w}{\partial x \partial t}(0,t) \right|^2 \mathrm{d}t \geqslant \left(\|w_0\|^2_{\mathcal{H}^3(0,\pi)} + \|w_1\|^2_{\mathcal{H}^1_0(0,\pi)} \right) \qquad \forall \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in \mathcal{D}(A).$$

Note that we have derived the admissibility and the exact observability of this boundary observed hinged beam equation by reducing them to the corresponding properties of a (one-dimension) Schrödinger equation. If we go back to Example 6.7.3, we see that the admissibility and the exact observability of the Schrödinger equation have in turn been obtained from the corresponding properties of a (one-dimensional) wave equation. Thus, we have here a very indirect proof for the properties of the hinged beam, relying on nontrivial theorems for the reductions. The proof of the admissibility and the exact observability (in arbitrary positive time) for the hinged beam can also be approached directly, using the fact that the generator $\mathcal{A}$ is diagonalizable. Indeed, after computing the eigenvalues and the Fourier coefficients of the observation operator, the desired properties follow from a consequence of Ingham's theorem, which appears later in this book as Proposition 8.1.3. We mention that the admissibility and the exact observability of this system in time 2π (but not in shorter times) can be shown also in a completely elementary fashion, as was done in Proposition 6.2.1 for the string equation.

6.9 Spectral conditions for exact observability with a skew-adjoint generator

Recall that in the finite-dimensional case the observability of (A, C) is equivalent to $C\phi \neq 0$ for every eigenvector ϕ of A (see Remark 1.5.2). The situation is much more complicated in the infinite-dimensional case. In this section we assume that A is skew-adjoint and has compact resolvents. We denote by $(\phi_k)_{k\in\Lambda}$ an orthonormal basis consisting of eigenvectors of A and by $(i\mu_k)_{k\in\Lambda}$, with $\mu_k \in \mathbb{R}$, the corresponding eigenvalues of A. The index set Λ is a subset of $\mathbb{Z}$. Y is another Hilbert space and $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for the unitary group $\mathbb{T}$ generated by A. We denote

$$c_k = C\phi_k \qquad \forall\, k \in \Lambda.$$

First we give a simple necessary and sufficient condition for approximate observability in infinite time (as defined in Definition 6.5.1).

Proposition 6.9.1. *The following conditions are equivalent:*

(C1) $c_k \neq 0$ *for all* $k \in \Lambda$,

(C2) (A, C) *is approximately observable in infinite time.*

Proof. First we show that (C1) implies (C2). Let $z \in X$ and let $y = \Psi z$, then according to Theorem 4.3.7 the Laplace transform of y is $\hat{y}(s) = C(sI - A)^{-1}z$, for all $s \in \mathbb{C}_0$. Let $\eta \in Y$ be fixed. It follows that

$$\langle \hat{y}(s), \eta\rangle = \langle C(sI - A)^{-1}z, \eta\rangle = \eta^* C(sI - A)^{-1}z \qquad \forall\, s \in \mathbb{C}_0\,,$$

where η^* is the linear functional on Y associated with η. Since $\eta^* C(sI - A)^{-1}$ is a bounded functional on X, it is represented in the orthonormal basis (ϕ_k) by a family $(v_k) \in l^2(\Lambda)$. Using formula (2.6.6) (with $\tilde{\phi}_k = \phi_k$ and with $i\mu_k$ in place of λ_k), it is easy to compute that $v_k = \frac{\langle c_k, \eta\rangle}{s - i\mu_k}$. It follows that

$$\langle \hat{y}(s), \eta\rangle = \sum_{k\in\Lambda} \langle c_k, \eta\rangle \frac{z_k}{s - i\mu_k} \qquad \forall\, s \in \mathbb{C}_0\,, \tag{6.9.1}$$

where $z_k = \langle z, \phi_k\rangle$, so that $(z_k) \in l^2(\Lambda)$. Since both (v_k) and (z_k) are in $l^2(\Lambda)$, it follows that $(v_k z_k) \in l^1(\Lambda)$, so that the series in (6.9.1) is absolutely convergent. Now it follows from (6.9.1) that

$$\langle c_k, \eta\rangle z_k = \lim_{\substack{cs \to i\mu_k \\ s\in\mathbb{C}_0}} (s - i\mu_k)\langle \hat{y}(s), \eta\rangle \qquad \forall\, k \in \Lambda.$$

If for some $z \in X$ we have $\Psi z = 0$, then it follows that $\langle c_k, \eta\rangle z_k = 0$ for all $\eta \in Y$ and for all $k \in \Lambda$. For each $k \in \Lambda$ we can argue as follows: Since by assumption $c_k \neq 0$, taking $\eta = c_k$, it follows that $z_k = 0$. Thus we have proved that $z = 0$, so that (C2) holds. The converse implication follows from Remark 6.1.8. $\square$

We remark that the last proposition can be generalized easily to diagonalizable semigroups whose generator has compact resolvents.

Now we turn our attention to exact observability. For $\omega \in \mathbb{R}$ and $r > 0$, set

$$J(\omega, r) = \{k \in \Lambda \text{ such that } |\mu_k - \omega| < r\}. \tag{6.9.2}$$

Note that $J(\omega, r)$ is finite. An important role will be played by elements $z \in X$ of the form

$$z = \sum_{k \in J(\omega,r)} z_k \phi_k, \quad z_k \in \mathbb{C}. \tag{6.9.3}$$

We call such an element z a *wave packet of A* of parameters ω and r. Notice that $z \in \mathcal{D}(A^\infty)$. We show that the admissibility of C can be verified using wave packets.

Proposition 6.9.2. *For A and $\mathbb{T}$ as above, assume that $C \in \mathcal{L}(X_1, Y)$. Then C is an admissible observation operator for $\mathbb{T}$ if and only if for some $\gamma \geqslant 0$,*

$$\|Cz\|_Y \leqslant \gamma \|z\|$$

for every z that is a wave packet of A of parameters n and 1, where $n \in \mathbb{Z}$.

Proof. To prove the "if" part, assume that the condition in the proposition holds. Take $\mathrm{v} \in Y$, then for every z as in (6.9.3), with $\omega = n \in \mathbb{Z}$ and $r = 1$,

$$\left| \sum_{k \in J(n,1)} z_k \langle c_k, \mathrm{v}\rangle_Y \right| = |\langle Cz, \mathrm{v}\rangle_Y| \leqslant \gamma \|z\| \cdot \|\mathrm{v}\|_Y . \tag{6.9.4}$$

Taking the supremum over all finite sequences (z_k) (where $k \in J(n, 1)$) with Euclidean norm $\|z\| = 1$, we obtain that

$$\left(\sum_{k \in J(n,1)} |\langle c_k, \mathrm{v}\rangle_Y|^2 \right)^{\frac{1}{2}} \leqslant \gamma \|\mathrm{v}\|_Y \qquad \forall\, n \in \mathbb{Z}. \tag{6.9.5}$$

Define the observation operator $C^{\mathrm{v}} \in \mathcal{L}(X_1, \mathbb{C})$ by $C^{\mathrm{v}} z = \langle Cz, \mathrm{v}\rangle$, so that it is represented by the scalar sequence $c_k^{\mathrm{v}} = C^{\mathrm{v}} \phi_k = \langle C\phi_k, \mathrm{v}\rangle = \langle c_k, \mathrm{v}\rangle$. Then (6.9.5) shows that

$$\sum_{\operatorname{Im} \lambda_k \in (n-1, n+1)} |c_k^{\mathrm{v}}|^2 \leqslant \gamma^2 \|\mathrm{v}\|_Y^2 .$$

It follows from Proposition 5.3.5 and Remark 5.3.6 that C^{v} is admissible for $\mathbb{T}$. Since this conclusion holds for every $\mathrm{v} \in Y$, it follows from Corollary 5.2.5 that C is an admissible observation operator for $\mathbb{T}$.

Conversely, suppose that C is admissible. For every $\mathrm{v} \in Y$ we define C^{v} and c_k^{v} as earlier, let Ψ_τ be the output maps corresponding to $\mathbb{T}$ and C and let

$\Psi_\tau^{\rm v}$ be the output maps corresponding to $\mathbb{T}$ and $C^{\rm v}$. Then it is easy to see that $\|\Psi_\tau^{\rm v}\| \leqslant \|\Psi_\tau\| \cdot \|{\rm v}\|_Y$. From the last part of Proposition 5.3.5 (with $a = 1$) we see that

$$\sum_{{\rm Im}\,\lambda_k \in [n,n+1)} |c_k^{\rm v}|^2 \leqslant \frac{25}{9(1-e^{-2})} \|\Psi_1^{\rm v}\|^2 \qquad \forall\, n \in \mathbb{Z},\ {\rm v} \in Y\,,$$

so that, for a suitable $\gamma > 0$,

$$\sum_{{\rm Im}\,\lambda_k \in [n,n+1)} |\langle c_k, {\rm v}\rangle|^2 \leqslant \gamma \|{\rm v}\|_Y^2 \qquad \forall\, n \in \mathbb{Z},\ {\rm v} \in Y\,,$$

which implies (6.9.5). With the finite-dimensional version of the Cauchy–Schwarz inequality we obtain that (6.9.4) holds for every z as in (6.9.3), with $\omega = n \in \mathbb{Z}$ and $r = 1$. Clearly this implies the condition in the proposition. □

It is clear that in the above proposition, the parameter 1 could be replaced with any positive number (by rescaling the time, and hence the frequency axis).

The main result of this section is the following:

Theorem 6.9.3. *For A and C as above, the following statements are equivalent:*

(S1) *There exist $r, \delta > 0$ such that for all $\omega \in \mathbb{R}$ and for every wave packet of A of parameters ω and r, denoted by z, we have*

$$\|Cz\|_Y \geqslant \delta \|z\|_X\,. \tag{6.9.6}$$

(S2) *There exist $r, \delta > 0$ such that* (6.9.6) *holds for every wave packet of A of parameters μ_n and $2r$, where $n \in \Lambda$.*

(S3) *(A, C) is exactly observable.*

Moreover, if (S1) *or* (S2) *holds for some $r, \delta > 0$, then (A, C) is exactly observable in any time*

$$\tau > \pi \sqrt{\frac{1}{r^2} + \frac{4K^2(r)}{r\delta^2}}\,, \tag{6.9.7}$$

where $K : (0,\infty) \to [0,\infty)$ is the non-increasing function defined by

$$K(r) = \sup_{s \in \mathbb{C}_r} \sqrt{{\rm Re}\, s}\, \|C(sI - A)^{-1}\|_{\mathcal{L}(X,Y)}\,. \tag{6.9.8}$$

Note that $K(r)$ is finite according to Theorem 4.3.7, and it is obviously non-increasing. In order to prove this theorem, we need a lemma.

Lemma 6.9.4. *For each $r > 0$ and $\omega \in \mathbb{R}$, we define the subspace $V(\omega, r) \subset X$ by*

$$V(\omega, r) = \{\phi_k \mid k \in J(\omega, r)\}^\perp, \tag{6.9.9}$$

where $J(\omega, r)$ is as in (6.9.2)*. Let $A_{\omega,r}$ be the part of A in $V(\omega, r)$* (*see Definition* 2.4.1)*. If K is the non-increasing positive function from* (6.9.8)*, then*

$$\|C(i\omega I - A_{\omega,r})^{-1}\|_{\mathcal{L}(V(\omega,r),Y)} \leqslant \frac{2K(r)}{\sqrt{r}} \qquad \forall\, \omega \in \mathbb{R}\,. \tag{6.9.10}$$

Proof. For $\omega \in \mathbb{R}$ and $r > 0$, set $s = r + i\omega$. Using the resolvent identity,

$$(i\omega I - A_{\omega,r})^{-1} = (sI - A_{\omega,r})^{-1}\left[I + r(i\omega I - A_{\omega,r})^{-1}\right]. \tag{6.9.11}$$

First we show that

$$\|(i\omega I - A_{\omega,r})^{-1}\|_{\mathcal{L}(V(\omega,r))} \leqslant \frac{1}{r}. \tag{6.9.12}$$

Indeed, let $f = \sum_{k\in\Lambda\setminus J(\omega,r)} f_k\phi_k$ be an element of $V(\omega,r)$. Then

$$\|(i\omega I - A_{\omega,r})^{-1}f\|^2 = \sum_{k\in\Lambda\setminus J(\omega,r)} \frac{|f_k|^2}{|\mu_k - \omega|^2}.$$

This and the fact that $|\mu_k - \omega| \geqslant r$ for all $k \in \Lambda \setminus J(\omega,r)$ imply (6.9.12).

On the other hand, clearly we have

$$\|C(sI - A_{\omega,r})^{-1}\|_{\mathcal{L}(V(\omega,r),Y)} \leqslant \|C(sI - A)^{-1}\|_{\mathcal{L}(X,Y)}.$$

Using (6.9.8) (in which we take $s = r + i\omega$) we obtain that

$$\|C(sI - A_{\omega,r})^{-1}\|_{\mathcal{L}(V(\omega,r),Y)} \leqslant \frac{K(r)}{\sqrt{r}} \qquad \forall\, \omega \in \mathbb{R}.$$

Applying C to both sides of (6.9.11) and using the last estimate, we obtain

$$\begin{aligned}\|C(i\omega I - A_{\omega,r})^{-1}\| &\leqslant \|C(sI - A_{\omega,r})^{-1}\| \cdot \left\|I + r(i\omega I - A_{\omega,r})^{-1}\right\| \\ &\leqslant \frac{K(r)}{\sqrt{r}}\left[1 + r\|(i\omega I - A_{\omega,r})^{-1}\|\right].\end{aligned}$$

Using (6.9.12), this reduces to (6.9.10). □

Proof of Theorem 6.9.3. First we show that the statements (S1) and (S2) are equivalent. It is clear that (S1) implies (S2) with $r/2$ in place of r (take $\omega = \mu_n$). Conversely, assume that (S2) holds for some $r, \delta > 0$, and let $\omega \in \mathbb{R}$. Then either $J(\omega,r)$ is empty, or there exists $n \in J(\omega,r)$. In the latter case, one can easily check that $J(\omega,r) \subset J(\mu_n, 2r)$. Consequently, in both cases (S1) holds for r and δ.

Now we show that (S3) implies (S1). Assume that (A,C) is exactly observable. By Theorem 6.6.1, there exist constants $M, m > 0$ such that (6.6.1) holds. For $r = \frac{1}{M\sqrt{2}}$ and $\omega \in \mathbb{R}$, let $z = \sum_{k\in J(\omega,r)} z_k\phi_k$, where $z_k \in \mathbb{C}$. Then we have

$$\|(i\omega I - A)z\|^2 = \sum_{k\in J(\omega,r)} |i(\omega - \mu_k)z_k|^2 \leqslant \frac{1}{2M^2}\|z\|^2.$$

The above and (6.6.1) imply that (S1) holds with $r = \frac{1}{M\sqrt{2}}$ and $\delta = \frac{1}{m\sqrt{2}}$.

Finally we prove that (S1) implies (S3), and also that (A,C) is exactly observable in any time τ satisfying (6.9.7). For this, we show that (S1) implies (6.6.1), and then we apply Theorem 6.6.1. Take $z \in \mathcal{D}(A)$ and represent it in the basis (ϕ_k): $z = \sum_{k\in\Lambda} z_k\phi_k$. Take $\omega \in \mathbb{R}$ and $r > 0$ and decompose $z = \zeta_1 + \zeta_2$, where

$$\zeta_1 = \sum_{k\in J(\omega,r)} z_k\phi_k, \qquad \zeta_2 = \sum_{k\notin J(\omega,r)} z_k\phi_k.$$

Then we have

$$\|Cz\|^2 = \|C\zeta_1\|^2 + \|C\zeta_2\|^2 + 2\,\mathrm{Re}\,\langle C\zeta_1, C\zeta_2\rangle .$$

The above implies, by using the elementary inequality

$$2\,\mathrm{Re}\,\langle C\zeta_1, C\zeta_2\rangle \geqslant -\eta\|C\zeta_1\|^2 - \frac{1}{\eta}\|C\zeta_2\|^2 \qquad \forall\, \eta > 0,$$

that

$$\|Cz\|^2 \geqslant (1-\eta)\|C\zeta_1\|^2 - \frac{1-\eta}{\eta}\|C\zeta_2\|^2 \qquad \forall\, \eta > 0. \tag{6.9.13}$$

According to (S1) we can choose r, $\delta > 0$ such that (6.9.6) holds for all $\omega \in \mathbb{R}$ and for every wave packet of A of parameters ω and r. Since ζ_1 is such a wave packet, from (6.9.13) and (6.9.6) we obtain that, for every $\eta > 0$,

$$\|Cz\|^2 \geqslant \delta^2(1-\eta)\|\zeta_1\|^2 - \frac{1-\eta}{\eta}\|C(i\omega I - A_{\omega,r})^{-1}(i\omega I - A_{\omega,r})\zeta_2\|^2 ,$$

where $A_{\omega,r}$ is as in Lemma 6.9.4. By Lemma 6.9.4 it follows that

$$\|Cz\|^2 \geqslant \delta^2(1-\eta)\|\zeta_1\|^2 - \frac{1-\eta}{\eta}\cdot\frac{4K^2(r)}{r}\,\|(i\omega I - A_{\omega,r})\zeta_2\|^2 \qquad \forall\, \eta \in (0,1). \tag{6.9.14}$$

On the other hand, we have

$$\|(i\omega I - A)z\|^2 \geqslant \|(i\omega I - A_{\omega,r})\zeta_2\|^2 .$$

The above relation and (6.9.14) imply that, for every $M, m > 0$,

$$\begin{aligned} &M^2\|(i\omega I - A)z\|^2 + m^2\|Cz\|^2 \\ &\quad \geqslant m^2\delta^2(1-\eta)\|\zeta_1\|^2 \left(M^2 - m^2\frac{1-\eta}{\eta}\cdot\frac{4K^2(r)}{r}\right)\|(i\omega I - A_{\omega,r})\zeta_2\|^2 . \end{aligned} \tag{6.9.15}$$

We have to be careful in choosing good values for M, η and m, in order to obtain (6.6.1) with M as small as possible. First we choose M such that

$$M > \sqrt{\frac{1}{r^2} + \frac{4K^2(r)}{r\delta^2}} . \tag{6.9.16}$$

Afterwards, we choose $\eta \in (0,1)$ sufficiently close to 1 such that

$$M^2 > \frac{1}{r^2} + \frac{4K^2(r)}{\eta r\delta^2} > \frac{1}{r^2} + \frac{4K^2(r)}{r\delta^2},$$

and we choose $m = \frac{1}{\delta\sqrt{1-\eta}}$. These choices imply that

$$M^2 - m^2\frac{1-\eta}{\eta}\cdot\frac{4K^2(r)}{r} > \frac{1}{r^2} .$$

With these choices, we can rewrite (6.9.15) as follows:

$$M^2\|(i\omega I - A)z\|^2 + m^2\|Cz\|^2 \geqslant \|\zeta_1\|^2 + \frac{1}{r^2}\|(i\omega I - A_{\omega,r})\zeta_2\|^2 .$$

The above estimate, the fact that $\|(i\omega I - A_{\omega,r})\zeta_2\|^2 \geqslant r^2\|\zeta_2\|^2$ and the orthogonality of ζ_1 and ζ_2 imply (6.6.1). According to Theorem 6.6.1, the pair (A, C) is exactly observable in any time $\tau > M\pi$. Since M can be any number satisfying (6.9.16), it follows that (A, C) is exactly observable in any time τ satisfying (6.9.7).

If (S2) holds for some r, $\delta > 0$ then, as we have seen earlier in this proof, (S1) holds with the same constants r, δ. Therefore, again it follows that (A, C) is exactly observable in any time τ satisfying (6.9.7). □

In some cases it is more convenient to check condition (S1) or (S2) only for "high frequencies". More precisely, the following result holds.

Proposition 6.9.5. *With the notation of this section, assume that*

1. *there exist* α, r, $\delta > 0$ *such that* (S2) *in Theorem* 6.9.3 *holds for for every* μ_k *with* $|\mu_k| > \alpha$*),*
2. *if* ϕ *is an eigenvector of* A*, then* $C\phi \neq 0$.

Then (A, C) *is exactly observable in any time* τ *satisfying* (6.9.7).

Proof. As in Lemma 6.9.4 we denote $V(0, \alpha) = \{\phi_k \mid k \in J(0, \alpha)\}^\perp$ and $A_{0,\alpha}$ is the part of A in $V(0, \alpha)$. We also introduce $C_{0,\alpha}$ as the restriction of C to $\mathcal{D}(A_{0,\alpha})$. Note that $C_{0,\alpha}$ is an admissible observation operator for the semigroup generated by $A_{0,\alpha}$. The first assumption in the proposition means that $(A_{0,\alpha}, C_{0,\alpha})$ satisfies condition (S2) in Theorem 6.9.3. According to this theorem, $(A_{0,\alpha}, C_{0,\alpha})$ is exactly observable in any time τ satisfying (6.9.7). Since $C\phi \neq 0$ for every eigenvector ϕ of A, all the assumptions in Proposition 6.4.4 are satisfied. Consequently, (A, C) is exactly observable in any time τ satisfying (6.9.7). □

Corollary 6.9.6. *Assume that the eigenvalues* $(i\mu_k)_{k\in\Lambda}$ *of* A *are simple and that they are ordered such that the sequence* $(\mu_k)_{k\in\Lambda}$ *is strictly increasing. Moreover, assume that* $\lim_{|k|\to\infty}(\mu_{k+1} - \mu_k) = \infty$ *and that there exist* β_1, $\beta_2 > 0$ *such that*

$$\beta_1 \leqslant \|c_k\| \leqslant \beta_2 \qquad \forall\, k \in \Lambda .$$

Then C *is an admissible observation operator for* $\mathbb{T}$ *and the pair* (A, C) *is exactly observable in any time* $\tau > 0$.

Proof. The admissibility of C for $\mathbb{T}$ follows from Remark 5.3.8.

For an arbitrary $\tau > 0$ we choose $r > 0$ such that

$$\tau > \pi\sqrt{\frac{1}{r^2} + \frac{4K^2(r)}{r\beta_1^2}} ,$$

where $K(r)$ is as in (6.9.8). Since $\mu_{k+1} - \mu_k \to \infty$, there exists $\alpha > 0$ such that every wave packet of A of parameters μ_n and $2r$ with $|\mu_n| > \alpha$ is formed of only ϕ_n. Consequently, we can apply Proposition 6.9.5 with $\delta = \beta_1$ to get the conclusion. □

Remark 6.9.7. It is easy to check that the above corollary provides alternative proofs for the exact observability results from Examples 6.7.3 and 6.7.6.

6.10 The clamped Euler–Bernoulli beam with torque observation at an endpoint

In this section we consider a system modeling the vibrations of an Euler–Bernoulli beam clamped at both ends. The output is the torque at the left end. Due to the boundary conditions, the fourth-order derivative operator appearing here is not the square of a second-order derivative operator, as it was in Examples 6.8.3 and 6.8.4. Thus, unlike in those examples, the study of this system cannot be based on properties of a corresponding system governed by the Schrödinger equation.

The system we study is described by the equations

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^4 w}{\partial x^4}(x,t) = 0, \quad (x,t) \in (0,1) \times [0,\infty), \tag{6.10.1}$$

$$w(0,t) = w(1,t) = 0, \quad t \geqslant 0, \tag{6.10.2}$$

$$\frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(1,t) = 0, \quad t \geqslant 0, \tag{6.10.3}$$

$$w(x,0) = w_0(x), \quad \frac{\partial w}{\partial t}(x,0) = w_1(x), \quad x \in [0,1], \tag{6.10.4}$$

where w stands for the transverse displacement of the beam. The output is

$$y(t) = \frac{\partial^2 w}{\partial x^2}(0,t) \qquad \forall\, t \geqslant 0.$$

Let $H = L^2[0,1]$ and let $A_0 : \mathcal{D}(A_0) \to H$ be the strictly positive fourth derivative operator defined in Example 3.4.13. Recall that

$$H_1 = \mathcal{H}^4(0,1) \cap \mathcal{H}_0^2(0,1), \quad H_{\frac{1}{2}} = \mathcal{H}_0^2(0,1).$$

Denote $X = H_{\frac{1}{2}} \times H$ and let $A : X_1 \to X$ be the operator defined by

$$X_1 = H_1 \times H_{\frac{1}{2}}, \quad A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}.$$

We know from Proposition 3.7.6 that A is skew-adjoint, so that it generates a unitary group $\mathbb{T}$ on X. Let $C \in \mathcal{L}(X_1, \mathbb{C})$ be the observation operator defined by

$$C \begin{bmatrix} f \\ g \end{bmatrix} = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(0) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X_1. \tag{6.10.5}$$

The main result in this section is the following.

Proposition 6.10.1. *C is an admissible observation operator for $\mathbb{T}$ and (A, C) is exactly observable in any time $\tau > 0$. In PDEs terms, this means that if $\tau > 0$, then there exists $k_\tau > 0$ such that the solution w of (6.10.1)–(6.10.4) satisfies*

$$\int_0^\tau \left|\frac{\partial^2 w}{\partial x^2}(0,t)\right|^2 \mathrm{d}t \geqslant k_\tau^2 \left(\|w_0\|^2_{\mathcal{H}^2(0,1)} + \|w_1\|^2_{L^2[0,1]}\right) \qquad \forall \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in \mathcal{D}(A).$$

We know from Example 3.4.13 that there exists an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ in H formed of eigenvectors of A_0. In order to prove Proposition 6.10.1 we need more information on the eigenvalues and the eigenfunctions of A_0.

Lemma 6.10.2. *With the above notation, the eigenvalues of A_0 are simple and they can be ordered in a strictly increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ such that*

$$\lambda_k = \pi^4 \left(k - \frac{1}{2}\right)^4 + a_k, \tag{6.10.6}$$

where $(a_k)_{k\in\mathbb{R}}$ is a sequence converging exponentially to zero. Denote by φ_k a normalized eigenvector corresponding to λ_k. There exists $m > 0$ such that

$$\frac{1}{\sqrt{\lambda_k}} \left|\frac{\mathrm{d}^2\varphi_k}{\mathrm{d}x^2}(0)\right| \geqslant m \qquad \forall\, k \in \mathbb{N} \tag{6.10.7}$$

and

$$\lim \frac{1}{\sqrt{\lambda_k}} \left|\frac{\mathrm{d}^2\varphi_k}{\mathrm{d}x^2}(0)\right| = 2. \tag{6.10.8}$$

Proof. $\lambda > 0$ is an eigenvalue of A_0 iff there exists $f \in \mathcal{D}(A_0)$, $f \neq 0$, such that

$$\begin{cases} \dfrac{\mathrm{d}^4 f}{\mathrm{d}x^4}(x) = \lambda f(x), \quad x \in (0,1), \\ f(0) = f(1) = 0, \\ \dfrac{\mathrm{d}f}{\mathrm{d}x}(0) = \dfrac{\mathrm{d}f}{\mathrm{d}x}(1) = 0. \end{cases}$$

From the first equation above it follows that

$$f(x) = p_1 \cos(\xi x) + p_2 \sin(\xi x) + p_3 \cosh(\xi x) + p_4 \sinh(\xi x),$$

where $\xi = \lambda^{\frac{1}{4}}$ and $p_1, p_2, p_3, p_4 \in \mathbb{C}$. From $f(0) = 0$ we get that $p_1 + p_3 = 0$ while from $\frac{\mathrm{d}f}{\mathrm{d}x}(0) = 0$ we get that $p_2 + p_4 = 0$. Thus,

$$f(x) = p_1 [\cos(\xi x) - \cosh(\xi x)] + p_2 [\sin(\xi x) - \sinh(\xi x)]\,. \tag{6.10.9}$$

This and the boundary conditions $f(1) = 0$ and $\frac{\mathrm{d}f}{\mathrm{d}x}(1) = 0$ yield

$$\begin{cases} \;\;[\cos(\xi) - \cosh(\xi)]p_1 + [\sin(\xi) - \sinh(\xi)]p_2 \,=\, 0\,, \\ -[\sin(\xi) + \sinh(\xi)]p_1 + [\cos(\xi) - \cosh(\xi)]p_2 \,=\, 0\,. \end{cases}$$

This homogeneous system of equations in the unknowns p_1 and p_2 admits a non-trivial solution iff the corresponding determinant is zero, i.e.,

$$[\cos(\xi) - \cosh(\xi)]^2 + [\sin(\xi) - \sinh(\xi)][\sin(\xi) + \sinh(\xi)] \,=\, 0\,,$$

which is equivalent to

$$\cos(\xi)\cosh(\xi) \,=\, 1\,. \tag{6.10.10}$$

If ξ satisfies (6.10.10), then, by solving the homogeneous system of two equations, we obtain

$$p_1 \,=\, \gamma(\cos(\xi) - \cosh(\xi))\,, \quad p_2 \,=\, \gamma(\sin(\xi) + \sinh(\xi))\,, \tag{6.10.11}$$

where $\gamma \in \mathbb{C} \setminus \{0\}$ is arbitrary. It follows that the eigenvalues of A_0 are simple.

On the other hand, it is not difficult to check that the set formed by all the positive solutions of (6.10.10) can be ordered to form strictly increasing sequence $(\xi_k)_{k\geqslant 1}$ such that

$$\xi_k \,=\, \pi\left(k - \frac{1}{2}\right) + \widetilde{a}_k\,, \tag{6.10.12}$$

where $(\widetilde{a}_k)_{k\geqslant 1}$ is a sequence converging exponentially to zero. From this we clearly obtain (6.10.6), since $\lambda_k = \xi_k^4$.

We still have to show (6.10.7). By combining (6.10.9) and (6.10.11) it follows that

$$\varphi_k(x) \,=\, \gamma_k \psi_k(x) \qquad \forall\, k \in \mathbb{N}\,, \tag{6.10.13}$$

where

$$\begin{aligned} \psi_k(x) \,=\, & [\cos(\xi_k) - \cosh(\xi_k)][\cos(\xi_k x) - \cosh(\xi_k x)] \\ & + [\sin(\xi_k) + \sinh(\xi_k)][\sin(\xi_k x) - \sinh(\xi_k x)] \qquad \forall\, k \in \mathbb{N} \end{aligned} \tag{6.10.14}$$

and $\gamma_k > 0$ is chosen such that $\|\varphi_k\|_H = 1$. From (6.10.14) it follows that

$$\psi_k(x) \,=\, g_k(x) + h_k(x)\,, \tag{6.10.15}$$

where the significant term is

$$g_k(x) \,=\, \frac{1}{2} e^{\xi_k}[\sin(\xi_k x) - \cos(\xi_k x)]\,,$$

in the sense that $\lim e^{-\xi_k}\|g_k\|_H \,=\, \frac{1}{2}$, while $\lim e^{-\xi_k}\|h_k\|_H \,=\, 0$. Therefore, the condition $\|\varphi_k\|_H = 1$ implies that

$$\lim \gamma_k e^{\xi_k} \,=\, 2\,. \tag{6.10.16}$$

On the other hand, from (6.10.13) and (6.10.14) it follows that

$$\frac{\mathrm{d}^2\varphi_k}{\mathrm{d}x^2}(0) = 2\gamma_k\xi_k^2\left[\cosh(\xi_k) - \cos(\xi_k)\right] \qquad \forall\, k \in \mathbb{N}. \tag{6.10.17}$$

From the above, (6.10.16) and the fact that $\lim\cos(\xi_k) = 0$, it follows that (6.10.8) holds. Therefore, in order to get the conclusion (6.10.7) it suffices to show that

$$\frac{\mathrm{d}^2\varphi_k}{\mathrm{d}x^2}(0) \neq 0 \qquad \forall\, k \in \mathbb{N}. \tag{6.10.18}$$

If we had that $\frac{\mathrm{d}^2\varphi_k}{\mathrm{d}x^2}(0) = 0$ for some $k \in \mathbb{N}$, then from (6.10.10) and (6.10.17) it would follow that

$$\cos(\xi_k)\cosh(\xi_k) = 1 \quad \text{and} \quad \cos(\xi_k) = \cosh(\xi_k).$$

These equations imply that either $\cos(\xi_k) = \cosh(\xi_k) = 1$ or $\cos(\xi_k) = \cosh(\xi_k) = -1$, with $\xi_k > 0$, which is not possible. We have thus shown (6.10.18). □

We are now in a position to prove the main result in this section.

Proof of Proposition 6.10.1. . Denote $\mu_k = \sqrt{\lambda_k}$. For all $k \in \mathbb{N}$ we define $\varphi_{-k} = -\varphi_k$ and $\mu_{-k} = -\mu_k$. We know from Proposition 3.7.7 that A is diagonalizable, with the eigenvalues $(i\mu_k)_{k\in\mathbb{Z}^*}$ corresponding to the orthonormal basis of eigenvectors

$$\phi_k = \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{i\mu_k}\varphi_k \\ \varphi_k\end{bmatrix} \qquad \forall\, k \in \mathbb{Z}^*. \tag{6.10.19}$$

Therefore, by applying Lemma 6.10.2 it follows that the eigenvalues $(i\mu_k)_{k\in\mathbb{Z}^*}$ of A are simple with $\lim_{|k|\to\infty|}|\mu_{k+1} - \mu_k| = \infty$. Moreover, from (6.10.19) it follows that

$$C\phi_k = \frac{1}{i\mu_k\sqrt{2}}\frac{\mathrm{d}^2\varphi_k}{\mathrm{d}x^2}(0) \qquad \forall\, k \in \mathbb{Z}^*.$$

From the above formula, together with (6.10.7) and (6.10.8), it follows, by applying Corollary 6.9.6, that C is an admissible observation operator for $\mathbb{T}$ and that the pair (A, C) is exactly observable in any time $\tau > 0$. □

6.11 Remarks and bibliographical notes on Chapter 6

General remarks. For finite-dimensional linear systems, the concept of observability has been introduced in the works of Rudolf Kalman around 1960. Besides being the dual property to controllability, an important motivation for studying this concept was that it implies the existence of state observers with any desired exponential decay rate of the estimation error. Various infinite-dimensional generalizations were soon formulated; see, for example, Delfour and Mitter [47], Fattorini and Russell [62], and we refer the reader to the survey paper by Russell [199] for an overview of (approximately) the first ten years of the development of this theory. In general, the approach was more of an ad hoc PDE nature, using eigenfunction expansions and moment problems. The general functional analytic formulation of infinite-dimensional control or observation problems was not well understood, except for bounded control or observation operators.

The next big step was the so-called HUM (Hilbert Uniqueness Method) approach, which by itself is a very simple idea (renorm the state space of an approximately observable system by $\|z\|_{HUM} = \|\Psi_\tau z\|$ and it becomes exactly observable), but coupled with a clever use of multipliers for specific PDEs, this yielded powerful new results for wave and plate equations; see Lions [156] and Lagnese and Lions [139]. We refer the reader to the survey paper of Lagnese [138] for an account of this development. The next big breakthrough was the application of microlocal analysis to observability problems initiated in Bardos, Lebeau and Rauch [15] (we say more on this in the bibliographic comments on Chapter 7). The functional analytic approach to exact observability started late and was overshadowed by the PDE developments. An important early paper is Dolecki and Russell [51], and the book by Avdonin and Ivanov [9] belongs to this stream (also Nikol'skii [178]).

Section 6.1. The material here is fairly standard. We are not aware of any reference that contains Proposition 6.1.9. Proposition 6.1.12 is taken from Tucsnak and Weiss [222]. A stronger version of Proposition 6.1.13 has appeared in Weiss and Rebarber [234, Proposition 5.5]. Corollary 6.1.14 has appeared in Datko [41] and has been generalized in various directions; see, for example, Weiss [227] and the references therein. Proposition 6.1.15 is due to Xu, Liu and Yung [238].

Section 6.2. This contains simple and well-known examples whose origin we cannot trace. Interesting problems which can be seen as extensions (still in one space dimension) of these examples concern networks of strings and of beams, which have been studied in the monographs Lagnese and Leugering [140] and Dager and Zuazua [40].

Section 6.3. It contains new results inspired by the results of Hadd [84].

Section 6.4. Simultaneous exact observability is the dual concept of simultaneous exact controllability. The latter concept will be studied in Chapter 11 and relevant bibliographic comments on it are contained in Section 11.7. The results in this section are taken from from Tucsnak and Weiss [222], except the simple Proposition 6.4.4, which is new. (The paper [222] had a mistake in the statement of the main result, which of course has been rectified here.)

Section 6.5. The material in this section (except for the last proposition) is based on Russell and Weiss [201] (the Hautus test (6.5.2) appeared for the first time in [201]). Proposition 6.5.7 is due to Jacob and Zwart [122, Section 4], and it is the strengthening of an earlier result by Grabowski and Callier [75].

We mention that in [201] the following result has also been derived:

Theorem 6.11.1. *Suppose that* $A \in \mathcal{L}(X)$ *and* $C \in \mathcal{L}(X, Y)$*. If for every* $s \in \mathbb{C}_-$ *there is an* $m_s > 0$ *such that for each* $z \in X$*,*

$$\|(sI - A)z\|^2 + \|Cz\|^2 \geqslant m_s \cdot \|z\|^2,$$

then (A, C) *is exactly observable in infinite time.*

This theorem follows also from results in Rodman [191]. We mention that Hautus-type necessary conditions for estimatability (a weaker property than exact observability) were given in Weiss and Rebarber [234]. The paper Hadd and Zhong [85] contains some necessary as well as some sufficient Hautus-type conditions for the stabilizability of systems with delays.

Assume that A is diagonalizable and its eigenvalues are *properly spaced*, which means that $|\lambda_j - \lambda_k| \geqslant \delta \cdot |\mathrm{Re}\,\lambda_k|$ for all $j, k \in \mathbb{N}$ with $j \neq k$, where $\delta > 0$. We denote $c_k = C\varphi_k$, where (φ_k) is an orthonormal basis in X such that $A\varphi_k = \lambda_k \varphi_k$. It has been shown in [201, Section 4] that under these assumptions, the estimate (6.5.2) is equivalent to the existence of a $\kappa > 0$ such that

$$\|c_k\|_Y^2 \geqslant \kappa |\mathrm{Re}\,\lambda_k| \qquad \forall\, n \in \mathbb{N}.$$

It has been conjectured in [201] that the Hautus test (6.5.2) is a sufficient condition for exact observability. This turned out to be false, a counterexample has been constructed in Jacob and Zwart [121] (with an analytic semigroup). Another counterexamle in Jacob and Zwart [122] shows that (6.5.2) does not even imply approximate observability, if we weaken the exponential stability assumption to strong stability. (More details on [122] are at the end of this section.) Today, we have a good hope that the conjecture from [201] may be true for normal semigroups, and a weaker hope that it may be true for contraction semigroups.

The paper Grabowski and Callier [75] contains the following theorem.

Theorem 6.11.2. *With the notation of Section* 6.5, (A, C) *is exactly observable if and only if there exists* $H \in \mathcal{L}(X)$*,* $H > 0$*, such that for all* $s \in \mathbb{C}$ *and* $z \in \mathcal{D}(A)$*,*

$$\frac{1}{|\mathrm{Re}\,s|^2} \langle (sI - A)z, H(sI - A)z \rangle + \frac{1}{|\mathrm{Re}\,s|} \|Cz\|^2 \geqslant \langle z, Hz \rangle. \tag{6.11.1}$$

This theorem implies (by taking $H = I$) that if $\mathbb{T}$ is a contraction semigroup and the Hautus condition (6.5.2) holds with $m = 1$, then (A, C) is exactly observable. Indeed, for $s \in \mathbb{C}_-$ the estimate (6.11.1) follows from (6.5.2), while for $s \in \mathbb{C}_0$, (6.11.1) follows from (3.1.2). The above theorem also implies Theorem 6.5.3 (see [75] for the details). The same paper [75] also gives interesting (but difficult to verify) necessary and sufficient conditions for admissibility.

Section 6.6. Most of the results in Sections 6.6 and 6.9 have been proved first for bounded observation operators and without specifying the exact observability time. This was done by using the equivalence of the exact observability property to the exponential stability of a certain semigroup obtained by feedback (a particular case of this implication has been used to prove Proposition 7.4.5). For example, earlier versions of Theorem 6.6.1 with bounded C (and without information on the observability time) were published in Zhou and Yamamoto [243], Chen et al. [34] and Liu [160].

The sufficiency of the Hautus-type condition in Theorem 6.6.1 for skew-adjoint generators with an unbounded admissible C and the estimates of the observability time have been obtained first in Burq and Zworski [27], with some additional technical assumptions on A and C. Our presentation of Theorem 6.6.1 follows closely Miller [170], who simplified the argument and generalized the result from [27]. The result in Proposition 6.6.4 is new, as far as we know.

Section 6.7. The derivation of exact observability results for abstract Schrödinger-type equations from properties of abstract wave-type equations are due to Miller [170]. Our contribution is a simpler proof, using instead of the "transmutation method" from [170] our simultaneous observability approach from Proposition 6.6.4. Moreover, we have made precise the state spaces and proved the admissibility results in Propositions 6.7.1 and 6.7.4.

Section 6.8. In its abstract form, the result in Proposition 6.8.2 is new, but its proof is essentially based on ideas used in Lebeau [150] in the study of the exact observability of the Euler–Bernoulli plate equation.

Section 6.9. Proposition 6.9.2 was given, with a different proof, in Ervedoza, Zheng and Zuazua [58]. As far as we know, the estimate of the observability time τ in Theorem 6.9.3 is new. Theorem 6.9.3 without information on τ was proved in Ramdani et al. [186]. Earlier versions with bounded C are in Chen et al. [34] and in Liu, Liu and Rao [161].

Section 6.10. The material here is standard and we cannot trace its origins. Haraux [94] investigated the exact observability of a clamped beam with distributed observation and we have used this reference for the computation of the eigenfunctions. Related material is also in Lagnese and Lions [139], Zhao and Weiss [242] and various papers by Guo; see, for example, [79] and the references therein.

The observability results of Jacob and Zwart. Recently, Jacob and Zwart have obtained the following results, contained in [122].

Theorem 6.11.3. *Let A be the generator of a strongly continuous group $\mathbb{T}$ on X satisfying*

$$M_1 e^{\alpha_1 t}\|z\| \leqslant \|\mathbb{T}_t z\| \leqslant M_2 e^{\alpha_2 t}\|z\| \qquad \forall\, z \in X\,,\ t \geqslant 0$$

for some constants $M_1, M_2 > 0$ and $\alpha_1 < \alpha_2 < 0$.

Assume that there exists $m > 0$ such that

$$\|((\alpha_2 + i\omega)I - A)z\|^2 + |\alpha_2| \cdot \|Cz\|^2 \geqslant m|\alpha_2|^2 \cdot \|z\|^2 \qquad \forall\, z \in \mathcal{D}(A),\ \omega \in \mathbb{R},$$

$$\frac{\alpha_2 - \alpha_1}{|\alpha_2|} < \frac{\sqrt{m}M_1}{4eM_2}.$$

Then (A, C) is exactly observable in time $\tau = 1/(\alpha_2 - \alpha_1)$.

Note that the second condition in the above theorem is the Hautus test (6.5.2) restricted to the vertical line where $\operatorname{Re} s = \alpha_2$. The above theorem is used in the proof of the following result about final state observability.

Theorem 6.11.4. *Let A be the generator of an exponentially stable and normal semigroup $\mathbb{T}$. Let C be an admissible observation operator for $\mathbb{T}$. Then the Hautus test* (6.5.2) *is sufficient for the final state observability of (A, C).*

From this theorem and Theorem 6.5.3 we can easily obtain the following generalization of Theorem 6.6.1 (a partial converse of Theorem 6.5.3).

Corollary 6.11.5. *Let A be the generator of an exponentially stable normal group $\mathbb{T}$. Let C be an admissible observation operator for $\mathbb{T}$. Then the Hautus test* (6.5.2) *is equivalent to the exact observability of (A, C).*

The above (not yet published) results from [122] are a natural continuation of important earlier results by the same authors. The main result of the paper of Jacob and Zwart [119], partially reproduced below, refers to systems with a diagonalizable semigroup and a finite-dimensional output space.

Theorem 6.11.6. *Assume that A is diagonalizable, it generates a strongly stable semigroup $\mathbb{T}$ and $Y = \mathbb{C}^n$. Let $C \in \mathcal{L}(X_1, Y)$ be infinite-time admissible. Then the following conditions are equivalent:*

(1) *(A, C) is exactly observable in infinite time.*
(2) *(A, C) satisfies the Hautus test* (6.5.2).
(3) *There exists $\mu > 0$ such that for every $(n+1)$-dimensional subspace $V \subset X$ that is invariant under $\mathbb{T}$, the solution P_V of the Lyapunov equation*

$$A_V^* P_V + P_V A_V \;=\; -\, C_V^* C_V$$

(which is unique, see Theorem 5.1.1*) satisfies $P_V > \mu I$. Here, A_V and C_V denote the restrictions of A and C to $\mathcal{D}(A) \cap V$.*

We mention that the one-dimensional version ($n = 1$) of the above result was obtained earlier by the same authors in [120], using different techniques.

Chapter 7

Observation for the Wave Equation

Notation. Throughout this chapter, Ω denotes a bounded open connected set in $\mathbb{R}^n$, where $n \in \mathbb{N}$. We assume that either the boundary $\partial\Omega$ is of class C^2 or Ω is a rectangular domain. The remaining part of the notation described below is used in the whole chapter, with the exception of Section 7.6, where some notation (like X and A) will have a different meaning.

Let A_0 be the Dirichlet Laplacian on Ω as defined in (3.6.3), so that $A_0 : \mathcal{D}(A_0) \to L^2(\Omega)$. Recall from Section 3.6 that A_0 is strictly positive. As usual, we denote $H = L^2(\Omega)$, $H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}})$ and $H_1 = \mathcal{D}(A_0)$, while $H_{-\frac{1}{2}}$ is the dual of $H_{\frac{1}{2}}$ with respect to the pivot space H.

According to Proposition 3.6.1 we have $H_{\frac{1}{2}} = \mathcal{H}_0^1(\Omega)$, $H_{-\frac{1}{2}} = \mathcal{H}^{-1}(\Omega)$. According to Theorem 3.6.2 and Remark 3.6.6, our assumptions on Ω imply that

$$H_1 = \mathcal{D}(A_0) = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega).$$

The norm in H will be simply denoted by $\|\cdot\|$.

We define $X = H_{\frac{1}{2}} \times H$, which is a Hilbert space with the inner product

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_X = \langle A_0^{\frac{1}{2}} f_1, A_0^{\frac{1}{2}} f_2 \rangle + \langle g_1, g_2 \rangle.$$

We define $\mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}$ (this is a dense subspace of X) and we define the operator $A : \mathcal{D}(A) \to X$ by

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad \text{i.e.,} \quad A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -A_0 f \end{bmatrix}. \tag{7.0.1}$$

Recall from Proposition 3.7.6 that A is skew-adjoint, so that it generates a unitary group $\mathbb{T}$ on X. In this chapter (as in Section 3.9) we denote by $v \cdot w$ the bilinear product of $v, w \in \mathbb{C}^n$ defined by $v \cdot w = v_1 w_1 + \cdots + v_n w_n$, and by $|\cdot|$ the Euclidean norm on $\mathbb{C}^n$.

For some fixed $x_0 \in \mathbb{R}^n$ we denote

$$m(x) = x - x_0 \qquad \forall\, x \in \mathbb{R}^n, \tag{7.0.2}$$

and we set (see Figure 7.1)

$$\Gamma(x_0) = \{x \in \partial\Omega \mid m(x)\cdot\nu(x) > 0\}, \qquad r(x_0) = \sup_{x\in\Omega} |m(x)|. \tag{7.0.3}$$

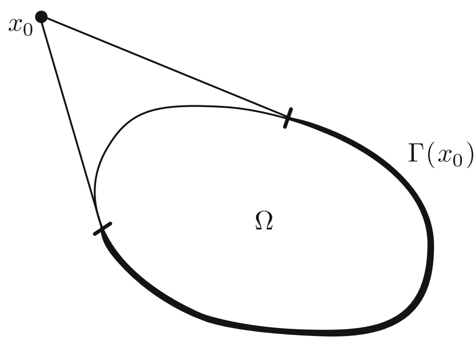

Figure 7.1: The set $\Gamma(x_0)$ is an open part of the boundary $\partial\Omega$.

7.1 An admissibility result for boundary observation

In this section we denote $Y = L^2(\Gamma)$, where Γ is an open subset of $\partial\Omega$ and we consider the operator $C \in \mathcal{L}(X_1, Y)$ defined by

$$C\begin{bmatrix} f \\ g \end{bmatrix} = \frac{\partial f}{\partial \nu}|_\Gamma \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X_1 = H_1 \times H_{\frac{1}{2}}, \tag{7.1.1}$$

where ν is the unit outward normal vector field on $\partial\Omega$. For the definition of the *normal derivative* $\frac{\partial f}{\partial \nu}$ see Section 13.6 in Appendix II.

Consider the following initial and boundary value problem:

$$\frac{\partial^2\eta}{\partial t^2} - \Delta\eta = 0 \ \text{ in } \ \Omega\times(0,\infty), \tag{7.1.2}$$

$$\eta = 0 \quad \text{ on } \ \partial\Omega\times(0,\infty), \tag{7.1.3}$$

$$\eta(x,0) = f(x), \quad \frac{\partial\eta}{\partial t}(x,0) = g(x) \ \text{ for } \ x\in\Omega. \tag{7.1.4}$$

By applying Proposition 3.8.7 with A_0 chosen as at the beginning of this chapter, we obtain the following result.

Proposition 7.1.1. *If $f \in H_1 = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$ and $g \in H_{\frac{1}{2}} = \mathcal{H}_0^1(\Omega)$, then the initial and boundary value problem (7.1.2)–(7.1.4) has a unique solution*

$$\eta \in C([0,\infty), H_1) \cap C^1([0,\infty), H_{\frac{1}{2}}) \cap C^2([0,\infty), H), \tag{7.1.5}$$

and this solution satisfies

$$\|\nabla \eta(\cdot,t)\|^2 + \left\| \frac{\partial \eta}{\partial t}(\cdot,t) \right\|^2 = \|\nabla f\|^2 + \|g\|^2 \qquad \forall\, t \geqslant 0. \tag{7.1.6}$$

Remark 7.1.2. Recall from Proposition 3.4.3 and Remark 3.4.4 that $A_0^{\frac{1}{2}}$ is unitary from $H_{\frac{1}{2}}$ to H and from H to $H_{-\frac{1}{2}}$. This fact, combined with (7.1.6), implies that the solution η from Proposition 7.1.1 satisfies

$$\|\eta(\cdot,t)\|^2 + \left\| \frac{\partial \eta}{\partial t}(\cdot,t) \right\|^2_{\mathcal{H}^{-1}(\Omega)} = \|f\|^2 + \|g\|^2_{\mathcal{H}^{-1}(\Omega)} \qquad \forall\, t \geqslant 0. \tag{7.1.7}$$

The main result of this section is the following.

Theorem 7.1.3. *For every $\tau > 0$ there exists a constant $K_\tau > 0$ such that for every $f \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$, $g \in \mathcal{H}_0^1(\Omega)$, the solution η of (7.1.2)–(7.1.4) satisfies*

$$\int_0^T \int_{\partial\Omega} \left| \frac{\partial \eta}{\partial \nu} \right|^2 \mathrm{d}\sigma \;\leqslant\; K_\tau^2 \left(\|\nabla f\|^2 + \|g\|^2 \right), \tag{7.1.8}$$

where $\mathrm{d}\sigma$ is the surface measure on $\partial\Omega$.

In other words, C is an admissible observation operator for $\mathbb{T}$.

The integral identities given in the next two lemmas are important tools for the proof of the above theorem and of other results in later sections.

Lemma 7.1.4. *Let $\varphi \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$ and $q \in C^1(\mathrm{clos}\ \Omega, \mathbb{R}^n)$. Then*

$$\begin{aligned} \mathrm{Re} \int_\Omega (q \cdot \nabla \overline{\varphi})\, \Delta\varphi \,\mathrm{d}x = \frac{1}{2} \int_{\partial\Omega} (q\cdot\nu) \left| \frac{\partial \varphi}{\partial \nu} \right|^2 \mathrm{d}\sigma + \frac{1}{2} \int_\Omega (\operatorname{div} q) |\nabla\varphi|^2 \,\mathrm{d}x \\ - \sum_{l,k=1}^n \mathrm{Re} \int_\Omega \frac{\partial q_k}{\partial x_l} \frac{\partial \overline{\varphi}}{\partial x_k} \frac{\partial \varphi}{\partial x_l} \,\mathrm{d}x. \end{aligned} \tag{7.1.9}$$

Proof. By an integration by parts (see Theorem 13.7.1 in Appendix II) we obtain (taking $f = \frac{\partial\varphi}{\partial x_l}$, $g = q_k \frac{\partial\overline{\varphi}}{\partial x_k}$, and then summing with respect to all the indices l, k) that

$$\begin{aligned} \mathrm{Re} \int_\Omega (q \cdot \nabla \overline{\varphi})\, \Delta\varphi \,\mathrm{d}x &= \sum_{k,l=1}^n \mathrm{Re} \int_\Omega q_k \frac{\partial \overline{\varphi}}{\partial x_k} \frac{\partial^2 \varphi}{\partial x_l^2} \,\mathrm{d}x \\ &= -\sum_{k,l=1}^n \mathrm{Re} \int_\Omega \frac{\partial}{\partial x_l} \left(q_k \frac{\partial \overline{\varphi}}{\partial x_k} \right) \frac{\partial \varphi}{\partial x_l} \,\mathrm{d}x + \mathrm{Re} \int_{\partial\Omega} (q\cdot\nabla\overline{\varphi}) \frac{\partial\varphi}{\partial\nu} \,\mathrm{d}\sigma. \end{aligned}$$

This formula and the fact that $\nabla\overline{\varphi}_{|\partial\Omega} = \frac{\partial\overline{\varphi}}{\partial\nu}\nu_{|\partial\Omega}$ (which follows from the fact that φ is vanishing on $\partial\Omega$) imply that

$$\operatorname{Re}\int_\Omega (q\cdot\nabla\overline{\varphi})\,\Delta\varphi\,\mathrm{d}x = -\frac{1}{2}\int_\Omega q\cdot\nabla(|\nabla\varphi|^2)\,\mathrm{d}x - \sum_{l,k=1}^{n}\operatorname{Re}\int_\Omega \frac{\partial q_k}{\partial x_l}\frac{\partial\overline{\varphi}}{\partial x_k}\frac{\partial\varphi}{\partial x_l}\,\mathrm{d}x + \int_{\partial\Omega}(q\cdot\nu)\left|\frac{\partial\varphi}{\partial\nu}\right|^2\mathrm{d}\sigma. \quad (7.1.10)$$

From formula (13.3.1) we see that

$$q\cdot\nabla(|\nabla\varphi|^2) = \operatorname{div}\left(|\nabla\varphi|^2 q\right) - (\operatorname{div} q)|\nabla\varphi|^2.$$

This, combined with the Gauss formula (13.7.3) and (7.1.10), leads to (7.1.9). □

Lemma 7.1.5. *Let*

$$w\in C\left([0,\infty);\mathcal{H}^2(\Omega)\cap\mathcal{H}^1_0(\Omega)\right)\cap C^1\left([0,\infty);\mathcal{H}^1_0(\Omega)\right)\cap C^2\left([0,\infty);L^2(\Omega)\right),$$

let $q\in C^1(\operatorname{clos}\Omega,\mathbb{R}^n)$ and let $G\in C^1([0,\infty);\mathbb{R})$. If we denote

$$\frac{\partial^2 w}{\partial t^2} - \Delta w = F, \quad (7.1.11)$$

then for every $\tau\geqslant 0$,

$$\int_0^\tau G\int_{\partial\Omega}(q\cdot\nu)\left|\frac{\partial w}{\partial\nu}\right|^2\mathrm{d}\sigma\,\mathrm{d}t = 2\operatorname{Re}\left[G\int_\Omega\frac{\partial w}{\partial t}(q\cdot\nabla\overline{w})\,\mathrm{d}x\right]_{t=0}^{t=\tau}$$
$$+2\sum_{k,l=1}^{n}\operatorname{Re}\int_0^\tau G\int_\Omega\frac{\partial q_k}{\partial x_l}\frac{\partial\overline{w}}{\partial x_k}\frac{\partial w}{\partial x_l}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau G\int_\Omega(\operatorname{div} q)\left(\left|\frac{\partial w}{\partial t}\right|^2 - |\nabla w|^2\right)\mathrm{d}x\,\mathrm{d}t$$
$$-2\operatorname{Re}\int_0^\tau G\int_\Omega F(q\cdot\nabla\overline{w})\,\mathrm{d}x\,\mathrm{d}t - 2\operatorname{Re}\int_0^\tau\frac{\mathrm{d}G}{\mathrm{d}t}\int_\Omega\frac{\partial w}{\partial t}(q\cdot\nabla\overline{w})\,\mathrm{d}x\,\mathrm{d}t. \quad (7.1.12)$$

Proof. We take the inner products in $L^2([0,\tau];L^2(\Omega))$ of both sides of (7.1.11) with $Gq\cdot\nabla w$. For the first term we integrate by parts with respect to time:

$$\int_0^\tau\int_\Omega G\frac{\partial^2 w}{\partial t^2}(q\cdot\nabla\overline{w})\,\mathrm{d}x\,\mathrm{d}t = \left[G\int_\Omega\frac{\partial w}{\partial t}(q\cdot\nabla\overline{w})\,\mathrm{d}x\right]_{t=0}^{t=\tau}$$
$$-\int_0^\tau G\int_\Omega\frac{\partial w}{\partial t}\left[q\cdot\nabla\left(\frac{\partial\overline{w}}{\partial t}\right)\right]\mathrm{d}x\,\mathrm{d}t - \int_0^\tau\frac{\mathrm{d}G}{\mathrm{d}t}\int_\Omega\frac{\partial w}{\partial t}(q\cdot\nabla\overline{w})\,\mathrm{d}x\,\mathrm{d}t.$$

From here, using an integration by parts in space applied to the second term on the right (the Green formula (13.7.2)) and using that $\frac{\partial w}{\partial t}=0$ on $\partial\Omega$, we obtain

$$\operatorname{Re}\int_0^\tau G\int_\Omega\frac{\partial^2 w}{\partial t^2}(q\cdot\nabla\overline{w})\,\mathrm{d}x\,\mathrm{d}t = \operatorname{Re}\left[G\int_\Omega\frac{\partial w}{\partial t}(q\cdot\nabla\overline{w})\,\mathrm{d}x\right]_{t=0}^{t=\tau}$$
$$+\frac{1}{2}\int_0^\tau G\int_\Omega\left|\frac{\partial w}{\partial t}\right|^2(\operatorname{div} q)\,\mathrm{d}x\,\mathrm{d}t - \operatorname{Re}\int_0^\tau\frac{\mathrm{d}G}{\mathrm{d}t}\int_\Omega\frac{\partial w}{\partial t}(q\cdot\nabla\overline{w})\,\mathrm{d}x\,\mathrm{d}t. \quad (7.1.13)$$

By applying Lemma 7.1.4 we obtain that the contribution of the second term from the left-hand side of (7.1.11) is

$$\begin{aligned}\operatorname{Re} \int_0^\tau G \int_\Omega (q \cdot \nabla \overline{w})\, \Delta w \,\mathrm{d}x\,\mathrm{d}t &= \frac{1}{2} \int_0^\tau G \int_{\partial\Omega} (q \cdot \nu) \left| \frac{\partial w}{\partial \nu} \right|^2 \mathrm{d}\sigma\,\mathrm{d}t \\ + \frac{1}{2} \int_0^\tau G \int_\Omega (\mathrm{div}\ q) |\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t &- \sum_{k,l=1}^n \operatorname{Re} \int_0^\tau G \int_\Omega \frac{\partial q_k}{\partial x_l} \frac{\partial \overline{w}}{\partial x_k} \frac{\partial w}{\partial x_l} \,\mathrm{d}x\,\mathrm{d}t .\end{aligned}$$

The above relation, combined with (7.1.11) and (7.1.13), implies (7.1.12). □

The proof of Theorem 7.1.3 is based on the above lemma, applied to a particular vector field q, which is constructed below.

Lemma 7.1.6. *Assume that the boundary $\partial\Omega$ is of class C^2. Then there exists a vector field $h \in C^1(\operatorname{clos}\Omega, \mathbb{R}^n)$ such that $h(x) = \nu(x)$ for all $x \in \partial\Omega$.*

Proof. The compactness of $\partial\Omega$ implies that there exists a finite set $\{x_1, \ldots, x_m\} \subset \partial\Omega$ such that for every $k \in \{1, \ldots, m\}$ there exists a neighborhood V_k of x_k in $\mathbb{R}^n$ and a system of orthonormal coordinates denoted by $(y_{k,1}, \ldots, y_{k,n})$ such that, in these coordinates

$$V_k = \{(y_{k,1}, \ldots, y_{k,n}) \mid -a_{k,j} < y_{k,j} < a_{k,j},\ 1 \leqslant j \leqslant n\}$$

and the sets V_k cover $\partial\Omega$ (i.e., $\partial\Omega$ is contained in their union). The fact that $\partial\Omega$ is of class C^2 (see Definition 13.5.2 in Appendix II) implies that for every $k \in \{1, \ldots, m\}$ there exists a C^2 function φ_k defined on

$$V_k' = \{(y_{k,1}, \ldots, y_{k,n-1}) \mid -a_{k,j} < y_j < a_{k,j},\ 1 \leqslant j \leqslant n-1\},$$

such that

$$\begin{aligned} |\varphi_k(y_k')| &\leqslant \frac{a_{k,n}}{2} \quad \text{for every} \quad y_k' = (y_{k,1}, \ldots, y_{k,n-1}) \in V_k', \\ \Omega \cap V_k &= \{y_k = (y_k', y_{k,n}) \in V_k \mid y_{k,n} < \varphi_k(y_k')\}, \\ \partial\Omega \cap V_k &= \{y_k = (y_k', y_{k,n}) \in V_k \mid y_n = \varphi_k(y')\}. \end{aligned}$$

Moreover, from the definition of the outward normal field ν in Appendix II it follows that

$$\nu(x) = \psi_k(x) \qquad \forall\, x \in \partial\Omega \cap V_k,$$

where, for every $k \in \{1, \ldots, m\}$ and $x \in V_k$ we have

$$\psi_k(x) = \frac{1}{\sqrt{1 + \left[\frac{\partial \varphi_k}{\partial y_{k,1}}(y_k')\right]^2 + \cdots + \left[\frac{\partial \varphi_k}{\partial y_{n-1,k}}(y_k')\right]^2}} \begin{pmatrix} -\frac{\partial \varphi_k}{\partial y_{k,1}}(y_k') \\ \vdots \\ -\frac{\partial \varphi_k}{\partial y_{k,n-1}}(y_k') \\ 1 \end{pmatrix}.$$

Let V_0 be an open set such that

$$\operatorname{clos} V_0 \subset \Omega, \quad \Omega \subset \bigcup_{k=0}^{m} V_k .$$

Let K be the compact set

$$K = \operatorname{clos} \bigcup_{k=0}^{m} V_k ,$$

and let $(\phi_k)_{0\leqslant k\leqslant m} \subset \mathcal{D}(\mathbb{R}^n)$ be a real-valued partition of unity subordinated to the covering $(V_k)_{0\leqslant k\leqslant m}$ of K (see Proposition 13.1.6). We extend ψ_k by 0 outside V_k and we denote

$$h(x) = \sum_{k=0}^{m} \phi_k(x)\psi_k(x) \qquad \forall\, x \in \mathbb{R}^n .$$

Then clearly $h \in C^1(\operatorname{clos}\Omega; \mathbb{R}^n)$ and $h(x) = \nu(x)$ on $\partial\Omega$. $\square$

We are now in a position to prove the main result of this section.

Proof of Theorem 7.1.3. First we consider the case when $\partial\Omega$ is of class C^2. Let h be the vector field from Lemma 7.1.6. By applying Lemma 7.1.5 with $w = \eta$, where η is the solution of (7.1.2)–(7.1.4) (so that $F = 0$), $q = h$ and $G = 1$ we obtain that

$$\int_0^\tau \int_{\partial\Omega} \left|\frac{\partial \eta}{\partial \nu}\right|^2 \mathrm{d}\sigma\, \mathrm{d}t = 2\mathrm{Re} \int_\Omega \frac{\partial \eta}{\partial t}\,(h\cdot\nabla\overline{\eta})\,\mathrm{d}x \Big|_{t=0}^{t=\tau}$$
$$+\, 2\sum_{k,l=1}^{n} \mathrm{Re} \int_0^\tau \int_\Omega \frac{\partial h_k}{\partial x_l}\frac{\partial \overline{\eta}}{\partial x_k}\frac{\partial \eta}{\partial x_l}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \operatorname{div}(h) \left(\left|\frac{\partial \eta}{\partial t}\right|^2 - |\nabla \eta|^2\right) \mathrm{d}x\,\mathrm{d}t .$$

The second term on the right-hand side can be estimated using that for each $t \geqslant 0$,

$$\left|\sum_{k,l=1}^{n} \mathrm{Re} \int_\Omega \frac{\partial h_k}{\partial x_l}\frac{\partial \overline{\eta}}{\partial x_k}\frac{\partial \eta}{\partial x_l}\,\mathrm{d}x\right| \leqslant n\|h\|_{C^1(\overline{\Omega})} \int_\Omega |\nabla \eta|^2\,\mathrm{d}x .$$

The other terms on the right-hand side can be estimated similarly, leading to

$$\int_0^\tau \int_{\partial\Omega} \left|\frac{\partial \eta}{\partial \nu}\right|^2 \mathrm{d}\sigma\,\mathrm{d}t \leqslant M^2 \int_0^\tau \int_\Omega \left(\left|\frac{\partial \eta}{\partial t}\right|^2 + |\nabla \eta|^2\right) \mathrm{d}x\,\mathrm{d}t$$
$$+\, M^2 \int_\Omega \left(\left|\frac{\partial \eta}{\partial t}(x,0)\right|^2 + |\nabla \eta(x,0)|^2\right) \mathrm{d}x$$
$$+\, M^2 \int_\Omega \left(\left|\frac{\partial \eta}{\partial t}(x,\tau)\right|^2 + |\nabla \eta(x,\tau)|^2\right) \mathrm{d}x ,$$

where $M > 0$ is a constant depending only on $\|h\|_{C^1(\overline{\Omega})}$. This, combined with (7.1.6), implies that (7.1.8) holds for $K_\tau^2 = M^2(\tau+2)$.

If Ω is rectangular we can assume, without loss of generality, that it is centered at zero and aligned with the coordinate system. We do similar calculations, but with $q = x_j e_j$, where (e_j) is the jth vector in the standard basis of $\mathbb{R}^n$. We obtain an estimate similar to (7.1.8) but instead of integration on $\partial\Omega$ we now have integration on the two faces perpendicular to e_j only. Adding these estimates for $j = 1, \ldots, n$ we obtain the desired estimate. $\square$

7.2 Boundary exact observability

In this section we study the exact observability of the wave equation with Neumann boundary observation. Recall that the particular case of the wave equation in one space dimension has been already investigated in Example 6.2.1. In other terms, this section is devoted to the exact observability of the pair (A, C), with A given by (7.0.1) and C given by (7.1.1). We first show that the observed part Γ of the boundary cannot be chosen arbitrarily.

Proposition 7.2.1. *Assume that $n = q + r$ with $q, r \in \mathbb{N}$ and that $\Omega = \Omega_1 \times \Omega_2$, where Ω_1 (respectively, Ω_2) is an open bounded set in $\mathbb{R}^q$ (respectively $\mathbb{R}^r$). Assume that there exists a non-empty open set $\mathcal{O}_1 \subset \Omega_1$ such that $\Gamma \cap \operatorname{clos}(\mathcal{O}_1 \times \Omega_2) = \emptyset$. Then the pair (A, C) is not exactly observable.*

Proof. Indeed, let $(\omega_p^2)_{p\in\mathbb{N}}$ be the strictly increasing sequence of the eigenvalues of the Dirichlet Laplacian on Ω_2 and let $(\psi_p)_{p\in\mathbb{N}}$ be a corresponding sequence of orthonormal (in $L^2(\Omega_2)$) eigenvectors. Choose a fixed $f \in \mathcal{D}(\mathcal{O}_1)$ such that $\|f\|_{L^2(\Omega_1)} = 1$. For $x \in \mathbb{R}^n$ we denote $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$, where $x_1 \in \mathbb{R}^q$ and $x_2 \in \mathbb{R}^r$. For all $p \in \mathbb{N}$ we set

$$\varphi_p(x) = \psi_p(x_2) f(x_1), \qquad z_p = \begin{bmatrix} \varphi_p \\ i\omega_p \varphi_p \end{bmatrix}.$$

From our assumption on Γ it follows that

$$C z_p = \frac{\partial \varphi_p}{\partial \nu}|_\Gamma = 0 \qquad \forall\, p \in \mathbb{N}. \tag{7.2.1}$$

On the other hand,

$$\begin{aligned} \|(i\omega_p - A) z_p\|_X^2 &= \|(\omega_p^2 - A_0)\varphi_p\|_H^2 \\ &= \int_\Omega |\psi_p(x_2)\Delta f(x_1)|^2 \,\mathrm{d}x_1\,\mathrm{d}x_2 = \|\Delta f\|_{L^2(\Omega_1)}^2 . \end{aligned} \tag{7.2.2}$$

Relations (7.2.1) and (7.2.2), together with the fact that $\lim_{p\to\infty} \|z_p\|_X = \infty$, show that the pair (A, C) does not satisfy condition (6.6.1) in Theorem 6.6.1. Consequently the pair (A, C) is not exactly observable. $\square$

In order to give a sufficient condition for the exact observability of the pair (A, C), first we derive an integral relation.

Lemma 7.2.2. *Let $\tau > 0$, $x_0 \in \mathbb{R}^n$ and let m be defined by* (7.0.2)*. Let*

$$w \in C\left([0,\tau]; \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)\right) \cap C^1\left([0,\tau]; \mathcal{H}_0^1(\Omega)\right) \cap C^2\left([0,\tau]; L^2(\Omega)\right)$$

and denote

$$\frac{\partial^2 w}{\partial t^2} - \Delta w = F\,. \tag{7.2.3}$$

Then

$$\begin{aligned}\int_0^\tau \int_{\partial\Omega} (m\cdot\nu)\left|\frac{\partial w}{\partial \nu}\right|^2 \mathrm{d}\sigma\,\mathrm{d}t \ =\ & \int_0^\tau \int_\Omega \left(\left|\frac{\partial w}{\partial t}\right|^2 + |\nabla w|^2\right) \mathrm{d}x\,\mathrm{d}t \\ & + \mathrm{Re}\left[\int_\Omega \left[2m\cdot\nabla\overline{w} + (n-1)\overline{w}\right]\frac{\partial w}{\partial t}\,\mathrm{d}x\right]_{t=0}^{t=\tau} \\ & - 2\mathrm{Re}\int_0^\tau\int_\Omega F(m\cdot\nabla\overline{w})\,\mathrm{d}x\,\mathrm{d}t - (n-1)\mathrm{Re}\int_0^\tau\int_\Omega F\overline{w}\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned} \tag{7.2.4}$$

Proof. We apply Lemma 7.1.5 with $q = m$ and $G = 1$. By using the facts that $\operatorname{div} m = n$ and that $\frac{\partial m_k}{\partial x_l} = \delta_{kl}$ (the Kronecker symbol), relation (7.1.12) yields

$$\begin{aligned}\int_0^\tau \int_{\partial\Omega} (m\cdot\nu)\left|\frac{\partial w}{\partial \nu}\right|^2 \mathrm{d}\sigma\,\mathrm{d}t \ =\ & \int_0^\tau \int_\Omega \left(\left|\frac{\partial w}{\partial t}\right|^2 + |\nabla w|^2\right) \mathrm{d}x\,\mathrm{d}t \\ + (n-1)\int_0^\tau\int_\Omega \left(\left|\frac{\partial w}{\partial t}\right|^2 - |\nabla w|^2\right)\mathrm{d}x\,\mathrm{d}t & + 2\mathrm{Re}\int_\Omega (m\cdot\nabla\overline{w})\frac{\partial w}{\partial t}\,\mathrm{d}x\Big|_{t=0}^{t=\tau} \\ & - 2\mathrm{Re}\int_0^\tau\int_\Omega F(m\cdot\nabla\overline{w})\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned} \tag{7.2.5}$$

On the other hand, by taking the inner product in $L^2([0,\tau]; L^2(\Omega))$ of both sides of (7.2.3) with w it follows (integrating by parts using (13.7.2)) that

$$\int_0^\tau\int_\Omega \left(\left|\frac{\partial w}{\partial t}\right|^2 - |\nabla w|^2\right)\mathrm{d}x\,\mathrm{d}t = \ \mathrm{Re}\int_\Omega \frac{\partial w}{\partial t}\,\overline{w}\,\mathrm{d}x\Big|_0^\tau - \mathrm{Re}\int_0^\tau\int_\Omega F\overline{w}\,\mathrm{d}x\,\mathrm{d}t\,.$$

From the above relation and (7.2.5) we obtain the conclusion (7.2.4). □

We shall also need the following technical lemma.

Lemma 7.2.3. *Let $x_0 \in \mathbb{R}^n$, let the vector field m and the number $r(x_0)$ be as in* (7.0.2) *and* (7.0.3)*. Then we have*

$$\left|\int_\Omega g\left[2m\cdot\nabla f + (n-1)f\right]\mathrm{d}x\right| \leqslant r(x_0)\left(\|\nabla f\|^2 + \|g\|^2\right) \qquad \forall \begin{bmatrix} f \\ g\end{bmatrix} \in X\,. \tag{7.2.6}$$

Proof. We have, using (13.3.1) in the last step,

$$\begin{aligned}\|2m\cdot\nabla f+(n-1)f\|^2 \;&=\; \int_\Omega |2m\cdot\nabla f|^2\,\mathrm{d}x+(n-1)^2\int_\Omega |f|^2\,\mathrm{d}x\\ +\,2(n-1)\int_\Omega m\cdot\nabla(|f|^2)\,\mathrm{d}x \;&=\; \int_\Omega |2m\cdot\nabla f|^2\,\mathrm{d}x+(n-1)^2\int_\Omega |f|^2\,\mathrm{d}x\\ &\quad+2(n-1)\int_\Omega \operatorname{div}(|f|^2 m)\,\mathrm{d}x-2n(n-1)\int_\Omega |f|^2\,\mathrm{d}x\,.\end{aligned}$$

By the Gauss formula (13.7.3) and since $f=0$ on $\partial\Omega$, the above formula yields

$$\|2m\cdot\nabla f+(n-1)f\|^2 \;=\; \|2m\cdot\nabla f\|^2-(n^2-1)\|f\|^2,$$

so that

$$\|2m\cdot\nabla f+(n-1)f\| \;\leqslant\; \|2m\cdot\nabla f\|\,.$$

From the above and the Cauchy–Schwarz inequality it follows that

$$\begin{aligned}&\left|\int_\Omega g\,[2m\cdot\nabla f+(n-1)f)]\,\mathrm{d}x\right|\leqslant 2\|g\|\cdot\|m\cdot\nabla f\|\\ &\leqslant r(x_0)\|g\|^2+\frac{1}{r(x_0)}\,\|m\cdot\nabla f\|^2_{L^2(\Omega)}\leqslant r(x_0)\left(\|\nabla f\|^2+\|g\|^2\right),\end{aligned}$$

where we have used that $|m(x)|\leqslant r(x_0)$ for all $x\in\Omega$. □

For the following theorem, recall the definition of $\Gamma(x_0)$ from (7.0.3).

Theorem 7.2.4. *Assume that Γ is an open subset of $\partial\Omega$ such that $\Gamma\supset\Gamma(x_0)$ for some $x_0\in\mathbb{R}^n$ and that $\tau>2r(x_0)$. Then for every $f\in\mathcal{H}^2(\Omega)\cap\mathcal{H}^1_0(\Omega)$, $g\in\mathcal{H}^1_0(\Omega)$, the solution η of* (7.1.2)–(7.1.4) *satisfies*

$$\int_0^\tau\int_\Gamma\left|\frac{\partial\eta}{\partial\nu}\right|^2\mathrm{d}\sigma\,\mathrm{d}t \;\geqslant\; \frac{\tau-2r(x_0)}{r(x_0)}(\|\nabla f\|^2+\|g\|^2), \tag{7.2.7}$$

so that the pair (A,C) is exactly observable in any time $\tau>2r(x_0)$.

Proof. We apply Lemma 7.2.2 with $w=\eta$. By using the facts that (7.2.3) holds with $F=0$ and that, by (7.1.6),

$$\int_0^\tau\int_\Omega\left(\left|\frac{\partial w}{\partial t}\right|^2+|\nabla w|^2\right)\mathrm{d}x\,\mathrm{d}t \;=\; \tau(\|\nabla f\|^2+\|g\|^2),$$

relation (7.2.4) yields

$$\begin{aligned}\int_0^\tau\int_{\partial\Omega}(m\cdot\nu)\left|\frac{\partial\eta}{\partial\nu}\right|^2\mathrm{d}\sigma\,\mathrm{d}t \;&=\; \tau(\|\nabla f\|^2+\|g\|^2)\\ &\quad+\operatorname{Re}\left[\int_\Omega[2m\cdot\nabla\overline{\eta}+(n-1)\overline{\eta}]\,\frac{\partial\eta}{\partial t}\,\mathrm{d}x\right]_{t=0}^{t=\tau}.\end{aligned} \tag{7.2.8}$$

On the other hand, by applying Lemma 7.2.3 and (7.1.6) it follows that, for every $t \geqslant 0$, we have

$$\left|\int_\Omega [2m\cdot\nabla\overline{\eta}+(n-1)\overline{\eta}]\,\frac{\partial\eta}{\partial t}\,\mathrm{d}x\right| \leqslant r(x_0)\left(\|\nabla\eta\|^2+\left\|\frac{\partial\eta}{\partial t}\right\|^2\right)$$
$$= r(x_0)\left(\|\nabla f\|^2+\|g\|^2\right).$$

Consequently,

$$\left|\left[\int_\Omega \frac{\partial\eta}{\partial t}\,[2m\cdot\nabla\overline{\eta}+(n-1)\overline{\eta}]\,\mathrm{d}x\right]_{t=0}^{t=\tau}\right| \leqslant 2r(x_0)\left(\|\nabla f\|^2+\|g\|^2\right).$$

By using the above estimate in (7.2.8) we obtain that

$$\int_0^\tau\int_{\partial\Omega}(m\cdot\nu)\left|\frac{\partial\eta}{\partial\nu}\right|^2\mathrm{d}\sigma\,\mathrm{d}t \;\geqslant\; (\tau-2r(x_0))\left(\|\nabla f\|+\|g\|^2\right).$$

Finally, by using the fact that $m(x)\cdot\nu(x)\leqslant 0$ for $x\in\partial\Omega\setminus\Gamma$ and then the fact that $|m(x)\cdot\nu(x)|\leqslant r(x_0)$ for all $x\in\Gamma$, we obtain the conclusion (7.2.7). □

Remark 7.2.5. The assumption $\Gamma\supset\Gamma(x_0)$ in Theorem 7.2.4 is a simple sufficient condition for the observability inequality (7.2.7). This condition is not necessary: there are also other open subsets Γ of $\partial\Omega$ for which the exact observability estimate

$$\int_0^\tau\int_\Gamma\left|\frac{\partial\eta}{\partial\nu}\right|^2\mathrm{d}\sigma\,\mathrm{d}t \;\geqslant\; k_\tau^2(\|\nabla f\|^2+\|g\|^2) \tag{7.2.9}$$

holds for some $\tau>0$, $k_\tau>0$ and every solution η of (7.1.2)–(7.1.4). For a discussion of other sufficient conditions, of which one is "almost" necessary, see Section 7.7.

Remark 7.2.6. According to Remark 6.1.3, the estimate (7.2.9) is equivalent to

$$\int_0^\tau\int_\Gamma\left|\frac{\partial\dot\eta}{\partial\nu}\right|^2\mathrm{d}\sigma\,\mathrm{d}t \;\geqslant\; k_\tau^2(\|\Delta f\|^2+\|\nabla g\|^2) \qquad \forall\begin{bmatrix} f\\ g\end{bmatrix}\in\mathcal{D}(A^2). \tag{7.2.10}$$

7.3 A perturbed wave equation

In this section we consider the following perturbation of the initial and boundary value problem (7.1.2)–(7.1.4):

$$\frac{\partial^2\eta}{\partial t^2}-\Delta\eta+a\eta = 0 \;\text{ in }\; \Omega\times(0,\infty), \tag{7.3.1}$$

$$\eta = 0 \quad\text{on}\quad \partial\Omega\times(0,\infty), \tag{7.3.2}$$

$$\eta(x,0) = f(x),\quad \frac{\partial\eta}{\partial t}(x,0) = g(x) \;\text{ for }\; x\in\Omega, \tag{7.3.3}$$

where $a\in L^\infty(\Omega)$ is a real-valued function.

Recall the notation $H = L^2(\Omega)$, $H_1 = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$, $A_0 : H_1 \to H$, $A_0 = -\Delta$, $H_{\frac{1}{2}} = \mathcal{H}_0^1(\Omega)$, $X = H_{\frac{1}{2}} \times H$, $A = \left[\begin{smallmatrix} 0 & I \\ -A_0 & 0 \end{smallmatrix}\right]$ and $\mathcal{D}(A) = X_1 = H_1 \times H_{\frac{1}{2}}$ introduced at the beginning of this chapter. Recall that $\|\cdot\|$ (without subscripts) stands for the norm in $L^2(\Omega)$. As in Section 7.1, Γ is an open subset of $\partial\Omega$ and $Y = L^2(\Gamma)$. The operator $C \in \mathcal{L}(X_1, Y)$ corresponds to Neumann boundary observation on Γ, as in (7.1.1). In order to study (7.3.1)–(7.3.3) we introduce several operators. First we define $P_0 \in \mathcal{L}(H)$ by

$$P_0 f = -af \qquad \forall\, f \in H. \tag{7.3.4}$$

We define $P \in \mathcal{L}(X)$ by $P = \left[\begin{smallmatrix} 0 & 0 \\ P_0 & 0 \end{smallmatrix}\right]$ and $A_P : \mathcal{D}(A_P) \to X$ by

$$\mathcal{D}(A_P) = \mathcal{D}(A), \qquad A_P = A + P. \tag{7.3.5}$$

Clearly we have $\|P\|_{\mathcal{L}(X)} \leqslant \|a\|_\infty$.

By combining Theorem 2.11.2 and Proposition 2.3.5 we obtain the following.

Proposition 7.3.1. *The operator A_P defined by* (7.3.5) *is the generator of a strongly continuous semigroup $\mathbb{T}$ on X with $\|\mathbb{T}_t\| \leqslant e^{t\|a\|_\infty}$ for every $t \geqslant 0$. In other words, if $f \in H_1$ and $g \in H_{\frac{1}{2}}$, then the initial and boundary value problem* (7.3.1)–(7.3.3) *has a unique solution*

$$\eta \in C([0,\infty), H_1) \cap C^1([0,\infty), H_{\frac{1}{2}}) \cap C^2([0,\infty), H),$$

and this solution satisfies

$$\left\| \frac{\partial \eta}{\partial t}(\cdot, t) \right\|^2 + \|\nabla \eta(\cdot, t)\|^2 \leqslant e^{2t\|a\|_\infty} \left(\|g\|^2 + \|\nabla f\|^2 \right) \qquad \forall\, t \geqslant 0. \tag{7.3.6}$$

We mention that the following identity is easy to prove (by checking that for any initial state in $\mathcal{D}(A)$, the time-derivative of the left-hand side is zero):

$$\int_\Omega \left(\left| \frac{\partial \eta}{\partial t} \right|^2 + |\nabla \eta|^2 + a|\eta|^2 \right) \mathrm{d}x = \int_\Omega \left(|g|^2 + |\nabla f|^2 + a|f|^2 \right) \mathrm{d}x.$$

The main result of this section is the following.

Theorem 7.3.2. *Assume that Γ is such that (A, C) is exactly observable in time τ_0. Then (A_P, C) is exactly observable in any time $\tau > \tau_0$. In other words, for every $\tau > \tau_0$, there exists $k_{P,\tau} > 0$ such that the solution η of* (7.3.1)–(7.3.3) *satisfies*

$$\int_0^\tau \int_\Gamma \left| \frac{\partial \eta}{\partial \nu} \right|^2 \mathrm{d}\sigma\, \mathrm{d}t \;\geqslant\; k_{P,\tau}^2 \left(\|\nabla f\|^2 + \|g\|^2 \right) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A_P). \tag{7.3.7}$$

To prove the above theorem, we will use an appropriate decomposition of X as a direct sum of invariant subspaces. To obtain this decomposition, we need the following characterization of the eigenvalues and eigenvectors of A_P.

Proposition 7.3.3. *With the above notation, $\phi = \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix}\right] \in \mathcal{D}(A_P)$ is an eigenvector of A_P, associated with the eigenvalue $i\mu$, if and only if φ is an eigenvector of $A_0 - P_0$, associated with the eigenvalue μ^2, and $\psi = i\mu\varphi$.*

Note that the number μ appearing above does not have to be real.

Proof. Suppose that $\mu \in \mathbb{C}$ and $\left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix}\right] \in X \setminus \{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\}$ are such that $A_P \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix}\right] = i\mu \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix}\right]$. According to the definition of A_P this is equivalent to

$$\psi = i\mu\varphi \quad \text{and} \quad (-A_0 + P_0)\varphi = i\mu\psi.$$

The above conditions hold iff

$$(A_0 - P_0)\varphi = \mu^2\varphi \quad \text{and} \quad \psi = i\mu\varphi. \qquad \square$$

Clearly, $A_0 - P_0$ is self-adjoint. By Remarks 2.11.3 and 3.6.4, it has compact resolvents, so that we may apply Proposition 3.2.12. We obtain that $A_0 - P_0$ is diagonalizable with an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ of eigenvectors and the corresponding family of real eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ satisfies $\lim_{k\to\infty} |\lambda_k| = \infty$. Since $A_0 - P_0 + \|P_0\| I \geqslant 0$, it follows that all the eigenvalues λ of $A_0 - P_0$ satisfy $\lambda > -\|P_0\|$. Hence, $\lim_{k\to\infty} \lambda_k = \infty$. Without loss of generality, we may assume that the sequence $(\lambda_k)_{k\in\mathbb{N}}$ is non-decreasing. We extend the sequence (φ_k) to a sequence indexed by $\mathbb{Z}^*$ by setting $\varphi_k = -\varphi_{-k}$ for every $k \in \mathbb{Z}_-$. We introduce the real sequence $(\mu_k)_{k\in\mathbb{Z}^*}$ by

$$\mu_k = \sqrt{|\lambda_k|} \ \text{ if } \ k > 0 \quad \text{and} \quad \mu_k = -\mu_{-k} \ \text{ if } \ k < 0.$$

We denote

$$W_0 = \operatorname{span}\left\{ \left[\begin{matrix} \frac{1}{i\operatorname{sign}(k)}\varphi_k \\ \varphi_k \end{matrix}\right] \middle| \ k \in \mathbb{Z}^*, \ \mu_k = 0 \right\}.$$

If $\operatorname{Ker}(A_0 - P_0) = \{0\}$, then of course W_0 is the zero subspace of X. Let $N \in \mathbb{N}$ be such that $\lambda_N > 0$. We denote

$$W_N = \operatorname{span}\left\{ \left[\begin{matrix} \frac{1}{i\mu_k}\varphi_k \\ \varphi_k \end{matrix}\right] \middle| \ k \in \mathbb{Z}^*, \ |k| < N, \ \mu_k \neq 0 \right\},$$

and define $Y_N = W_0 + W_N$. We also introduce the space

$$V_N = \operatorname{clos\,span}\left\{ \left[\begin{matrix} \frac{1}{i\mu_k}\varphi_k \\ \varphi_k \end{matrix}\right] \middle| \ |k| \geqslant N \right\}. \qquad (7.3.8)$$

Lemma 7.3.4. *We have $X = Y_N \oplus V_N$ and Y_N, V_N are invariant under $\mathbb{T}$.*

By $X = Y_N \oplus V_N$ we mean that $X = Y_N + V_N$ and $Y_N \cap V_N = \{0\}$.

Proof. Let $A_1 : \mathcal{D}(A_0) \to H$ be defined by

$$A_1 f = \sum_{\lambda_k = 0} \langle f, \varphi_k \rangle \varphi_k + \sum_{\lambda_k \neq 0} |\lambda_k| \langle f, \varphi_k \rangle \varphi_k \qquad \forall\, f \in \mathcal{D}(A_0).$$

Since the family $(\varphi_k)_{k \in \mathbb{N}}$ is an orthonormal basis in H and each φ_k is an eigenvector of A_1, it follows that A_1 is diagonalizable. Moreover, since the eigenvalues of A_1 are strictly positive, it follows that $A_1 > 0$. According to Proposition 3.4.9, the inner product on X, defined by

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_1 = \langle A_1^{\frac{1}{2}} f_1, A_1^{\frac{1}{2}} f_2 \rangle + \langle g_1, g_2 \rangle \qquad \forall \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \in X,$$

is equivalent to the original one (meaning that it induces a norm equivalent to the original norm). Let $\mathcal{A}_1$ be the operator on X defined by

$$\mathcal{D}(\mathcal{A}_1) = H_1 \times H_{\frac{1}{2}}, \quad \mathcal{A}_1 = \begin{bmatrix} 0 & I \\ -A_1 & 0 \end{bmatrix}.$$

According to Proposition 3.7.6, $\mathcal{A}_1$ is skew-adjoint on X (if endowed with the inner product $\langle \cdot, \cdot \rangle_1$). Consequently, by applying Proposition 3.7.7 we obtain that $Y_N = V_N^\perp$ (with respect to this inner product $\langle \cdot, \cdot \rangle_1$). It follows that $X = Y_N \oplus V_N$.

We still have to show that V_N and Y_N are invariant subspaces under $\mathbb{T}$. Since V_N is the closed span of a set of eigenvectors of A_P, its invariance under the action of $\mathbb{T}$ is clear. If $\mu_k = 0$, then

$$A_P \begin{bmatrix} \frac{1}{i\,\mathrm{sign}(k)} \varphi_k \\ \varphi_k \end{bmatrix} = \begin{bmatrix} \varphi_k \\ 0 \end{bmatrix} = \frac{1}{2} \left(\begin{bmatrix} \frac{1}{i\,\mathrm{sign}(k)} \varphi_k \\ \varphi_k \end{bmatrix} + \begin{bmatrix} \frac{1}{i\,\mathrm{sign}(-k)} \varphi_{-k} \\ \varphi_{-k} \end{bmatrix} \right) \in W_0,$$

so that W_0 is invariant under $\mathbb{T}$. If $|k| < N$ and $\lambda_k < 0$, then

$$(A_0 - P_0)\varphi_k = -\mu_k^2 \varphi_k,$$

so that

$$A_P \begin{bmatrix} \frac{1}{i\mu_k} \varphi_k \\ \varphi_k \end{bmatrix} = \begin{bmatrix} \varphi_k \\ \frac{\mu_k}{i} \varphi_k \end{bmatrix} = i\mu_k \begin{bmatrix} \frac{1}{i\mu_k} \varphi_k \\ -\varphi_k \end{bmatrix} = i\mu_k \begin{bmatrix} \frac{1}{i\mu_{-k}} \varphi_{-k} \\ \varphi_{-k} \end{bmatrix} \in W_N.$$

If $|k| < N$ and $\lambda_k > 0$, then

$$A_P \begin{bmatrix} \frac{1}{i\mu_k} \varphi_k \\ \varphi_k \end{bmatrix} = i\mu_k \begin{bmatrix} \frac{1}{i\mu_k} \varphi_k \\ \varphi_k \end{bmatrix} \in W_N.$$

Thus W_N, and hence also $Y_N = W_0 + W_N$, are invariant for $\mathbb{T}$. $\square$

Lemma 7.3.5. *With the notation from the beginning of this section and* (7.3.8)*, let* $N \in \mathbb{N}$ *be such that* $\lambda_N > \|a\|_{L^\infty}$. *Let us denote by* $P_{V_N} \in \mathcal{L}(V_N, X)$ *the restriction of* P *to* V_N. *Then*

$$\|P_{V_N}\| \leqslant \frac{\|a\|_{L^\infty}}{\sqrt{\lambda_N - \|a\|_{L^\infty}}}.$$

Proof. Take a finite linear combination of the vectors φ_k with $k \geqslant N$:

$$f = \sum_{k=N}^{M} \alpha_k \varphi_k \,, \tag{7.3.9}$$

so that $\|f\|^2 = \sum_{k=N}^M |\alpha_k|^2$. It is easy to see that

$$\|\nabla f\|^2 + \langle af, f\rangle = 2\mathrm{Re} \sum_{N \leqslant k,j \leqslant M} \alpha_k \overline{\alpha}_j \langle (-\Delta + a)\varphi_k, \varphi_j\rangle = \sum_{k=N}^{M} \lambda_k |\alpha_k|^2 \geqslant \lambda_N \|f\|^2 .$$

From here we see that

$$\|\nabla f\|^2 \geqslant (\lambda_N - \|a\|_{L^\infty}) \|f\|^2 .$$

Now take z to be a finite linear combination of the eigenvectors of A_P in V_N:

$$z \in \operatorname{span} \left\{ \begin{bmatrix} \frac{1}{i\mu_k}\varphi_k \\ \varphi_k \end{bmatrix} \middle| \ |k| \geqslant N \right\},$$

so that in particular $z \in V_N$ and $z = \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$, with f as in (7.3.9). Therefore

$$\begin{aligned} \|P_{V_N} z\|_X = \|Pz\|_X &= \|af\| \leqslant \|a\|_{L^\infty} \|f\| \\ &\leqslant \frac{\|a\|_{L^\infty}}{\sqrt{\lambda_N - \|a\|_\infty}} \|\nabla f\| \leqslant \frac{\|a\|_{L^\infty}}{\sqrt{\lambda_N - \|a\|_\infty}} \|z\|_X . \end{aligned}$$

Since all the vectors like our z are dense in V_N, it follows that the above estimate holds for all $z \in V_N$, and this implies the estimate in the lemma. □

Proof of Theorem 7.3.2. Let $N \in \mathbb{N}$ be such that $\lambda_N > 0$ and let A_N and C_N be the parts of A_P and of C in V_N, where V_N has been defined in (7.3.8). (Thus, $A_N = (A + P)|_{V_N}$ and $C_N = C|_{V_N}$.) We claim that for $N \in \mathbb{N}$ large enough the pair (A_N, C_N) (with state space V_N) is exactly observable in time τ_0.

By assumption there exists $k_{\tau_0} > 0$ such that

$$\int_0^{\tau_0} \int_\Gamma \left| \frac{\partial \eta}{\partial \nu} \right|^2 \mathrm{d}\sigma \, \mathrm{d}t \geqslant k_{\tau_0}^2 \left(\|\nabla f\|^2 + \|g\|^2 \right) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A),$$

where η is a solution of the unperturbed wave equation (7.1.2)–(7.1.4), which corresponds to the pair (A, C). As in Section 6.3, we denote

$$|\!|\!| C |\!|\!|_{\tau_0} = \|\Psi_{\tau_0}\|_{\mathcal{L}(X, L^2([0,\tau_0];Y)},$$

where Ψ_{τ_0} is the output map for time τ_0 of the unperturbed pair (A, C). According to Proposition 6.3.3, (A_N, C_N) is exactly observable in time τ_0 if

$$\|P_{V_N}\| \leqslant \frac{k_{\tau_0}}{\tau_0 M |\!|\!| C |\!|\!|_{\tau_0}} ,$$

where $M = \sup_{t\in[0,\tau_0]} \|\mathbb{T}_t\|$. Notice that the right-hand side above is independent of N. Thus, according to Lemma 7.3.5, for N large enough the above condition will be satisfied. Hence, for N large enough, (A_N, C_N) is exactly observable in time τ_0.

On the other hand, if $\phi = \left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right] \in \mathcal{D}(A_P)$ is an eigenvector of A_P, associated with the eigenvalue $i\mu$, such that $C\phi = 0$, then, according to Proposition 7.3.3, $\varphi \in H_1$ is an eigenvector of $A_0 - P_0$, associated with the eigenvalue μ^2, i.e., $\varphi \in H_1$ satisfies

$$\Delta\varphi - a\varphi + \mu^2\varphi = 0. \tag{7.3.10}$$

Moreover, the condition $C\phi = 0$ is equivalent to

$$\frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \Gamma. \tag{7.3.11}$$

As shown in Corollary 15.2.2 from Appendix III, the only function $\varphi \in H_1$ satisfying (7.3.10) and (7.3.11) is $\varphi = 0$. Since, by Proposition 7.3.3, $\psi = i\mu\varphi = 0$ we obtain that $\phi = 0$. By the finite-dimensional version of the Hautus test in Remark 1.5.2, it follows that the pair $(\widetilde{A}_N, \widetilde{C}_N)$, where $\widetilde{A}_N$ and $\widetilde{C}_N$ are the parts of A_P and of C in Y_N, is observable. Since $\widetilde{A}_N$ and A_N have no common eigenvalues and (A_N, C_N) is exactly observable in time τ_0, according to Theorem 6.4.2, (A, C) is exactly observable in any time $\tau > \tau_0$. □

Remark 7.3.6. The class of perturbations considered in the last proposition can be enlarged to consider bounded perturbations of A of the form

$$P = \begin{bmatrix} 0 & 0 \\ -a - b\cdot\nabla & -c \end{bmatrix},$$

so that $A + P$ corresponds to the perturbed wave equation

$$\frac{\partial^2\eta}{\partial t^2} - \Delta\eta + c\frac{\partial\eta}{\partial t} + b\cdot\nabla\eta + a\eta = 0 \quad \text{in} \quad \Omega\times(0,\infty).$$

The assumptions on $a,\ b,\ c$ are that

$$a \in L^\infty(\Omega;\mathbb{R}), \quad b \in L^\infty(\Omega;\mathbb{C}^n), \quad c \in L^\infty(\Omega),$$

with the L^∞ norms of b and c sufficiently small, as indicated in Theorem 6.3.2. There is no size restriction on a, as shown in Theorem 7.3.2. The size restrictions on b and c can be removed; see the comments in Section 7.7.

7.4 The wave equation with distributed observation

In this section we show that if a portion Γ of $\partial\Omega$ is a good region for the exact observability of the wave equation by Neumann boundary observation, then any open neighborhood of Γ intersected with Ω is also a good region for exact observability, this time by distributed observation of the velocity.

Let $\Omega \subset \mathbb{R}^n$ be as at the beginning of the chapter and let Γ be an open subset of $\partial\Omega$. For every $\varepsilon > 0$, we denote

$$\mathcal{N}_\varepsilon(\Gamma) = \{x \in \Omega \mid \mathrm{d}(x,\Gamma) < \varepsilon\}, \tag{7.4.1}$$

where $\mathrm{d}(x,\Gamma) = \inf\{|x-y| \mid y \in \Gamma\}$; see Figure 7.2.

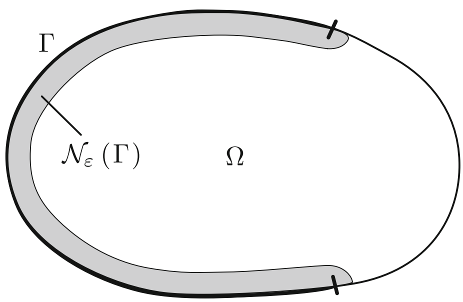

Figure 7.2: The set $\mathcal{N}_\varepsilon(\Gamma)$, which is an open neighborhood of Γ intersected with Ω.

Recall from the beginning of the chapter that we denote $X = \mathcal{H}_0^1(\Omega) \times L^2(\Omega)$ and that A is the operator defined in (7.0.1). In this section we denote $Y = L^2(\Omega)$, $\mathcal{O}$ is an open subset of Ω and the observation operator $C \in \mathcal{L}(X,Y)$ is given by

$$C\begin{bmatrix} f \\ g \end{bmatrix} = g\chi_{\mathcal{O}} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X,$$

where $\chi_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$. The main result of this section is:

Theorem 7.4.1. *Assume that there exists $\tau_0 > 0$ such that the estimate* (7.2.9) *holds for $\tau = \tau_0$. Assume that $\mathcal{O}$ is such that $\mathcal{N}_\varepsilon(\Gamma) \subset \operatorname{clos} \mathcal{O}$ for some $\varepsilon > 0$. Then for every $\tau > \tau_0$ there exists $k_\tau > 0$ such that the solutions η of* (7.1.2)–(7.1.4) *satisfy*

$$\int_0^\tau \int_{\mathcal{O}} \left|\frac{\partial \eta}{\partial t}\right|^2 \mathrm{d}x\,\mathrm{d}t \;\geqslant\; k_\tau^2\left(\|\nabla f\|^2 + \|g\|^2\right) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A). \tag{7.4.2}$$

Thus, the pair (A,C) is exactly observable in any time $\tau > \tau_0$.

In order to prove the above result we need two lemmas.

Lemma 7.4.2. *With the assumptions of Theorem* 7.4.1, *let* $\tau > \tau_0$ *and let* $\alpha > 0$ *be such that* $\tau - 4\alpha > \tau_0$. *Then there exists* $c_{\tau,\alpha} > 0$ *such that the solutions* η *of* (7.1.2)–(7.1.4) *satisfy*

$$\int_\alpha^{\tau-\alpha}\int_{\mathcal{N}_{\varepsilon/2}(\Gamma)}\left(\left|\frac{\partial\eta}{\partial t}\right|^2 + |\nabla\eta|^2 + |\eta|^2\right)\mathrm{d}x\,\mathrm{d}t \geqslant c_{\tau,\alpha}^2(\|\nabla f\|^2 + \|g\|^2) \qquad \forall \begin{bmatrix} f \\ g\end{bmatrix} \in \mathcal{D}(A). \quad (7.4.3)$$

Proof. Let $\Gamma_0 = \partial\mathcal{N}_{\varepsilon/4}(\Gamma)\cap\partial\Omega$. Clearly we have $\Gamma\subset\Gamma_0$ so that, by the assumption in Theorem 7.4.1 and by (7.1.6), it follows that

$$\int_{2\alpha}^{\tau-2\alpha}\int_{\Gamma_0}\left|\frac{\partial\eta}{\partial\nu}\right|^2\mathrm{d}\sigma\,\mathrm{d}t \;\geqslant\; k_\tau^2(\|\nabla f\|^2 + \|g\|^2) \qquad \forall \begin{bmatrix} f \\ g\end{bmatrix} \in \mathcal{D}(A).$$

Thus, it suffices to show that there exists $c > 0$ such that

$$c\int_{2\alpha}^{\tau-2\alpha}\int_{\Gamma_0}\left|\frac{\partial\eta}{\partial\nu}\right|^2\mathrm{d}\sigma\,\mathrm{d}t \leqslant \int_\alpha^{\tau-\alpha}\int_{\mathcal{N}_{\varepsilon/2}(\Gamma)}\left(\left|\frac{\partial\eta}{\partial t}\right|^2 + |\nabla\eta|^2 + |\eta|^2\right)\mathrm{d}x\,\mathrm{d}t \qquad \forall \begin{bmatrix} f \\ g\end{bmatrix} \in \mathcal{D}(A). \quad (7.4.4)$$

Let $\psi \in C^\infty(\operatorname{clos}\Omega)$ be such that $\psi = 1$ on $\mathcal{N}_{\varepsilon/4}(\Gamma)$, $\psi = 0$ on $\Omega\backslash\mathcal{N}_{\varepsilon/2}(\Gamma)$ and $\psi(x) \geqslant 0$ for all $x \in \operatorname{clos}\Omega$. For $x\in\Omega$ and $t \geqslant 0$ we denote $w(x,t) = \psi(x)\eta(x,t)$. Then clearly

$$w \in C\left([0,\tau];\mathcal{H}^2(\Omega)\cap\mathcal{H}_0^1(\Omega)\right)\cap C^1\left([0,\tau];\mathcal{H}_0^1(\Omega)\right)$$

and

$$\frac{\partial^2 w}{\partial t^2} - \Delta w = F, \quad (7.4.5)$$

where, by (13.3.5),

$$F = -2\nabla\psi\cdot\nabla\eta - \eta\Delta\psi. \quad (7.4.6)$$

By applying Lemma 7.1.5 with $q\in C^1(\operatorname{clos}\Omega)$ and $G(t) = (t-\alpha)(\tau - t - \alpha)$, it follows that

$$\int_\alpha^{\tau-\alpha} G\int_{\partial\Omega}(q\cdot\nu)\left|\frac{\partial w}{\partial\nu}\right|^2\mathrm{d}\sigma\,\mathrm{d}t = 2\sum_{k,l=1}^n \mathrm{Re}\int_\alpha^{\tau-\alpha} G\int_\Omega \frac{\partial q_k}{\partial x_l}\frac{\partial\overline{w}}{\partial x_k}\frac{\partial w}{\partial x_l}\mathrm{d}x\,\mathrm{d}t$$
$$+\int_\alpha^{\tau-\alpha} G\int_\Omega(\operatorname{div} q)\left(\left|\frac{\partial w}{\partial t}\right|^2 - |\nabla w|^2\right)\mathrm{d}x\,\mathrm{d}t$$
$$-2\mathrm{Re}\int_\alpha^{\tau-\alpha} G\int_\Omega F(q\cdot\nabla\overline{w})\,\mathrm{d}x\,\mathrm{d}t - 2\mathrm{Re}\int_\alpha^{\tau-\alpha}\frac{\mathrm{d}G}{\mathrm{d}t}\int_\Omega\frac{\partial w}{\partial t}(q\cdot\nabla\overline{w})\,\mathrm{d}x\,\mathrm{d}t. \quad (7.4.7)$$

On the other hand, from (7.4.6) it follows that there exists a constant $K_\psi > 0$ such that

$$\|F(\cdot,t)\|^2 \leqslant K_\psi \int_{\mathcal{N}_{\varepsilon/2}(\Gamma)} \left(|\nabla\eta(\cdot,t)|^2 + |\eta(\cdot,t)|^2\right) \mathrm{d}x \qquad \forall\, t \geqslant 0. \tag{7.4.8}$$

On the other hand, for every $t \in [0,\tau]$,

$$\begin{aligned}\int_\Omega |\nabla w|^2 \,\mathrm{d}x = \int_{\mathcal{N}_{\varepsilon/2}(\Gamma)} |\nabla w|^2 \,\mathrm{d}x &\leqslant \int_{\mathcal{N}_{\varepsilon/2}(\Gamma)} \left(|\psi|^2 \cdot |\nabla\eta|^2 + |\eta|^2 \cdot |\nabla\psi|^2\right) \mathrm{d}x \\ &\leqslant \left(\|\psi\|^2_{L^\infty(\Omega)} + \|\nabla\psi\|^2_{L^\infty(\Omega)}\right) \int_{\mathcal{N}_{\varepsilon/2}(\Gamma)} \left(|\nabla\eta|^2 + |\eta|^2\right) \mathrm{d}x. \end{aligned} \tag{7.4.9}$$

From the above inequality and from (7.4.8) it follows that, for every $t \geqslant 0$,

$$\begin{aligned}\left|\int_\Omega F(q\cdot\nabla\overline{w})\,\mathrm{d}x\right| &\leqslant \frac{\|q\|_{L^\infty(\Omega)}}{2}\|F\|^2 + \frac{\|q\|_{L^\infty(\Omega)}}{2}\|\nabla w\|^2 \\ &\leqslant \frac{\|q\|_{L^\infty(\Omega)}}{2}\left(K_\psi + \|\psi\|^2_{L^\infty(\Omega)} + \|\nabla\psi\|^2_{L^\infty(\Omega)}\right) \int_{\mathcal{N}_{\varepsilon/2}(\Gamma)} \left(|\nabla\eta|^2 + |\eta|^2\right) \mathrm{d}x. \end{aligned}$$

The above inequality, combined with (7.4.7), (7.4.9) and with the fact that

$$\int_\Omega \left|\frac{\partial w}{\partial t}\right|^2 \mathrm{d}x = \int_{\mathcal{N}_{\varepsilon/2}(\Gamma)} \left|\frac{\partial w}{\partial t}\right|^2 \mathrm{d}x \leqslant \|\psi\|^2_{L^\infty(\Omega)} \int_{\mathcal{N}_{\varepsilon/2}(\Gamma)} \left|\frac{\partial \eta}{\partial t}\right|^2 \mathrm{d}x \qquad \forall\, t \geqslant 0,$$

implies that there exists a constant $\widetilde{K}_{\psi,\tau,\alpha} > 0$ such that

$$\begin{aligned}&\int_\alpha^{\tau-\alpha} G \int_{\partial\Omega} (q\cdot\nu) \left|\frac{\partial w}{\partial \nu}\right|^2 \mathrm{d}\sigma\,\mathrm{d}t \\ &\qquad \leqslant \widetilde{K}_{\psi,\tau,\alpha} \int_\alpha^{\tau-\alpha} \int_{\mathcal{N}_{\varepsilon/2}(\Gamma)} \left(\left|\frac{\partial\eta}{\partial t}\right|^2 + |\nabla\eta|^2 + |\eta|^2\right) \mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

By using the fact that $G(t) \geqslant \alpha(\tau - 3\alpha)$ for every $t \in [2\alpha, \tau - 2\alpha]$, it follows that

$$\begin{aligned}&\alpha(\tau-3\alpha) \int_{2\alpha}^{\tau-2\alpha} \int_{\partial\Omega} (q\cdot\nu) \left|\frac{\partial w}{\partial \nu}\right|^2 \mathrm{d}\sigma\,\mathrm{d}t \\ &\qquad \leqslant \widetilde{K}_{\psi,\tau,\alpha} \int_\alpha^{\tau-\alpha} \int_{\mathcal{N}_{\varepsilon/2}(\Gamma)} \left(\left|\frac{\partial\eta}{\partial t}\right|^2 + |\nabla\eta|^2 + |\eta|^2\right) \mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{7.4.10}$$

Let us first consider the case when $\partial\Omega$ is of class C^2. We take $q = \psi h$, where h is the vector field in Lemma 7.1.6. Inequality (7.4.10) combined with the facts that $q\cdot\nu \geqslant 0$ on $\partial\Omega$, $q\cdot\nu = 1$ on Γ_0 and $\frac{\partial w}{\partial \nu} = \frac{\partial \eta}{\partial \nu}$ on Γ_0 imply that

$$\int_{2\alpha}^{\tau-2\alpha}\int_{\Gamma_0}\left|\frac{\partial \eta}{\partial \nu}\right|^2 \mathrm{d}\sigma\,\mathrm{d}t \leqslant \frac{\widetilde{K}_{\psi,\tau,\alpha}}{\alpha(\tau-3\alpha)}\int_{\alpha}^{\tau-\alpha}\int_{\mathcal{N}_{\varepsilon/2}(\Gamma)}\left(\left|\frac{\partial \eta}{\partial t}\right|^2 + |\nabla\eta|^2 + |\eta|^2\right)\mathrm{d}x\,\mathrm{d}t\,,$$

so that (7.4.4) holds. As mentioned, this implies the conclusion of the lemma for domains with a C^2 boundary.

Now consider Ω to be an n-dimensional rectangle. Without loss of generality, we can assume that this rectangle is centered at zero. In this case, we take $q(x) = \psi(x)x$ in (7.4.10). The argument is similar to the previous case, using that $q\cdot\nu \geqslant 0$ on $\partial\Omega$ and bounded from below on Γ_0. $\square$

In order to prove Theorem 7.4.1, we have to get rid of the integrals of $|\nabla\eta|^2$ and $|\eta|^2$ on the left-hand side of (7.4.3). The lemma below gives un upper bound for the integral of $|\nabla\eta|^2$ over $\mathcal{N}_{\varepsilon/2}(\Gamma)$.

Lemma 7.4.3. *Let $\tau > 0$ and $\alpha \in [0,\tau/2)$. Let Γ be an open subset of $\partial\Omega$ and let $\varepsilon > 0$. Then there exists $c > 0$, depending on τ, α and ε, such that the solution η of* (7.1.2)–(7.1.4) *satisfies*

$$\int_{\alpha}^{\tau-\alpha}\int_{\mathcal{N}_{\varepsilon/2}(\Gamma)}|\nabla\eta|^2\mathrm{d}\sigma\,\mathrm{d}t \;\leqslant\; c^2\int_0^{\tau}\int_{\mathcal{N}_{\varepsilon}(\Gamma)}\left(\left|\frac{\partial \eta}{\partial t}\right|^2 + |\eta|^2\right)\mathrm{d}\sigma\,\mathrm{d}t \quad \forall \begin{bmatrix} f\\ g\end{bmatrix} \in \mathcal{D}(A). \tag{7.4.11}$$

Proof. We take the inner product in $L^2([0,\tau];L^2(\Omega))$ of (7.1.2) with $\xi(x,t) = t(\tau-t)\psi(x)\eta(x,t)$, where $\psi \in C^\infty(\operatorname{clos}\Omega)$ is a $[0,1]$-valued function with $\psi = 1$ on $\mathcal{N}_{\varepsilon/2}(\Gamma)$ and $\psi = 0$ on $\Omega\setminus\mathcal{N}_{\varepsilon}(\Gamma)$. For the first term we obtain, by integrating by parts with respect to t,

$$\begin{aligned}\int_0^{\tau}\int_{\Omega}\frac{\partial^2\eta}{\partial t^2}\overline{\xi}\,\mathrm{d}x\,\mathrm{d}t = &-\int_0^{\tau} t(\tau-t)\int_{\Omega}\left|\frac{\partial \eta}{\partial t}\right|^2\psi(x)\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^{\tau}(\tau-2t)\int_{\Omega}\frac{\partial \eta}{\partial t}\,\overline{\eta}\,\psi(x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{7.4.12}$$

For the second term we have, using the fact that $\eta(\cdot,t) \in \mathcal{H}^2(\Omega)\cap\mathcal{H}_0^1(\Omega)$ and formulas (3.6.5) and (13.3.2), we have

$$\begin{aligned}\int_0^{\tau}\int_{\Omega}\Delta\eta\,\overline{\xi}\,\mathrm{d}x\,\mathrm{d}t = &-\int_0^{\tau} t(\tau-t)\int_{\Omega}|\nabla\eta|^2\,\psi(x)\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^{\tau} t(\tau-t)\int_{\Omega}(\nabla\eta\cdot\nabla\psi)\overline{\eta}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Because we started from (7.1.2), the above expression is equal to the one in (7.4.12). It follows that

$$\int_0^\tau t(\tau-t)\int_\Omega |\nabla\eta|^2\,\psi(x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau t(\tau-t)\int_\Omega \left|\frac{\partial\eta}{\partial t}\right|^2 \psi(x)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^\tau(\tau-2t)\int_\Omega \frac{\partial\eta}{\partial t}\,\overline{\eta}\,\psi(x)\,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau t(\tau-t)\int_\Omega (\nabla\eta\cdot\nabla\psi)\overline{\eta}\,\mathrm{d}x\,\mathrm{d}t.$$

Taking real parts and using (3.6.5) we obtain

$$\int_0^\tau t(\tau-t)\int_\Omega |\nabla\eta|^2\,\psi(x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau t(\tau-t)\int_\Omega \left|\frac{\partial\eta}{\partial t}\right|^2 \psi(x)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\,\mathrm{Re}\int_0^\tau(\tau-2t)\int_\Omega \frac{\partial\eta}{\partial t}\,\overline{\eta}\,\psi(x)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^\tau t(\tau-t)\int_\Omega |\eta|^2\Delta\psi\,\mathrm{d}x\,\mathrm{d}t.$$

It follows that

$$\alpha(\tau-\alpha)\int_\alpha^{\tau-\alpha}\int_{\mathcal{N}_{\varepsilon/2}(\Gamma)} |\nabla\eta|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{\tau^2}{4}\int_0^\tau\int_{\mathcal{N}_\varepsilon(\Gamma)}\left|\frac{\partial\eta}{\partial t}\right|^2\mathrm{d}x\,\mathrm{d}t$$
$$+\frac{\tau\|\psi\|_{L^\infty(\Omega)}}{2}\int_0^\tau\int_{\mathcal{N}_\varepsilon(\Gamma)}\left(\left|\frac{\partial\eta}{\partial t}\right|^2+|\eta|^2\right)\mathrm{d}x\,\mathrm{d}t + \frac{\tau^2\|\Delta\psi\|_{L^\infty(\Omega)}}{4}\int_0^\tau\int_{\mathcal{N}_\varepsilon(\Gamma)}|\eta|^2\,\mathrm{d}x\,\mathrm{d}t.$$

The above estimate clearly implies the conclusion (7.4.11). □

We are now in a position to prove the main result of this section.

Proof of Theorem 7.4.1. By combining Lemmas 7.4.2 and 7.4.3, it follows that for every $\tau > \tau_0$ there exists $m_\tau > 0$ such that

$$\int_0^\tau\int_{\mathcal{O}}\left(\left|\frac{\partial\eta}{\partial t}\right|^2+|\eta|^2\right)\mathrm{d}\sigma\,\mathrm{d}t \geqslant m_\tau\left(\|\nabla f\|^2+\|g\|^2\right) \qquad \forall \begin{bmatrix} f\\ g\end{bmatrix}\in\mathcal{D}(A). \tag{7.4.13}$$

We have seen in Remark 3.6.4 that the Dirichlet Laplacian A_0 is diagonalizable with an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ of eigenvectors and the corresponding family of positive eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ which satisfies $\lim_{k\to\infty}\lambda_k=\infty$. We extend the sequence (φ_k) to a sequence indexed by $\mathbb{Z}^*$ by setting $\varphi_k=-\varphi_{-k}$ for every $k\in\mathbb{Z}_-$. We introduce the real sequence $(\mu_k)_{k\in\mathbb{Z}^*}$ defined by

$$\mu_k = \sqrt{\lambda_k}\ \text{ if }\ k>0 \quad\text{and}\quad \mu_k = -\,\mu_{-k}\ \text{ if }\ k<0.$$

According to Proposition 3.7.7 the skew-adjoint operator A is diagonalizable, with the orthonormal basis of eigenvectors $(\phi_k)_{k\in\mathbb{Z}^*}$ given by

$$\phi_k = \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{i\mu_k}\varphi_k\\ \varphi_k\end{bmatrix} \qquad \forall\, k\in\mathbb{Z}^*,$$

and the corresponding eigenvalues are $(i\mu_k)_{k\in\mathbb{Z}^*}$. Note that

$$\|\nabla h\|^2 = \sum_{k\in\mathbb{N}} \lambda_k |\langle h, \varphi_k\rangle|^2 \qquad \forall\, h \in \mathcal{H}_0^1(\Omega). \tag{7.4.14}$$

For $\omega > 0$ we denote

$$V_\omega = \operatorname{span}\{\phi_k \mid |\mu_k| \leqslant \omega\}^\perp.$$

For $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{D}(A) \cap V_\omega$ we have $\eta(\cdot,t) \in \operatorname{span}\{\varphi_k \mid \lambda_k \leqslant \omega^2\}^\perp$, so that, by using (7.1.6) and (7.4.14), we have

$$\omega^2\|\eta(\cdot,t)\|^2 \leqslant \|\nabla\eta(\cdot,t)\|^2 \leqslant \|\nabla f\|^2 + \|g\|^2 \qquad \forall\, t \in [0,\tau].$$

From the above inequality and (7.4.13) we obtain that for ω large enough there exists $c_{\tau,\omega} > 0$ such that

$$\int_0^\tau \int_{\mathcal{O}} \left|\frac{\partial\eta}{\partial t}\right|^2 \mathrm{d}x\,\mathrm{d}t \geqslant c_{\tau,\omega}\left(\|\nabla f\|^2 + \|g\|^2\right) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A) \cap V_\omega. \tag{7.4.15}$$

If we denote by A_ω the part of A in V_ω and by C_ω the restriction of C to V_ω, inequality (7.4.15) means that the pair (A_ω, C_ω) is exactly observable in any time $\tau > \tau_0$, provided that ω is large enough.

On the other hand, assume that $\phi = \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix}\right] \in \mathcal{D}(A)$ is an eigenvector of A, associated with the eigenvalue $i\mu$. According to Proposition 3.7.7,

$$\Delta\varphi + \mu^2\varphi = 0. \tag{7.4.16}$$

If we assume that $C\phi = 0$, then $\psi_{|\mathcal{O}} = 0$ and by using the facts that $\psi = i\mu\varphi$ (see Proposition 3.7.7) and $\mu \neq 0$, we obtain that the function $\varphi \in \mathcal{D}(A_0)$ satisfies

$$\varphi = 0 \quad \text{on } \mathcal{O}. \tag{7.4.17}$$

As shown in Theorem 15.2.1 in Appendix III, the only function $\varphi \in \mathcal{H}^2(\Omega)\cap\mathcal{H}_0^1(\Omega)$ satisfying (7.4.16) and (7.4.17) is $\varphi = 0$. Since $\psi = i\mu\varphi = 0$, we obtain that $\phi = 0$. This contradiction shows that $C\phi \neq 0$ for every eigenvector ϕ of A. This fact and the exact observability in any time $\tau > \tau_0$ of (A_ω, C_ω) implies, by Proposition 6.4.4, that (A, C) is exactly observable in any time $\tau > \tau_0$. $\square$

Note that the observability condition imposed on Γ in Theorem 7.4.1 is satisfied, in particular, if Γ is as in Theorem 7.2.4. Other sets Γ satisfying the observability condition can be found using the references cited in Section 7.7.

Remark 7.4.4. The main result in this section can be generalized by replacing the generator A with a perturbed generator $A+P$, where P is as described in Remark 7.3.6. Thus P depends on three L^∞ functions a, b and c. The fact that there is no size restriction on a can be shown as in the proof of Theorem 7.3.2, except that now (at the end of the proof) we apply Theorem 15.2.1 instead of Corollary 15.2.2. The functions b and c have to be small, as indicated in Theorem 6.3.2. However the size restrictions on b and c can be removed by more sophisticated methods; see the comments in Section 7.7.

The exact observability result of this section can be used to derive an exponential stability result for some of the perturbed semigroups described in the last remark. These semigroups are associated with damped wave equations.

Proposition 7.4.5. *With the assumptions and the notation in Theorem* 7.4.1 *let* $a, c \in L^\infty(\Omega)$ *be such that*

$$a(x) \geqslant 0, \qquad c(x) \geqslant 0 \qquad (x \in \Omega),$$

and $c(x) \geqslant \delta > 0$ *for* $x \in \mathcal{O}$. *Then the semigroup* $\mathbb{S}$ *generated by* $A + P$, *where*

$$P = \begin{bmatrix} 0 & 0 \\ -a & -c \end{bmatrix},$$

is exponentially stable. In terms of PDEs this means that the solutions η *of*

$$\frac{\partial^2 \eta}{\partial t^2} - \Delta \eta + c \frac{\partial \eta}{\partial t} + a\eta = 0 \quad \text{in} \quad \Omega \times (0, \infty), \tag{7.4.18}$$

$$\eta = 0 \quad \text{on} \ \partial\Omega \times (0, \infty), \tag{7.4.19}$$

$$\eta(x, 0) = f(x), \qquad \frac{\partial \eta}{\partial t}(x, 0) = g(x) \quad \text{for} \quad x \in \Omega, \tag{7.4.20}$$

satisfy, for some $M, \omega > 0$,

$$\left\| \frac{\partial \eta}{\partial t}(\cdot, t) \right\|^2 + \|\nabla \eta(\cdot, t)\|^2 \leqslant M e^{-\omega t} \left(\|g\|^2 + \|\nabla f\|^2 \right) \qquad \forall\, t \geqslant 0.$$

Proof. Let $\widetilde{A} = A + \left[\begin{smallmatrix} 0 & I \\ -a & 0 \end{smallmatrix}\right]$ and let $C_1 \in \mathcal{L}(X, H)$ be defined by

$$C_1 \begin{bmatrix} f \\ g \end{bmatrix} = \sqrt{c}\, g \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X.$$

Since c is bounded from below by the positive constant δ on $\mathcal{O}$, according to Remark 7.4.4, the pair $(\widetilde{A}, C_1)$ is exactly observable in some time τ. Since

$$A + P = \widetilde{A} + \begin{bmatrix} 0 \\ -\sqrt{c} \end{bmatrix} C_1,$$

we can apply Theorem 6.3.2 to get that (A, C_1) is exactly observable in time τ; i.e., there exists a constant $k_\tau > 0$ such that

$$\int_0^\tau \left\| C_1 \mathbb{S}_t \begin{bmatrix} f \\ g \end{bmatrix} \right\|^2 \mathrm{d}t \geqslant k_\tau^2 \left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|_X^2 \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X. \tag{7.4.21}$$

Without loss of generality, we can assume that $k_\tau \in (0, 1)$.

On the other hand, it is easy to see that, for all $\begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A)$, $\mathbb{S}_t \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \eta(\cdot,t) \\ \dot\eta(\cdot,t) \end{bmatrix}$, with η satisfying (7.4.18)–(7.4.20). Therefore, if we take the inner product in $L^2([0,\tau];H)$ of (7.4.18) with $\dot\eta$, it follows that

$$\left\|\begin{bmatrix} f \\ g \end{bmatrix}\right\|_X^2 - \left\|\mathbb{S}_\tau \begin{bmatrix} f \\ g \end{bmatrix}\right\|_X^2 = \int_0^\tau \|\sqrt{c(\cdot)}\,\dot\eta(\cdot,t)\|^2 \,\mathrm{d}t = \int_0^\tau \left\|C_1\mathbb{S}_t \begin{bmatrix} f \\ g \end{bmatrix}\right\|^2 \mathrm{d}t.$$

From the above and (7.4.21) it follows that

$$\left\|\mathbb{S}_\tau \begin{bmatrix} f \\ g \end{bmatrix}\right\|_X^2 \leqslant (1-k_\tau^2) \left\|\begin{bmatrix} f \\ g \end{bmatrix}\right\|_X^2 \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A),$$

which implies that $\|\mathbb{S}_\tau\|_{\mathcal{L}(X)} < 1$. According to the definition (2.1.3) of the growth bound, it follows that $\mathbb{S}$ is exponentially stable. $\square$

7.5 Some consequences for the Schrödinger and plate equations

Here we derive exact observability results for the Schrödinger and plate equations by combining the exact observability results for the wave equation obtained in Sections 7.2 and 7.4 with the results in Sections 6.7 and 6.8. More results on the exact observability of the Schrödinger and plate equations will be given in Section 8.5.

Notation and preliminaries. Recall, from the beginning of this chapter, that Ω stands for a bounded open connected set in $\mathbb{R}^n$, where $n \in \mathbb{N}$, and $\partial\Omega$ is supposed to be of class C^2 or Ω is supposed to be a rectangular domain, $H = L^2(\Omega)$ and $\mathcal{D}(A_0) = H_1$ is the Sobolev space $\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$. The strictly positive operator $A_0 : \mathcal{D}(A_0) \to H$ is defined by $A_0\varphi = -\Delta\varphi$ for all $\varphi \in \mathcal{D}(A_0)$. The norm on H is denoted by $\|\cdot\|$. Recall that $H_{\frac12} = \mathcal{H}_0^1(\Omega)$, $H_{-\frac12} = \mathcal{H}^{-1}(\Omega)$ and that $X = H_{\frac12} \times H$. As before, we define $X_1 = H_1 \times H_{\frac12}$ and the skew-adjoint operator $A : X_1 \to X$ is given by

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad \text{i.e.,} \quad A\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -A_0 f \end{bmatrix}.$$

For some fixed $x_0 \in \mathbb{R}^n$, the function m, the set $\Gamma(x_0)$ and the number $r(x_0)$ are defined as at the beginning of this chapter.

Throughout this section, we denote by $\mathcal{X}$ the Hilbert space $H_1 \times H$, with the scalar product

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_{\mathcal{X}} = \langle A_0 f_1, A_0 f_2\rangle + \langle g_1, g_2\rangle.$$

We introduce the dense subspace of $\mathcal{X}$ defined by $\mathcal{D}(\mathcal{A}) = H_2 \times H_1$ and the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ defined by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0^2 & 0 \end{bmatrix}, \quad \text{i.e.,} \quad \mathcal{A} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -A_0^2 f \end{bmatrix}. \tag{7.5.1}$$

By using the strict positivity of A_0 and Proposition 3.3.6, it follows that $A_0^2 > 0$ so that, by Proposition 3.7.6, we have that $\mathcal{A}$ is skew-adjoint. By Stone's theorem it follows that $\mathcal{A}$ generates a unitary group on $\mathcal{X}$. We denote by $\mathcal{X}_1$ the space $\mathcal{D}(\mathcal{A})$ endowed with the graph norm.

Let Γ be an open subset of $\partial\Omega$, $\mathcal{O}$ an open subset of Ω, let $Y = L^2(\Gamma)$ and consider $C_1 \in \mathcal{L}(H_1, Y)$, $C_0 \in \mathcal{L}(H)$, $C \in \mathcal{L}(X_1, Y)$ and $\widetilde{C} \in \mathcal{L}(X, Y)$ defined by

$$C_1 f = \frac{\partial f}{\partial \nu}\Big|_{\Gamma} \qquad \forall\, f \in H_1, \qquad C_0 g = g\chi_{\mathcal{O}} \qquad \forall\, g \in H,$$

$$C = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} 0 & C_0 \end{bmatrix},$$

where $\chi_{\mathcal{O}}$ stands for the characteristic function of $\mathcal{O}$.

The first result concerns the Schrödinger equation with Neumann observation.

Proposition 7.5.1. *The operator C_1 is an admissible observation operator for the unitary group generated by iA_0 on $\mathcal{H}_0^1(\Omega)$. Moreover, if Γ is such that the pair (A, C) is exactly observable, then the pair (iA_0, C_1), with state space $\mathcal{H}_0^1(\Omega)$, is exactly observable in any time $\tau > 0$.*

Proof. We know from Theorem 7.1.3 that C is an admissible observation operator for the semigroup generated by A so that, by Proposition 6.7.1, it follows that C_1 is an admissible observation operator for the unitary group generated by iA_0.

If (A, C) is exactly observable, then it follows from Theorem 6.7.2 that the pair (iA_0, C_1), with state space $H_{\frac{1}{2}} = \mathcal{H}_0^1(\Omega)$, is exactly observable in any time $\tau > 0$. □

Remark 7.5.2. In terms of PDEs, the result in Proposition 7.5.1 means that if Γ is such that (7.2.9) holds for $\tau = \tau_0$, then for every $\tau > 0$ there exists $k_\tau > 0$ such that the solution z of the Schrödinger equation

$$\frac{\partial z}{\partial t}(x, t) = -i\Delta z(x, t) \qquad \forall\, (x, t) \in \Omega \times [0, \infty),$$

with

$$z(x, t) = 0 \qquad \forall\, (x, t) \in \partial\Omega \times [0, \infty),$$

and $z(\cdot, 0) = z_0 \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$, satisfies

$$\int_0^\tau \int_\Gamma \left| \frac{\partial z}{\partial \nu}(x, t) \right|^2 \mathrm{d}\sigma\, \mathrm{d}t \geqslant k_\tau^2 \|z_0\|^2_{\mathcal{H}_0^1(\Omega)} \qquad \forall\, z_0 \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega).$$

Recall that a sufficient condition for Γ to satisfy the above requirement has been given in Theorem 7.2.4.

Now we consider the Schrödinger equation with distributed observation.

Proposition 7.5.3. *Let $\mathcal{O}$ be an open subset of Ω such that the pair $(A, \widetilde{C})$ is exactly observable. Then the pair (iA_0, C_0) is exactly observable in any time $\tau > 0$.*

Proof. It suffices to apply Theorem 6.7.5. □

Remark 7.5.4. In terms of PDEs, the result in Proposition 7.5.3 means that if $\mathcal{O}$ is such that (7.4.2) holds for $\tau = \tau_0$, then for every $\tau > 0$ there exists $k_\tau > 0$ such that the solution z of the Schrödinger equation

$$\frac{\partial z}{\partial t}(x,t) = -i\Delta z(x,t) \qquad \forall\, (x,t) \in \Omega \times [0,\infty),$$

with

$$z(x,t) = 0 \qquad \forall\, (x,t) \in \partial\Omega \times [0,\infty),$$

and $z(\cdot, 0) = z_0 \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$, satisfies

$$\int_0^\tau \int_{\mathcal{O}} |z(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \geqslant k_\tau^2 \|z_0\|^2 \qquad \forall\, z_0 \in L^2(\Omega).$$

We next consider the two exact observability problems for the Euler–Bernoulli plate equation. First we tackle a boundary observability problem.

Proposition 7.5.5. *Assume that Γ is such that the pair (A, C) is exactly observable and let*

$$\mathcal{C}_1 \in \mathcal{L}(H_{\frac{5}{2}} \times H_{\frac{3}{2}}, Y)$$

be defined by

$$\mathcal{C}_1 \begin{bmatrix} f \\ g \end{bmatrix} = \frac{\partial g}{\partial \nu} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A_0^{\frac{5}{2}}) \times \mathcal{D}(A_0^{\frac{3}{2}}).$$

Then $\mathcal{C}_1$ is an admissible observation operator for the unitary group generated by $\mathcal{A}$ on $H_{\frac{3}{2}} \times H_{\frac{1}{2}}$ and the pair $(\mathcal{A}, \mathcal{C}_1)$, with state space $H_{\frac{3}{2}} \times H_{\frac{1}{2}}$, is exactly observable in any time $\tau > 0$.

Proof. We know from Proposition 7.5.1 that the pair (iA_0, C_1), with state space $H_{\frac{1}{2}}$ is exactly observable in any time $\tau > 0$. Moreover, by using Proposition 3.6.9, the eigenvalues of the Dirichlet Laplacian satisfy the condition (6.8.8) for an appropriate $d > 0$. By applying Proposition 6.8.2, it follows that the pair $(\mathcal{A}, \mathcal{C}_1)$ is exactly observable in any time $\tau > 0$. □

Remark 7.5.6. In terms of PDEs, the result in Proposition 7.5.5 means that for every $\tau > 0$ there exists $k_\tau > 0$ such that if Γ is such that (7.2.9) holds for $\tau = \tau_0$, then the solution w of the Euler–Bernoulli plate equation

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \Delta^2 w(x,t) = 0, \quad (x,t) \in \Omega \times [0,\infty),$$

with

$$w_{|\partial\Omega\times[0,\infty)} = \Delta w_{|\partial\Omega\times[0,\infty)} = 0,$$

and $w(\cdot,0) = w_0 \in \mathcal{D}(A_0^2)$, $\frac{\partial w}{\partial t}(\cdot,0) = w_1 \in \mathcal{D}(A_0)$, satisfies

$$\int_0^\tau \int_\Gamma \left|\frac{\partial^2 w}{\partial\nu\partial t}\right|^2 \mathrm{d}\sigma\,\mathrm{d}t \geqslant k_\tau^2 \left(\|w_0\|^2_{\mathcal{H}^3(\Omega)} + \|w_1\|^2_{\mathcal{H}^1_0(\Omega)}\right) \qquad \forall \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in \mathcal{D}(\mathcal{A}).$$

Proposition 7.5.7. *Let $\mathcal{O}$ be an open subset of Ω such that the pair $(A, \widetilde{C})$ is exactly observable, and let $\mathcal{C}_0 \in \mathcal{L}(\mathcal{X}, H)$ be defined by*

$$\mathcal{C}_0 \begin{bmatrix} f \\ g \end{bmatrix} = g\chi_{\mathcal{O}} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{X}.$$

Then $\mathcal{C}_0$ is an admissible observation operator for the unitary group generated by $\mathcal{A}$ and the pair $(\mathcal{A}, \mathcal{C}_0)$, with state space $\mathcal{X} = \mathcal{D}(A_0) \times X$, is exactly observable in any time $\tau > 0$.

Proof. We know from Proposition 7.5.3 that the pair (iA_0, C_0) is exactly observable in any time $\tau > 0$. Moreover, by using Proposition 3.6.9, the eigenvalues of the Dirichlet Laplacian satisfy condition (6.8.8) for an appropriate $d > 0$.

By applying Proposition 6.8.2, it follows that the pair $(\mathcal{A}, \mathcal{C}_0)$ is exactly observable in any time $\tau > 0$. □

Remark 7.5.8. In terms of PDEs, the result in Proposition 7.5.7 means that if $\mathcal{O}$ is such that (7.4.2) holds for $\tau = \tau_0$, then for every $\tau > 0$ there exists $k_\tau > 0$ such that the solution w of the Euler–Bernoulli plate equation

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \Delta^2 w(x,t) = 0, \quad (x,t) \in (0,\pi) \times [0,\infty),$$

with

$$w_{|\partial\Omega\times[0,\infty)} = \Delta w_{|\partial\Omega\times[0,\infty)} = 0,$$

and $w(\cdot,0) = w_0 \in \mathcal{D}(A_0^2)$, $\frac{\partial w}{\partial t}(\cdot,0) = w_1 \in \mathcal{D}(A_0)$, satisfies

$$\int_0^\tau \int_{\mathcal{O}} \left|\frac{\partial w}{\partial t}\right|^2 \mathrm{d}x\,\mathrm{d}t \geqslant k_\tau^2 \left(\|w_0\|^2_{\mathcal{H}^2(\Omega)} + \|w_1\|^2_{L^2(\Omega)}\right) \qquad \forall \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in \mathcal{D}(\mathcal{A}).$$

Remark 7.5.9. By using Theorem 7.2.4 it follows that the conclusions in Propositions 7.5.1 and 7.5.5 hold if $\Gamma \supset \Gamma(x_0)$ for some $x_0 \in \mathbb{R}^n$. According to Theorem 7.4.1 we have that the conclusions in Propositions 7.5.3 and 7.5.7 hold if Γ is as above and $\mathcal{N}_\varepsilon(\Gamma) \subset \overline{\mathcal{O}}$ for some $\varepsilon > 0$.

7.6 The wave equation with boundary damping and boundary velocity observation

In this section we give a sufficient condition for the exponential stability of the semigroup constructed in Section 3.9 (the wave equation with boundary damping) and we show that this implies an exact boundary observability result for the same semigroup with boundary observation of the velocity.

Notation and preliminaries. We use the notation from Section 3.9, but with stronger assumptions on Ω, Γ_0 and Γ_1. More precisely, $\Omega \subset \mathbb{R}^n$ is supposed to be bounded, connected and with C^2 boundary $\partial\Omega$. The sets Γ_0 and Γ_1 are defined by

$$\begin{aligned} \Gamma_0 &= \{x \in \partial\Omega \mid m(x)\cdot\nu(x) < 0\}, \\ \Gamma_1 &= \{x \in \partial\Omega \mid m(x)\cdot\nu(x) > 0\}, \end{aligned} \tag{7.6.1}$$

where ν is the outer normal field to $\partial\Omega$ and $m(x) = x - x_0$ for some $x_0 \in \mathbb{R}^n$. Thus Γ_0 and Γ_1 are disjoint open subsets of $\partial\Omega$. We assume that

$$\Gamma_0 \neq \emptyset, \quad \Gamma_1 \neq \emptyset, \qquad \Gamma_0 \cup \Gamma_1 = \partial\Omega. \tag{7.6.2}$$

Note that this implies

$$\operatorname{clos}\Gamma_0 = \Gamma_0, \quad \operatorname{clos}\Gamma_1 = \Gamma_1,$$

so that these assumptions clearly exclude simply connected domains. Intuitively, we imagine Γ_0 as the surface of a bubble inside the domain Ω, x_0 is in the bubble, while Γ_1 is the outer boundary. The space $\mathcal{H}^1_{\Gamma_0}(\Omega)$ consists of those functions in $\mathcal{H}^1(\Omega)$ whose trace vanishes on Γ_0 (this space is discussed in Section 13.6). We know from Section 13.6 that the induced norm on $\mathcal{H}^1_{\Gamma_0}(\Omega)$ (as a closed subspace of $\mathcal{H}^1(\Omega)$) is equivalent to the norm $\|\nabla f\|_{[L^2(\Omega)]^n}$. The state space is

$$X = \mathcal{H}^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$$

and it is endowed with the inner product

$$\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \right\rangle = \int_\Omega \nabla f \cdot \nabla\overline{\varphi}\,\mathrm{d}x + \int_\Omega g\overline{\psi}\,\mathrm{d}x \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in X.$$

The corresponding norm is denoted by $\|\cdot\|$. Let $b \in C^1(\Gamma_1)$ be a real-valued function and let $A : \mathcal{D}(A) \to X$ be the operator defined by

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in \left[\mathcal{H}^2(\Omega) \cap \mathcal{H}^1_{\Gamma_0}(\Omega)\right] \times \mathcal{H}^1_{\Gamma_0}(\Omega) \;\middle|\; \frac{\partial f}{\partial\nu}|_{\Gamma_1} = -\,b^2 g|_{\Gamma_1} \right\},$$

$$A\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \Delta f \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A).$$

We know from Propositions 3.9.1 and 3.9.2 that A is m-dissipative so that it generates a contraction semigroup $\mathbb{T}$ on X. Also recall from Section 3.9 that an alternative way of defining $\mathcal{D}(A)$ is to say that $\mathcal{D}(A)$ consists of those couples $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{H}^1_{\Gamma_0}(\Omega) \times \mathcal{H}^1_{\Gamma_0}(\Omega)$ such that $\Delta f \in L^2(\Omega)$ and

$$\langle \Delta f, \varphi \rangle_{L^2(\Omega)} + \langle \nabla f, \nabla \varphi \rangle_{[L^2(\Omega)]^n} = -\langle b^2 g, \varphi \rangle_{L^2(\Gamma_1)} \qquad \forall\, \varphi \in \mathcal{H}^1_{\Gamma_0}(\Omega). \quad (7.6.3)$$

Consider the initial and boundary value problem

$$\begin{cases} \ddot{z}(x,t) = \Delta z(x,t) & \text{on } \Omega \times [0,\infty), \\ z(x,t) = 0 & \text{on } \Gamma_0 \times [0,\infty), \\ \frac{\partial}{\partial \nu} z(x,t) + b^2(x)\dot{z}(x,t) = 0 & \text{on } \Gamma_1 \times [0,\infty), \\ z(x,0) = z_0(x), \quad \dot{z}(x,0) = w_0(x) & \text{on } \Omega. \end{cases} \quad (7.6.4)$$

We have seen in Corollary 3.9.3 that for every $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \mathcal{D}(A)$, problem (7.6.4) admits a unique strong solution z and that this solution satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla z(\cdot,t)\|^2_{[L^2(\Omega)]^n} + \|\dot{z}(\cdot,t)\|^2_{L^2(\Omega)} \right) = -2\int_{\Gamma_1} b^2(x)|\dot{z}(x,t)|^2\, \mathrm{d}\sigma. \quad (7.6.5)$$

The main result of this section is the following.

Theorem 7.6.1. *With the above notation, assume that* $\inf_{x\in\Gamma_1} |b(x)| > 0$. *Then there exist* $M \geqslant 1$ *and* $\omega > 0$ (*depending only on* Ω *and on* b) *such that the strong solutions of* (7.6.4) *satisfy, for every* $t \geqslant 0$,

$$\left\| \begin{bmatrix} z(\cdot,t) \\ \dot{z}(\cdot,t) \end{bmatrix} \right\| \leqslant M e^{-\omega t} \left\| \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \right\| \qquad \forall \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in \mathcal{D}(A). \quad (7.6.6)$$

In order to prove Theorem 7.6.1 we need some notation and two lemmas. If z is the strong solution of (7.6.4) and $\varepsilon \geqslant 0$, we set

$$\rho(t) = \mathrm{Re} \int_\Omega \dot{z}(x,t)\,[2m(x)\cdot\nabla\overline{z}(x,t) + (n-1)\overline{z}(x,t)]\,\mathrm{d}x \quad \forall\, t \geqslant 0, \quad (7.6.7)$$

$$V_\varepsilon(t) = \|\nabla z(\cdot,t)\|^2_{[L^2(\Omega)]^n} + \|\dot{z}(\cdot,t)\|^2_{L^2(\Omega)} + \varepsilon\rho(t) \quad \forall\, t \geqslant 0. \quad (7.6.8)$$

We also introduce the positive constants $r(x_0) = \|m\|_{L^\infty(\Omega)}$ and

$$\varepsilon_0 = \frac{1}{2r(x_0) + c(n-1)},$$

where c is the constant in the Poincaré inequality in Theorem 13.6.9.

Lemma 7.6.2. *With the above notation, assume that* $\varepsilon \in [0, \varepsilon_0)$. *Then*

$$\frac{1}{2}V_0(t) \leqslant V_\varepsilon(t) \leqslant \frac{3}{2}V_0(t) \qquad \forall\, t \geqslant 0.$$

Proof. From the Cauchy–Schwarz inequality and other elementary inequalities,

$$|\rho(t)| \leqslant \|\dot{z}(t)\|_{L^2(\Omega)} \left[2r(x_0)\|\nabla z(t)\|_{[L^2(\Omega)]^n} + (n-1)\|z(t)\|_{L^2(\Omega)}\right] \quad \forall t \geqslant 0.$$

By applying the Poincaré inequality in Theorem 13.6.9, it follows that

$$|\rho(t)| \leqslant [2r(x_0) + c(n-1)] \, \|\dot{z}(t)\|_{L^2(\Omega)} \, \|\nabla z(t)\|_{[L^2(\Omega)]^n} \leqslant \frac{1}{2\varepsilon_0} V_0(t) \qquad \forall t \geqslant 0.$$

The above inequality clearly implies the conclusion of the lemma. □

Lemma 7.6.3. *Let* $f \in \mathcal{H}^2(\Omega) \cap \mathcal{H}^1_{\Gamma_0}(\Omega)$. *Then*

$$\begin{aligned} 2\mathrm{Re} \int_\Omega (\Delta f)(m \cdot \nabla \overline{f})\,\mathrm{d}x = (n-2) \int_\Omega |\nabla f|^2 \\ + 2\mathrm{Re} \int_{\partial\Omega} \frac{\partial f}{\partial \nu}(m \cdot \nabla \overline{f})\mathrm{d}\sigma - \int_{\partial\Omega} (m\cdot\nu)|\nabla f|^2\,\mathrm{d}\sigma. \end{aligned}$$

Proof. By using integration by parts (see Remark 13.7.3) it follows that

$$\begin{aligned} 2\mathrm{Re} \int_\Omega (\Delta f)(m \cdot \nabla \overline{f}\,)\,\mathrm{d}x &= 2\mathrm{Re} \int_{\partial\Omega} \frac{\partial f}{\partial \nu}(m \cdot \nabla \overline{f})\mathrm{d}\sigma - \mathrm{Re} \int_\Omega \nabla f \cdot \nabla(2m \cdot \nabla \overline{f}\,)\,\mathrm{d}x \\ &= 2\mathrm{Re} \int_{\partial\Omega} \frac{\partial f}{\partial \nu}(m \cdot \nabla \overline{f})\mathrm{d}\sigma - 2\int_\Omega |\nabla f|^2\,\mathrm{d}x - \int_\Omega m \cdot \left(\nabla |\nabla f|^2\right)\,\mathrm{d}x. \end{aligned} \tag{7.6.9}$$

On the other hand, according to (13.3.1), we have

$$m \cdot \left(\nabla |\nabla f|^2\right) = \mathrm{div}\,\left(|\nabla f|^2 m\right) - n|\nabla f|^2,$$

so that by applying the Gauss formula (13.7.3) it follows that

$$\int_\Omega m \cdot \left(\nabla |\nabla f|^2\right)\,\mathrm{d}x = \int_{\partial\Omega} (m \cdot \nu)|\nabla f|^2\,\mathrm{d}\sigma - \int_\Omega |\nabla f|^2\,\mathrm{d}x.$$

The above formula and (7.6.9) clearly imply the conclusion of the lemma. □

We are now in a position to prove the main result of this section.

Proof of Theorem 7.6.1. Since z is a strong solution of (7.6.4), we have

$$z \in C([0,\infty), \mathcal{H}^2(\Omega)) \cap C^1([0,\infty), \mathcal{H}^1_{\Gamma_0}(\Omega)) \cap C^2([0,\infty), L^2(\Omega)),$$

and from Corollary 3.9.3 it follows that

$$\dot{V}_0(t) = -2\int_{\Gamma_1} b^2 |\dot{z}|^2\,\mathrm{d}x. \tag{7.6.10}$$

On the other hand, from (7.6.7) and the fact that z satisfies the first equation in (7.6.4), it follows that

$$\dot\rho(t) = 2\mathrm{Re}\int_\Omega \Delta z\,(m\cdot\nabla\overline{z})\,\mathrm{d}x + (n-1)\mathrm{Re}\int_\Omega \Delta z\,\overline{z}\,\mathrm{d}x + 2\mathrm{Re}\int_\Omega \dot z(m\cdot\nabla\dot{\overline{z}})\,\mathrm{d}x + (n-1)\int_\Omega |\dot z|^2\,\mathrm{d}x \qquad \forall\, t\geqslant 0. \tag{7.6.11}$$

For the first term on the right-hand side of the above formula we can use Lemma 7.6.3 to get

$$2\mathrm{Re}\int_\Omega \Delta z\,(m\cdot\nabla\overline{z})\,\mathrm{d}x = (n-2)\int_\Omega |\nabla z|^2\,\mathrm{d}x + 2\mathrm{Re}\int_{\partial\Omega}\frac{\partial z}{\partial\nu}\,(m\cdot\nabla\overline{z})\mathrm{d}\sigma - \int_{\partial\Omega}(m\cdot\nu)|\nabla z|^2\,\mathrm{d}\sigma \qquad \forall\, t\geqslant 0. \tag{7.6.12}$$

Using the facts that $\nabla z = \frac{\partial z}{\partial\nu}\nu$ on Γ_0 and $\frac{\partial z}{\partial\nu} = -b^2\dot z$ on Γ_1, the second and the third integral on the right-hand side of the above formula can be, respectively, written as

$$\int_{\partial\Omega}\frac{\partial z}{\partial\nu}\,(m\cdot\nabla\overline{z})\mathrm{d}\sigma = \int_{\Gamma_0}(m\cdot\nu)\left|\frac{\partial z}{\partial\nu}\right|^2\mathrm{d}\sigma - \int_{\Gamma_1} b^2\dot z(m\cdot\nabla\overline{z})\mathrm{d}\sigma \quad \forall\, t\geqslant 0, \tag{7.6.13}$$

$$\int_{\partial\Omega}(m\cdot\nu)|\nabla z|^2\,\mathrm{d}\sigma = \int_{\Gamma_0}(m\cdot\nu)\left|\frac{\partial z}{\partial\nu}\right|^2\mathrm{d}\sigma + \int_{\Gamma_1}(m\cdot\nu)|\nabla z|^2\,\mathrm{d}\sigma \quad \forall\, t\geqslant 0. \tag{7.6.14}$$

Using (7.6.12)–(7.6.14) and the fact that $m\cdot\nu<0$ on Γ_0, it follows that

$$2\mathrm{Re}\int_\Omega \Delta z\,(m\cdot\nabla\overline{z})\,\mathrm{d}x \leqslant (n-2)\int_\Omega |\nabla z|^2\,\mathrm{d}x - 2\mathrm{Re}\int_{\Gamma_1} b^2\dot z(m\cdot\nabla\overline{z})\mathrm{d}\sigma - \int_{\Gamma_1}(m\cdot\nu)|\nabla z|^2\,\mathrm{d}\sigma \qquad \forall\, t\geqslant 0. \tag{7.6.15}$$

For the second term on the right-hand side of (7.6.11) we note that

$$\int_\Omega \Delta z\,\overline{z}\mathrm{d}x = \int_{\partial\Omega}\frac{\partial z}{\partial\nu}\,\overline{z}\mathrm{d}\sigma - \int_\Omega|\nabla z|^2\,\mathrm{d}x \qquad \forall\, t\geqslant 0.$$

Since $z=0$ on Γ_0 and $\frac{\partial z}{\partial\nu} = -b^2\dot z$ on Γ_1, it follows that

$$\mathrm{Re}\int_\Omega \Delta z\,\overline{z}\mathrm{d}x = -\,\mathrm{Re}\int_{\Gamma_1} b^2\dot z\,\overline{z}\mathrm{d}\sigma - \int_\Omega|\nabla z|^2\,\mathrm{d}x \qquad \forall\, t\geqslant 0. \tag{7.6.16}$$

For the third term on the right-hand side of (7.6.11) we have

$$2\mathrm{Re}\int_\Omega \dot z(m\cdot\nabla\dot{\overline{z}})\,\mathrm{d}x = \int_\Omega m\cdot\nabla(|\dot z|^2)\,\mathrm{d}x = \int_\Omega \left[\mathrm{div}\,(|\dot z|^2\, m) - n|\dot z|^2\right]\mathrm{d}x \quad \forall t\geqslant 0.$$

Using the Gauss formula (13.7.3) together with the fact that $\dot z = 0$ on Γ_0, we obtain

$$2\mathrm{Re}\int_\Omega \dot z(m\cdot\nabla\dot{\overline{z}})\,\mathrm{d}x = \int_{\Gamma_1}(m\cdot\nu)|\dot z|^2\,\mathrm{d}\sigma - n\int_\Omega |\dot z|^2\,\mathrm{d}x \qquad \forall\, t\geqslant 0. \quad (7.6.17)$$

By combining (7.6.11) with (7.6.15)–(7.6.17) we obtain

$$\begin{aligned}\dot\rho(t) \leqslant &- V_0(t) + \int_{\Gamma_1}(m\cdot\nu)\left(|\dot z|^2 - |\nabla z|^2\right)\mathrm{d}\sigma\\ &- (n-1)\mathrm{Re}\int_{\Gamma_1} b^2\dot z\,\overline{z}\,\mathrm{d}\sigma - 2\mathrm{Re}\int_{\Gamma_1} b^2\dot z(m\cdot\nabla\overline{z})\mathrm{d}\sigma \qquad \forall\, t\geqslant 0.\end{aligned}$$

It follows that

$$\begin{aligned}\dot\rho(t) \leqslant &- V_0(t) + \frac{\|m\|_{L^\infty(\Gamma_1)}}{b_0^2}\int_{\Gamma_1} b^2|\dot z|^2\,\mathrm{d}\sigma - \int_{\Gamma_1}(m\cdot\nu)|\nabla z|^2\,\mathrm{d}\sigma\\ &- (n-1)\mathrm{Re}\int_{\Gamma_1} b^2\dot z\,\overline{z}\,\mathrm{d}\sigma - 2\mathrm{Re}\int_{\Gamma_1} b^2\dot z(m\cdot\nabla\overline{z})\mathrm{d}\sigma \quad \forall\, t\geqslant 0, \qquad (7.6.18)\end{aligned}$$

where $b_0 = \inf_{x\in\Gamma_1} b_0 > 0$. Let $\beta > 0$ be such that

$$\int_{\Gamma_1} b^2|f|^2\,\mathrm{d}x \leqslant \beta\int_\Omega |\nabla f|^2\,\mathrm{d}x \qquad \forall\, f\in\mathcal{H}^1_{\Gamma_0}(\Omega).$$

It is easy to see that

$$\begin{aligned}(n-1)\left|\int_{\Gamma_1} b^2\dot z\,\overline{z}\,\mathrm{d}\sigma\right| &\leqslant \frac{1}{2}\int_{\Gamma_1} b^2\left[(n-1)^2\beta|\dot z|^2 + \beta^{-1}|z|^2\right]\mathrm{d}x\\ &\leqslant \frac{1}{2}(n-1)^2\beta\int_{\Gamma_1} b^2|\dot z|^2\,\mathrm{d}x + \frac{1}{2}V_0(t) \qquad \forall\, t\geqslant 0.\end{aligned}$$

Moreover, denoting $\delta = \inf_{x\in\Gamma_1}\sqrt{m\cdot\nu} > 0$, we have

$$\begin{aligned}2\left|\int_{\Gamma_1} b^2\dot z(m\cdot\nabla\overline{z})\mathrm{d}\sigma\right| &\leqslant \int_{\Gamma_1}|b|\cdot|\dot z|\cdot\|bm\|_{L^\infty(\Gamma_1)}\,|\nabla z|\,\mathrm{d}\sigma\\ &\leqslant \frac{\|bm\|^2_{L^\infty(\Gamma_1)}}{2\delta^2}\int_{\Gamma_1} b^2|\dot z|^2\,\mathrm{d}\sigma + \frac{1}{2}\int_{\Gamma_1}(m\cdot\nu)|\nabla z|^2\,\mathrm{d}\sigma \qquad \forall\, t\geqslant 0.\end{aligned}$$

Using the last two formulas with (7.6.18) it follows that for every $t\geqslant 0$, we have

$$\dot\rho(t) \leqslant -\frac{V_0(t)}{2} + \frac{1}{2}\left[\frac{2\|m\|_{L^\infty(\Gamma_1)}}{b_0^2} + (n-1)^2\beta + \frac{\|bm\|^2_{L^\infty(\Gamma_1)}}{\delta^2}\right]\int_{\Gamma_1} b^2|\dot z|^2\,\mathrm{d}\sigma. \qquad (7.6.19)$$

Since, according to (7.6.8), for every $\varepsilon \geqslant 0$ we have $V_\varepsilon = V_0 + \varepsilon\rho$, we can combine (7.6.10) and (7.6.19) to obtain

$$\dot{V}_\varepsilon(t) \leqslant -\frac{\varepsilon}{2} V_0(t)$$
$$-\frac{1}{2}\left\{4 - \varepsilon\left[\frac{2\|m\|_{L^\infty(\Gamma_1)}}{b_0^2} + (n-1)^2\beta + \frac{\|bm\|^2_{L^\infty(\Gamma_1)}}{\delta^2}\right]\right\}\int_{\Gamma_1} b^2|\dot{z}|^2 \mathrm{d}\sigma \quad \forall t \geqslant 0.$$

It follows that there exists $\varepsilon_1 > 0$, depending only on b and on Ω, such that

$$\dot{V}_\varepsilon(t) \leqslant -\frac{\varepsilon}{2} V_0(t) \qquad \forall\, \varepsilon \in (0, \varepsilon_1),\ t \geqslant 0. \tag{7.6.20}$$

Let $\varepsilon_2 = \min\left\{\frac{\varepsilon_0}{2}, \frac{\varepsilon_1}{2}\right\}$, where ε_0 is the constant in Lemma 7.6.2. It is clear that $\varepsilon_2 > 0$ depends only on b and on Ω. By combining Lemma 7.6.2 and (7.6.20) it follows that

$$\dot{V}_{\varepsilon_2}(t) \leqslant -\frac{\varepsilon_2}{3} V_{\varepsilon_2}(t) \qquad \forall\, t \geqslant 0,$$

which implies that

$$V_{\varepsilon_2}(t) \leqslant e^{-\frac{t\varepsilon_2}{3}} V_{\varepsilon_2}(0) \qquad \forall\, t \geqslant 0.$$

Using again Lemma 7.6.2 it follows that (7.6.6) holds with $M = 3$ and $\omega = \frac{\varepsilon_2}{3}$. □

Corollary 7.6.4. *With the above notation and with the assumption in Theorem* 7.6.1*, the semigroup* $\mathbb{T}$ *is exponentially stable.*

Proof. We have seen in Section 3.9 that the strong solutions of (7.6.4) satisfy

$$\begin{bmatrix} z(\cdot, t) \\ \dot{z}(\cdot, t) \end{bmatrix} = \mathbb{T}_t \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \qquad \forall\, t \geqslant 0. \tag{7.6.21}$$

Therefore, the conclusion of Theorem 7.6.1 can be rewritten as

$$\left\|\mathbb{T}_t \begin{bmatrix} z_0 \\ w_0 \end{bmatrix}\right\| \leqslant Me^{-\omega t}\left\|\begin{bmatrix} z_0 \\ w_0 \end{bmatrix}\right\| \qquad \forall \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in \mathcal{D}(A), \quad t \geqslant 0.$$

Using the density of $\mathcal{D}(A)$ in X, it follows that the above estimate holds for every $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in X$, so that $\mathbb{T}$ is an exponentially stable semigroup. □

Below, as usual, X_1 stands for $\mathcal{D}(A)$ endowed with the graph norm.

Corollary 7.6.5. *With the assumptions in Theorem* 7.6.1*, let* $C \in \mathcal{L}(X_1, L^2(\Gamma_1))$ *be defined by*

$$C \begin{bmatrix} f \\ g \end{bmatrix} = b\frac{\partial g}{\partial \nu}\Big|_{\Gamma_1} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A).$$

Then C *is admissible for* $\mathbb{T}$ *and* (A, C) *is exactly observable.*

Proof. Integrating (7.6.5) with respect to time, it follows that for every $\tau > 0$,

$$\|\nabla z_0\|^2_{[L^2(\Omega)]^n} + \|w_0\|^2_{L^2(\Omega)} - \left(\|\nabla z(\cdot,\tau)\|^2_{[L^2(\Omega)]^n} + \|\dot z(\cdot,\tau)\|^2_{L^2(\Omega)}\right)$$
$$= 2\int_0^\tau \int_{\Gamma_1} b^2 |\dot z(\cdot,t)|^2 \,\mathrm{d}\sigma\,\mathrm{d}t = 2\int_0^\tau \left\| C \begin{bmatrix} z(\cdot,t) \\ \dot z(\cdot,t) \end{bmatrix} \right\|^2 \mathrm{d}t. \quad (7.6.22)$$

Because of (7.6.21), this implies that C is an admissible observation operator for $\mathbb{T}$.

On the other hand, Theorem 7.6.1 implies that for $\tau > 0$ large enough we have

$$\|\nabla z_0\|^2_{[L^2(\Omega)]^n} + \|w_0\|^2_{L^2(\Omega)} - \left(\|\nabla z(\cdot,\tau)\|^2_{[L^2(\Omega)]^n} + \|\dot z(\cdot,\tau)\|^2_{L^2(\Omega)}\right)$$
$$\geqslant \frac{1}{2}\left(\|\nabla z_0\|^2_{[L^2(\Omega)]^n} + \|w_0\|^2_{L^2(\Omega)}\right).$$

Combining the above estimate with (7.6.22) it follows that

$$\int_0^\tau \left\| C \begin{bmatrix} z(\cdot,t) \\ \dot z(\cdot,t) \end{bmatrix} \right\|^2 \mathrm{d}t \geqslant \frac{1}{4}\left(\|\nabla z_0\|^2_{[L^2(\Omega)]^n} + \|w_0\|^2_{L^2(\Omega)}\right) \qquad \forall \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in \mathcal{D}(A),$$

which means, according to (7.6.21), that the pair (A, C) is exactly observable. □

7.7 Remarks and bibliographical notes on Chapter 7

General remarks. An important idea which we aimed to explain in Chapter 7 is that the splitting of a system governed by PDEs into low- and high- frequency parts is an important step in understanding the observability properties of the system. The high-frequency part can be tackled by various methods (we used multiplier or perturbation techniques in our presentation) whereas low frequencies are tackled by using the finite-dimensional Hautus test combined with unique continuation for elliptic operators. The two parts are finally put together by using the simultaneous observability result in Theorem 6.4.2 and its consequences.

The multiplier method is a tool coming from the study of the wave equation in exterior domains and in particular from scattering problems (see Morawetz [172] and Strauss [212]). The use of multiplier methods for the exact observability of systems governed by wave equations or Euler–Bernoulli plate equations became very popular after the publication of the book by Lions in [156]. Since then, research in this area has flourished. The main advantage of the multiplier method is that it is very simple, being essentially based on integration by parts. This was the main motivation for choosing it in Chapter 7. Among the disadvantages of this method we mention that it cannot (in general) tackle lower-order terms or variable coefficients. This difficulty can be overcome in some cases (like in Section 7.3) by using the decomposition into low and high frequencies. A systematic method of tackling lower-order terms is provided by Carleman estimates, which can be seen as a sophisticated version of the multiplier method, the multiplier being constructed from an appropriate weight function. The calculations in this method can be very complex (see, for instance, Li and Zhang [153] and the references therein).

The important work of Bardos, Lebeau and Rauch [15] (see also Burq and Gérard [25]) brought in methods coming from micro-local analysis which gave sharp results for the minimal time required for exact observability and the choice of the observation region. Moreover, these methods are successful in tackling lower order terms. In their initial form, the micro-local analytic methods required a C^∞ boundary. This restriction has been relaxed in Burq [24]. A presentation of the methods introduced in [15] requires a solid background in pseudo-differential calculus and some basis in symplectic geometry, so that it lies outside the scope of this book.

A subject which is missing in our presentation is the approximate observability of systems governed by the wave equation. It turns out that, for the Neumann boundary observation, this property holds for any open subset Γ of $\partial\Omega$. In the case of analytical coefficients, this follows from Holmgren's uniqueness theorem (see, for instance, John [125, Section 3.5] or Lions [156, Section 1.8]). In the case of a wave equation with time-independent L^∞ coefficients in some of the lower terms, the corresponding results (much harder) have been obtained, with successive improvements of the observability time, in Robbiano [190], Hörmander [102] and Tataru [215].

Another issue of interest which has not been tackled in this work is the study of the relation between the observability of systems governed by the wave equation and the observability of finite-dimensional systems obtained by discretizing the system with respect to the space variable. More precisely, the observability constants of the finite-dimensional systems obtained by applying finite differences or finite elements schemes to a wave equation may blow up when the discretization step tends to zero, as it has been remarked in Infante and Zuazua [107]. This difficulty can be tackled, for instance, by filtering the spurious high frequencies. We refer the reader to Zuazua [245] and the references therein for more details on this question.

Section 7.1. The result in Theorem 7.1.3 has been called "hidden regularity property" by Lions and his coworkers. This terminology was motivated by the fact that (7.1.8) can be used to give a sense, by density, to the normal derivative on $\partial\Omega$ of the solution η of (7.1.2)–(7.1.4), for initial data $f \in \mathcal{H}_0^1(\Omega)$, $g \in L^2(\Omega)$. Note that, in this case, the trace of $\frac{\partial\eta}{\partial\nu}$ on $\partial\Omega$ makes no sense by the usual trace theorem since, at given $t > 0$, the regularity of the map $x \mapsto \eta(x,t)$ is, in general, only $\mathcal{H}_0^1(\Omega)$. Our proof of Theorem 7.1.3 is essentially the same as in Lasiecka, Lions and Triggiani [143] (see also Lions [156, p. 44] and Komornik [130, p. 20]).

Section 7.2. The main result in Theorem 7.2.4 has been first proved (with a less accurate estimate of the observability time) by Ho in [99]. Our proof follows [130, Chapter 3] and it yields the same observability time as in [130]. Note that the proof of Theorem 7.2.4 is quite elementary (only integration by parts).

As mentioned in Remark 7.2.5, the condition that $\Gamma \supset \Gamma(x_0)$ in Theorem 7.2.4 is not necessary for the exact observability of the wave equation with Neumann boundary observation. More general sufficient conditions have been given by versions of the multiplier method like the rotated multipliers from Osses [179] or the piecewise multipliers from Liu [160]. The most general known sufficient condition for exact observability in time τ has been given in [15]. This condition means, roughly speaking, that any light ray travelling in Ω at unit speed and reflected according to geometric optics laws when it hits $\partial\Omega$ in a point not belonging to Γ, will eventually hit Γ in time $\leqslant \tau$ (see [15] or [169] for more details on this condition). This condition is "almost" necessary in a sense made precise in [15] and we shall refer to it as the *geometric optics condition* of Bardos, Lebeau and Rauch.

Note that the minimum time for exact observability in Theorem 7.2.4 is, in general, far from being sharp (see [130, Remark 3.6] for the description of a situation in which $2r(x_0)$ is the optimal lower bound for the exact observability time).

Section 7.3. By using an approach based on Carleman estimates (for hyperbolic operators) as in Fursikov and Imanuvilov [69] or on microlocal analysis as in [15], it is possible to tackle directly the perturbed wave equation with Neumann boundary observation. Our aim in establishing Theorem 7.3.2 was to show that by a perturbation method the problem is reduced to the constant coefficients case from Theorem 7.2.4 without increasing the observability time. Note that Komornik in [128] has proved, by a multiplier based approach, the result in Theorem 7.3.2 in the particular case of Γ satisfying the assumptions in Theorem 7.2.4.

Section 7.4. The study of locally distributed observation for the wave equation seems to have been initiated by Lagnese in [136], who considered particular geometries (like one-dimensional or spherical). For a general n-dimensional bounded domain, Zuazua has shown in Chapter VII of [156] that the wave equation with distributed control in an ε-neighborhood of an appropriate part of the boundary is exactly observable. Our Theorem 7.4.1 improves the estimates on the observability time from [156]. Our proof of Theorem 7.4.1 combines methods from [156], Liu [160] and the decomposition of the system into low- and high-frequency parts. We mention that alternative ways of obtaining Theorem 7.4.1 are microlocal analysis or Carleman estimates. Our proof of Proposition 7.4.5 follows essentially Haraux [93]. For more general results which yield exponential stability from observability estimates we refer the reader to Tucsnak and Weiss [223] and to Ammari and Tucsnak [7].

Section 7.5. The first result on the exact boundary observability in arbitrarily small time for the Euler–Bernoulli plate equation has been obtained, by using multipliers and a compactness-uniqueness argument, by Zuazua in Appendix 1 of [156], who assumed that the observed part of the boundary satisfies the assumptions in Theorem 7.2.4. A different method for exact observability in arbitrarily small time has been applied for the Euler–Bernoulli plate equation with clamped or hinged boundary conditions in Komornik [130]. The microlocal approach to the Schrödinger and Euler–Bernoulli equations is due to Lebeau [150], who showed that we have exact boundary observability in arbitrarily small time for these equations provided that the observed part of the boundary satisfies the geometric optics condition of Bardos, Lebeau and Rauch (see the comments on Section 7.2). The approach based on microlocal analysis, without explicit reference to the wave equation, has been further developed in Burq and Zworski [27].

The fact that, with appropriate boundary conditions, an exact boundary observability result for the wave equation implies, with no need of repeating multipliers or microlocal analysis arguments, observability inequalities for the Schrödinger and plate equations has been remarked in Miller [170]. We were able to give very short proofs for Propositions 7.5.1 and 7.5.3 thanks to the use of the abstract results from Theorems 6.7.2 and 6.7.5. Note that the geometric optics condition is not necessary for the exact observability of the Schrödinger and plate equations, as it has been first remarked in Krabs, Leugering and Seidman [134] and then in Haraux [92]. Detailed results in this direction are given in Section 8.5.

If we consider the Euler–Bernoulli plate equations which correspond to clamped or free parts of the boundary, then the corresponding fourth-order differential operator is no longer the square of the Dirichlet Laplacian, so that the exact observability cannot be reduced to a problem for the wave equation. We refer the reader to Lasiecka and Triggiani [148], [147] for some results concerning this case.

Section 7.6. The study of the exponential stability of this damped wave equation has been initiated in Quinn and Russell [185]. Other early papers devoted to the same subject are Chen [30, 31, 32, 33] and Lagnese [137]. Our presentation follows closely Komornik and Zuazua [132]. An interesting feature of the method in [132] is that it allows, with $b^2 = m \cdot \nu$ and for $n \leqslant 3$, to avoid the second condition in (7.6.2) (which excludes simply connected domains). The fact that the second condition in (7.6.2) is not necessary for $n > 3$ (still with $b^2 = m \cdot \nu$) has been shown in Bey, Lohéac and Moussaoui [19]. The fact that condition (7.6.1) can be relaxed to

$$\Gamma_1 \supset \{x \in \partial\Omega \mid m(x) \cdot \nu(x) > 0\},$$

has been shown in Lasiecka and Triggiani [149]. Finally, let us mention that the exponential decay property has been established in Bardos, Lebeau and Rauch [15] assuming that $\partial\Omega$ is of class C^∞, that the second condition in (7.6.2) holds and that Γ_1 satisfies the geometric optics condition.

Chapter 8

Non-harmonic Fourier Series and Exact Observability

In this chapter we show how classical results on non-harmonic Fourier series imply exact observability for some systems governed by PDEs. The method of non-harmonic Fourier series for exact observability of PDEs is essentially limited to one space dimension or to rectangular domains in $\mathbb{R}^n$, since it uses that the eigenfunctions of the operator can be expressed (or approximated) by complex exponentials. We shall see that, in some of the above-mentioned cases, this method yields sharp estimates on the observability time and on the observation region.

Notation. In this chapter we denote by $|z|$ both the absolute value of a complex number z and the Euclidean norm of a vector $z \in \mathbb{R}^n$ (where $n \in \mathbb{N}$). The inner product of $z, w \in \mathbb{C}^n$ is denoted by $z \cdot w$. In this chapter we found it more convenient to use a definition for the Fourier transformation that differs by a constant factor from that in Section 12.4. More precisely, for $n \in \mathbb{N}$ and $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f, denoted by $\widehat{f}$ or $\mathcal{F}f$, is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} \exp(-ix \cdot \xi) f(x) \,\mathrm{d}x \qquad \forall\, \xi \in \mathbb{R}^n .$$

8.1 A theorem of Ingham

In this section we prove Ingham's theorem (shown below), widely used in the literature, in order to establish the exact observability of systems governed by PDEs. We also derive a consequence for systems with skew-adjoint generators.

Theorem 8.1.1. *Let $\mathcal{I} \subset \mathbb{Z}$, let $(\lambda_m)_{m\in\mathcal{I}}$ be a real sequence satisfying*

$$\inf_{\substack{m,l\in\mathcal{I} \\ m\neq l}} |\lambda_m - \lambda_l| = \gamma > 0 \tag{8.1.1}$$

and let $J \subset \mathbb{R}$ be a bounded interval. Then, for every sequence $(a_m) \in l^2(\mathcal{I}, \mathbb{C})$ the series $\sum_{m\in\mathcal{I}} a_m e^{i\lambda_m t}$ converges in $L^2(J)$ to a function f and there exists a constant $c_1 > 0$, depending only on γ and on the length of J, such that

$$\int_J |f(t)|^2 \, \mathrm{d}t \leqslant c_1 \sum_{m\in\mathcal{I}} |a_m|^2 . \tag{8.1.2}$$

Moreover, if the length of J is larger than $\frac{2\pi}{\gamma}$, then there exists $c_2 > 0$, depending only on γ and on the length of J, such that

$$c_2 \sum_{m\in\mathcal{I}} |a_m|^2 \leqslant \int_J |f(t)|^2 \, \mathrm{d}t . \tag{8.1.3}$$

The main ingredient of the proof of Theorem 8.1.1 is the following result.

Lemma 8.1.2. *Let $(\mu_m)_{m\in\mathcal{I}}$ be a sequence satisfying*

$$\inf_{\substack{m,l\in\mathcal{I}\\ m\neq l}} |\mu_m - \mu_l| = \gamma_0 > 1 . \tag{8.1.4}$$

Let $k : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$k(t) = \begin{cases} \cos\left(\frac{t}{2}\right) & \text{if} \quad |t| < \pi, \\ 0 & \text{if} \quad |t| \geqslant \pi . \end{cases}$$

Then the inequality

$$\begin{aligned} 4\left(1 - \frac{1}{\gamma_0^2}\right) \sum_{m\in\mathcal{I}} |a_m|^2 &\leqslant \int_{-\infty}^{+\infty} k(t) \left| \sum_{m\in\mathcal{I}} a_m e^{i\mu_m t} \right|^2 \mathrm{d}t \\ &\leqslant 4\left(1 + \frac{1}{\gamma_0^2}\right) \sum_{m\in\mathcal{I}} |a_m|^2 \end{aligned} \tag{8.1.5}$$

holds for every sequence $(a_m)_{m\in\mathcal{I}}$ with a finite number of non-vanishing terms.

Proof. Clearly we have that $k \in L^1(\mathbb{R})$ and

$$\int_{-\infty}^{+\infty} k(t) |f(t)|^2 \, \mathrm{d}t = \sum_{m,l\in\mathcal{I}} a_m \overline{a_l} \widehat{k}(\mu_m - \mu_l) . \tag{8.1.6}$$

It is easy to check that the Fourier transform of k is given by

$$\widehat{k}(\xi) = \frac{4\cos(\pi\xi)}{1 - 4\xi^2} \qquad \forall\, \xi \in \mathbb{R},$$

so that for every $m \in \mathcal{I}$ we have

$$\sum_{\substack{l\in\mathcal{I}\\ l\neq m}} \widehat{k}(\mu_m - \mu_l) \leqslant \sum_{\substack{l\in\mathcal{I}\\ l\neq m}} \frac{4}{4\gamma_0^2(m-l)^2 - 1} \leqslant \frac{8}{\gamma_0^2}\sum_{r=1}^{\infty}\frac{1}{4r^2-1}$$
$$= \frac{4}{\gamma_0^2}\sum_{r=1}^{\infty}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right) = \frac{4}{\gamma_0^2} = \frac{\widehat{k}(0)}{\gamma_0^2}.$$

The above inequality and the fact that $|a_m\overline{a_l}| \leqslant \frac{|a_m|^2+|a_n|^2}{2}$ for every $m,\ l \in \mathcal{I}$ imply that

$$\left|\sum_{\substack{m,l\in\mathcal{I}\\ m\neq l}} a_m\overline{a_l}\widehat{k}(\mu_m - \mu_l)\right|$$
$$\leqslant \frac{1}{2}\left(\sum_{m\in\mathcal{I}}|a_m|^2\sum_{l\neq m}|\widehat{k}(\mu_m-\mu_l)| + \sum_{l\in\mathcal{I}}|a_l|^2\sum_{m\neq l}|\widehat{k}(\mu_m-\mu_l)|\right)$$
$$= \sum_{m\in\mathcal{I}}|a_m|^2\sum_{l\neq m}|\widehat{k}(\mu_m-\mu_l)| \leqslant \frac{\widehat{k}(0)}{\gamma_0^2}\sum_{m\in\mathcal{I}}|a_m|^2.$$

By combining the above estimate and (8.1.6), we obtain the conclusion (8.1.5). $\square$

We are now in a position to prove the main result in this section.

Proof of Theorem 8.1.1. Suppose that the sequence $(a_m)_{m\in\mathcal{I}}$ has a finite number of non-vanishing terms. Let $\alpha > \frac{1}{\gamma}$. Then the sequence $(\mu_m)_{m\in\mathcal{I}}$, defined by $\mu_m = \alpha\lambda_m$ for every $m \in \mathcal{I}$, satisfies (8.1.4) with $\gamma_0 = \alpha\gamma$. By using (8.1.5) combined with the fact that

$$k(t) \geqslant \frac{\sqrt{2}}{2} \qquad \forall\, t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

it follows that

$$\frac{\sqrt{2}}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left|\sum_{m\in\mathcal{I}} a_m e^{i\mu_m t}\right|^2 \mathrm{d}t \leqslant 4\left(1 + \frac{1}{\gamma^2}\right)\sum_{m\in\mathcal{I}}|a_m|^2.$$

The above estimate, combined with the fact that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left|\sum_{m\in\mathcal{I}} a_m e^{i\mu_m t}\right|^2 \mathrm{d}t = \frac{1}{\alpha}\int_{-\frac{\alpha\pi}{2}}^{\frac{\alpha\pi}{2}}\left|\sum_{m\in\mathcal{I}} a_m e^{i\lambda_m t}\right|^2 \mathrm{d}t,$$

yields that

$$\int_{-\frac{\alpha\pi}{2}}^{\frac{\alpha\pi}{2}}\left|\sum_{m\in\mathcal{I}} a_m e^{i\lambda_m t}\right|^2 \mathrm{d}t \leqslant 4\alpha\sqrt{2}\left(1 + \frac{1}{\alpha^2\gamma^2}\right)\sum_{m\in\mathcal{I}}|a_m|^2.$$

By using a simple change of variables (a translation) it follows that

$$\int_J \left| \sum_{m\in\mathcal{I}} a_m e^{i\lambda_m t} \right|^2 dt \leqslant 4\alpha\sqrt{2}\left(1 + \frac{1}{\alpha^2\gamma^2}\right) \sum_{m\in\mathcal{I}} |a_m|^2$$

for every interval of length $\alpha\pi$. Since every bounded interval $J \subset \mathbb{R}$ can be covered by a finite number of intervals of length $\alpha\pi$, it follows that there exists a positive constant c_1, depending only on γ and on the length of J, such that

$$\int_J \left| \sum_{m\in\mathcal{I}} a_m e^{i\lambda_m t} \right|^2 dt \leqslant c_1 \sum_{m\in\mathcal{I}} |a_m|^2 .$$

This implies that for every bounded interval J and every l^2 sequence $(a_m)_{m\in\mathcal{I}}$, the series $\sum_{m\in\mathcal{I}} a_m e^{i\lambda_m t}$ converges in $L^2(J)$ to a function f and there exists a constant c_1, depending only on γ and on the length of J, such that f satisfies (8.1.2).

We still have to prove (8.1.3). By using (8.1.5) we obtain that for every sequence $(a_m)_{m\in\mathcal{I}}$ with a finite number of non-vanishing terms and for every $\alpha > \frac{1}{\gamma}$, we have

$$\begin{aligned} \int_{-\alpha\pi}^{\alpha\pi} \left| \sum_{m\in\mathcal{I}} a_m e^{i\lambda_m t} \right|^2 &= \alpha \int_{-\pi}^{\pi} \left| \sum_{m\in\mathcal{I}} a_m e^{i\mu_m t} \right|^2 dt \\ &\geqslant \alpha \int_{-\infty}^{+\infty} k(t) \left| \sum_{m\in\mathcal{I}} a_m e^{i\mu_m t} \right|^2 dt \geqslant 4\alpha\left(1 - \frac{1}{\alpha^2\gamma^2}\right) \sum_{m\in\mathcal{I}} |a_m|^2 . \end{aligned}$$

By using a simple change of variables (again a translation) we obtain that

$$\int_J \left| \sum_{m\in\mathcal{I}} a_m e^{i\lambda_m t} \right|^2 \geqslant 4\alpha\left(1 - \frac{1}{\alpha^2\gamma^2}\right) \sum_{m\in\mathcal{I}} |a_m|^2 \tag{8.1.7}$$

for every interval $J \subset \mathbb{R}$ of length $2\alpha\pi$ (which can be any real number strictly larger than $\frac{2\pi}{\gamma}$). We have already seen that, for every l^2 sequence (a_m) and every bounded interval $J \subset \mathbb{R}$, the series $\sum_{m\in\mathcal{I}} a_m e^{i\lambda_m t}$ converges to f in $L^2(J)$. This fact, combined with (8.1.7), implies that (8.1.3) holds for every interval J of finite length $|J| > \frac{2\pi}{\gamma}$, with

$$c_2 = \frac{2(\gamma^2|J|^2 - 4\pi^2)}{\pi\gamma^2|J|} .$$

□

One of the consequences of Ingham's theorem is the following result on systems with a skew-adjoint generator and scalar output.

Proposition 8.1.3. *Let $A : \mathcal{D}(A) \to X$ be a skew-adjoint operator generating the unitary group $\mathbb{T}$. Assume that A is diagonalizable with an orthonormal basis $(\phi_m)_{m\in\mathcal{I}}$ in X formed of eigenvectors and denote by $i\lambda_m \in i\mathbb{R}$ the eigenvalue* *corresponding to* ϕ_m*. Assume that the eigenvalues of* A *are simple and that there exists a bounded set* $J \subset i\mathbb{R}$ *such that*

$$\inf_{\substack{\lambda,\mu\in\sigma(A)\setminus J\\ \lambda\neq\mu}} |\lambda-\mu| = \gamma > 0. \tag{8.1.8}$$

Moreover, let $C \in \mathcal{L}(X_1, \mathbb{C})$ *be an observation operator for the semigroup generated by* A *such that*

$$\inf_{m\in\mathcal{I}} |C\phi_m| > 0 \quad \text{and} \quad \sup_{m\in\mathcal{I}} |C\phi_m| < \infty. \tag{8.1.9}$$

Then C *is an admissible observation operator for* $\mathbb{T}$ *and the pair* (A, C) *is exactly observable in any time* $\tau > \frac{2\pi}{\gamma}$*.*

Proof. Note first that for every $z \in X_1$ we have

$$C\mathbb{T}_t z = \sum_{m\in\mathcal{I}} \langle z, \phi_m\rangle C\phi_m e^{i\lambda_m t} \qquad \forall\, t \geqslant 0. \tag{8.1.10}$$

On the other hand, the fact that the eigenvalues of A are simple, combined with (8.1.8), implies that

$$\inf_{\substack{\lambda,\mu\in\sigma(A)\\ \lambda\neq\mu}} |\lambda-\mu| > 0.$$

The above property, combined with (8.1.10) and with the fact that $\sup_{m\in\mathcal{I}}|C\phi_m| < \infty$, implies, by using Theorem 8.1.1, that for every $\tau > 0$ there exists a constant $K_\tau > 0$ such that

$$\int_0^\tau |C\mathbb{T}_t z|^2 \,\mathrm{d}t \leqslant K_\tau^2 \|z\|^2 \qquad \forall\, z \in X_1.$$

We have thus shown that C is an admissible observation operator for $\mathbb{T}$. Denote

$$V = \operatorname{span}\{\phi_k \mid \lambda_k \in J\}^\perp.$$

For every $z \in X_1 \cap V$ we have

$$C\mathbb{T}_t z = \sum_{\substack{m\in\mathcal{I}\\ \lambda_m\notin J}} \langle z, \phi_m\rangle C\phi_m e^{i\lambda_m t} \qquad \forall\, t \geqslant 0.$$

From the above formula and (8.1.8) it follows, by using Theorem 8.1.1, that for every $\tau > \frac{2\pi}{\gamma}$ there exists $k_\tau > 0$ such that

$$\int_0^\tau |C\mathbb{T}_t z|^2 \,\mathrm{d}t \geqslant k_\tau^2 \|z\|^2 \qquad \forall\, z \in X_1 \cap V. \tag{8.1.11}$$

If we denote by A_V the part of A in V and by C_V the restriction of C to $\mathcal{D}(A_V)$, the last formula says that the pair (A_V, C_V) is exactly observable in any time $\tau > \frac{2\pi}{\gamma}$. Since $C\phi \neq 0$ for every eigenvector ϕ of A, we obtain (by applying Proposition 6.4.4) that the pair (A, C) is exactly observable in any time $\tau > \frac{2\pi}{\gamma}$. $\square$

8.2 Variable coefficients PDEs in one space dimension with boundary observation

Notation and preliminaries. Throughout this section, J denotes the interval $(0,1)$ and $a, b : J \to \mathbb{R}$ are two functions such that $a \in C^2(J)$, $b \in L^\infty(J)$ and a is bounded from below (i.e., there exists $m > 0$ such that $a(x) \geqslant m > 0$ for all $x \in J$). We denote by H the space $L^2(J)$ and $\mathcal{D}(A_0) = H_1$ is the Sobolev space $\mathcal{H}^2(J) \cap \mathcal{H}_0^1(J)$. The operator $A_0 : \mathcal{D}(A_0) \to H$ is defined by

$$\mathcal{D}(A_0) = \mathcal{H}^2(J) \cap \mathcal{H}_0^1(J), \qquad A_0 f = -\frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}f}{\mathrm{d}x}\right) + bf \qquad \forall\, f \in \mathcal{D}(A_0). \tag{8.2.1}$$

Recall from Proposition 3.5.2 that A_0 is self-adjoint, diagonalizable and that its simple eigenvalues can be ordered to form a strictly increasing sequence $(\lambda_k)_{k\geqslant 1}$. We have also seen in Proposition 3.5.2 that there exists an orthonormal basis $(\varphi_k)_{k\geqslant 1}$ of H formed by eigenvectors of A_0 and that if b is non-negative, then A_0 is strictly positive and $H_{\frac{1}{2}} = \mathcal{H}_0^1(J)$. In the case of a non-negative b we define $X = H_{\frac{1}{2}} \times H$, which is a Hilbert space with the inner product

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_X = \langle A_0^{\frac{1}{2}} f_1, A_0^{\frac{1}{2}} f_2 \rangle + \langle g_1, g_2 \rangle,$$

we set $X_1 = H_1 \times H_{\frac{1}{2}}$ and we define the linear operator $A : X_1 \to X$ by

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad \text{i.e.,} \quad A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -A_0 f \end{bmatrix}. \tag{8.2.2}$$

Recall from Proposition 3.7.6 that A is skew-adjoint on X so that it generates a unitary group $\mathbb{T}$ on X. Define $C_1 \in \mathcal{L}(H_1, \mathbb{C})$ and $C \in \mathcal{L}(X_1, \mathbb{C})$ by

$$C_1 z = \frac{\mathrm{d}z}{\mathrm{d}x}(0), \quad C = \begin{bmatrix} C_1 & 0 \end{bmatrix} \qquad \forall\, z \in H_1. \tag{8.2.3}$$

In this section we give some observability results for systems governed by the string or by the Schrödinger equation with variable coefficients. The basic tools for proving our results will be Ingham's theorem (with its consequences from Proposition 8.1.3) and the results on Sturm–Liouville operators from Section 3.5.

The following property of the eigenvalues and eigenvectors of A_0 plays an important role in the remaining part of this section.

Lemma 8.2.1. *Assume that b is non-negative. Then*

$$\sup_{n\geqslant 1} \frac{1}{\sqrt{\lambda_n}} \left| \frac{\mathrm{d}\varphi_n}{\mathrm{d}x}(0) \right| < \infty, \tag{8.2.4}$$

$$\inf_{n\geqslant 1} \frac{1}{\sqrt{\lambda_n}} \left| \frac{\mathrm{d}\varphi_n}{\mathrm{d}x}(0) \right| > 0. \tag{8.2.5}$$

Proof. We first note that, since the eigenvalues of A_0 are real and the coefficients a and b are real-valued functions, we have that the functions $(\varphi_n)_{n\geqslant 1}$ are real valued. Moreover, the fact that b is non-negative implies that $A_0 > 0$ so that $\lambda_n > 0$ for all $n \in \mathbb{N}$, hence the expression on the left-hand side of (8.2.5) is well defined. At this point it is convenient to use the change of variables introduced in Section 3.5. More precisely, we set

$$l = \int_0^1 \frac{\mathrm{d}x}{\sqrt{a(x)}}, \tag{8.2.6}$$

and we consider again the one-to-one function g from J onto $[0,l]$ defined by

$$g(x) = \int_0^x \frac{\mathrm{d}\xi}{\sqrt{a(\xi)}}\mathrm{d}\xi \qquad \forall\, x \in J, \tag{8.2.7}$$

and its inverse h which maps $[0,l]$ onto J. We know from Lemma 3.5.4 that the function ψ_n, defined by

$$\psi_n(s) = [a(h(s))]^{\frac{1}{4}}\, \varphi_n(h(s)) \qquad \forall\, s \in [0,l], \tag{8.2.8}$$

is in $\mathcal{H}^2(0,l) \cap \mathcal{H}_0^1(0,l)$ and it satisfies

$$-\frac{\mathrm{d}^2\psi_n}{\mathrm{d}s^2}(s) = (\lambda_n - r(s))\psi_n(s) \qquad \forall\, s \in [0,l], \tag{8.2.9}$$

where the function $r \in L^\infty(0,l)$ has been defined in (3.5.4). Moreover, it is easy to check, by using (8.2.7) and (8.2.8), that

$$\int_0^l \psi_n^2(s)\,\mathrm{d}s = 1 \qquad \forall\, n \in \mathbb{N}. \tag{8.2.10}$$

Taking next the inner product in $L^2[0,l]$ of both sides of (8.2.9) by ψ_n and using (8.2.10) we obtain that

$$\sup_{n\in\mathbb{N}} \frac{1}{\sqrt{\lambda_n}} \left\| \frac{\mathrm{d}\psi_n}{\mathrm{d}s} \right\|_{L^2[0,l]} < \infty. \tag{8.2.11}$$

Now we take the inner product in $L^2[0,l]$ of both sides of (8.2.9) by $(s-l)\frac{\mathrm{d}\psi_n}{\mathrm{d}s}$. For the left-hand side, we get

$$\int_0^l (s-l)\frac{\mathrm{d}^2\psi_n}{\mathrm{d}s^2}(s)\frac{\mathrm{d}\psi_n}{\mathrm{d}s}\,\mathrm{d}s = \frac{1}{2}\int_0^l (s-l)\frac{\mathrm{d}}{\mathrm{d}s}\left|\frac{\mathrm{d}\psi_n}{\mathrm{d}s}(s)\right|^2 \mathrm{d}s \tag{8.2.12}$$

$$= \frac{l}{2}\left|\frac{\mathrm{d}\psi_n}{\mathrm{d}s}(0)\right|^2 \mathrm{d}s - \frac{1}{2}\int_0^l \left|\frac{\mathrm{d}\psi_n}{\mathrm{d}s}(s)\right|^2 \mathrm{d}s. \tag{8.2.13}$$

For the right-hand side, we get

$$\begin{aligned}
&\int_0^l (s-l)\left[\lambda_n\psi_n(s) - r(s)\psi_n(s)\right]\frac{\mathrm{d}\psi_n}{\mathrm{d}s}\,\mathrm{d}s \\
&\quad = \frac{\lambda_n}{2}\int_0^l (s-l)\frac{\mathrm{d}}{\mathrm{d}s}(\psi_n^2(s))\,\mathrm{d}s - \int_0^l (s-l)r(s)\frac{\mathrm{d}\psi_n}{\mathrm{d}s}(s)\psi_n(s)\,\mathrm{d}s \\
&\quad = -\frac{\lambda_n}{2}\int_0^l \psi_n^2(s)\,\mathrm{d}s - \int_0^l (s-l)r(s)\frac{\mathrm{d}\psi_n}{\mathrm{d}s}(s)\psi_n(s)\,\mathrm{d}s\,.
\end{aligned}$$

By combining the above relation and (8.2.13) it follows that

$$\begin{aligned}
\frac{l}{\lambda_n}\left|\frac{\mathrm{d}\psi_n}{\mathrm{d}s}(0)\right|^2 \;=\; &\int_0^l \psi_n^2(s)\,\mathrm{d}s + \frac{1}{\lambda_n}\int_0^l\left|\frac{\mathrm{d}\psi_n}{\mathrm{d}s}(s)\right|^2 \\
&+ \frac{2}{\lambda_n}\int_0^l (s-l)r(s)\frac{\mathrm{d}\psi_n}{\mathrm{d}s}(s)\psi_n(s)\,\mathrm{d}s\,.
\end{aligned}$$

The above equality, together with (8.2.10), (8.2.11) and the fact that $\lim_{n\to\infty}\lambda_n = \infty$, implies (8.2.4).

The same ingredients yield that

$$\liminf_{\lambda_n\to\infty}\frac{l}{\lambda_n}\left|\frac{\mathrm{d}\psi_n}{\mathrm{d}s}(0)\right|^2 \;\geqslant 1\,.$$

Using next the fact (easy to check) that $\frac{\mathrm{d}\psi_n}{\mathrm{d}s}(0) \neq 0$ for every $n\in\mathbb{N}$, it follows that

$$\inf_{n\in\mathbb{N}}\frac{1}{\lambda_n}\left|\frac{\mathrm{d}\psi_n}{\mathrm{d}s}(0)\right|^2 \;> 0\,. \tag{8.2.14}$$

On the other hand, from (8.2.7) and (8.2.8) it follows that

$$\frac{\mathrm{d}\psi_n}{\mathrm{d}s}(0) \;=\; [a(0)]^{\frac{3}{4}}\,\frac{\mathrm{d}\varphi_n}{\mathrm{d}x}(0)\,.$$

The above relation and (8.2.14) imply the conclusion (8.2.5). □

Proposition 8.2.2. *Assume that b is non-negative. Then the operator C defined in* (8.2.3) *is admissible for $\mathbb{T}$. Moreover, the pair (A, C) is exactly observable in any time $\tau > 2l$, where l has been defined in* (8.2.6).

Proof. The proof is essentially based on Proposition 8.1.3 and on the above estimates on the spectral elements of A_0. More precisely, denote $\mu_k = \sqrt{\lambda_k}$, with $k\in\mathbb{N}$, and consider the family $(\phi_k)_{k\in\mathbb{Z}^*}$ defined by

$$\phi_k \;=\; \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{i\mu_k}\varphi_k \\ \varphi_k\end{bmatrix} \qquad \forall\, k\in\mathbb{Z}^*\,,$$

where, for all $k \in \mathbb{N}$, we define $\varphi_{-k} = -\varphi_k$ and $\mu_{-k} = -\mu_k$. According to Proposition 3.7.7, the eigenvalues of A are $(i\mu_k)_{k\in\mathbb{Z}^*}$ and they correspond to the orthonormal basis of eigenvectors $(\phi_k)_{k\in\mathbb{Z}^*}$. This fact, combined with Proposition 3.5.5, implies that assumption (8.1.8) in Proposition 8.1.3 holds with $\gamma = \frac{\pi}{l}$.

On the other hand, Lemma 8.2.1 implies that assumption (8.1.9) in Proposition 8.1.3 is also satisfied. Moreover, it is easy to check that $C\phi_k \neq 0$ for all $k \in \mathbb{Z}^*$, so that we can apply Proposition 8.1.3 to get the desired conclusion. □

Remark 8.2.3. In terms of PDEs, the above proposition can be restated as follows: For every $\tau > 2l$ there exists $k_\tau > 0$ such that the solution w of

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2}(x,t) = \dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial w}{\partial x}(x,t)\right) - b(x)w(x,t), & x \in J,\ t \geqslant 0, \\ w(0,t) = 0, \quad w(\pi,t) = 0, & t \in [0,\infty), \\ w(x,0) = f(x), \quad \dfrac{\partial w}{\partial t}(x,0) = g(x), & x \in J, \end{cases}$$

satisfies

$$\int_0^\tau \left|\frac{\partial w}{\partial x}(0,t)\right|^2 \mathrm{d}t \geqslant k_\tau^2 \left(\|f\|^2_{\mathcal{H}_0^1(J)} + \|g\|^2_{L^2(J)}\right) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A).$$

Moreover, according to Remark 6.1.3, the above estimate is equivalent to

$$\int_0^\tau \left|\frac{\partial^2 w}{\partial x \partial t}(0,t)\right|^2 \mathrm{d}t \geqslant k_\tau^2 \left(\|f\|^2_{\mathcal{H}^2(J)} + \|g\|^2_{\mathcal{H}_0^1(J)}\right) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A^2).$$

Corollary 8.2.4. *The observation operator C_1 is admissible for the group generated by iA_0 in $H_{\frac{1}{2}}$. Moreover, the pair (iA_0, C_1) is exactly observable (with state space $H_{\frac{1}{2}}$) in any time $\tau > 0$.*

Proof. It is easy to check, by a simple change of variables, that it suffices to consider the case of a non-negative b. In this case the result follows by simply combining Proposition 8.2.2 and Theorem 6.7.2. □

Remark 8.2.5. In terms of PDEs, the above proposition can be restated as follows: For every $\tau >$ there exists $k_\tau > 0$ such that the solution w of

$$\begin{cases} i\dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial w}{\partial x}(x,t)\right) - b(x)w(x,t), & x \in J,\ t \geqslant 0, \\ w(0,t) = 0, \quad w(1,t) = 0, & t \in [0,\infty), \\ w(x,0) = f(x), & x \in J, \end{cases}$$

satisfies

$$\int_0^\tau \left|\frac{\partial w}{\partial x}(0,t)\right|^2 \mathrm{d}t \geqslant k_\tau^2 \|f\|^2_{\mathcal{H}_0^1(J)} \qquad \forall\, f \in H_1.$$

8.3 Domains associated with a sequence

In this section we introduce the concept of domain associated with a sequence and we give some conditions, either necessary or sufficient, for an open bounded set $D \subset \mathbb{R}^n$ to be a domain associated with the sequence $\Lambda = (\lambda_m)$. These results will be used in Section 8.4 in order to obtain new estimates on non-harmonic Fourier series in several space dimensions.

Let $n \in \mathbb{N}$ and $\mathcal{I} \subset \mathbb{Z}$. We say that a sequence $\Lambda = (\lambda_m)_{m\in\mathcal{I}}$ in $\mathbb{R}^n$ is *regular* if

$$\inf_{\substack{m,l\in\mathcal{I}\\ m\neq l}} |\lambda_m - \lambda_l| = \gamma > 0. \tag{8.3.1}$$

In the remaining part of this section we denote by Λ a regular sequence in $\mathbb{R}^n$, $D \subset \mathbb{R}^n$ is a bounded open set and $L^2_\Lambda(D)$ is the closure in $L^2(D)$ of $\operatorname{span}\{e^{i\lambda_m\cdot x} \mid m \in \mathcal{I}\}$.

Definition 8.3.1. We call an open subset $D \subset \mathbb{R}^n$ *a domain associated with the regular sequence* Λ if there exist constants $\delta_1(D)$, $\delta_2(D) > 0$ such that, for every sequence of complex numbers $(a_m)_{m\in\mathcal{I}}$ with a finite number of non-vanishing terms, we have

$$\delta_2(D)\sum_{m\in\mathcal{I}}|a_m|^2 \leqslant \int_D \left|\sum_{m\in\mathcal{I}} a_m e^{i\lambda_m\cdot x}\right|^2 \leqslant \delta_1(D)\sum_{m\in\mathcal{I}}|a_m|^2\,\mathrm{d}x. \tag{8.3.2}$$

With the above definition, Theorem 8.1.1 can be rephrased as follows: If Λ is a real sequence satisfying (8.1.1), then every interval of length strictly larger than $\frac{2\pi}{\gamma}$ is a domain associated with Λ.

Remark 8.3.2. By using Proposition 2.5.3 we see that the open bounded set $D \subset \mathbb{R}^n$ is a domain associated with the regular sequence Λ iff the family $\left(e^{i\lambda_k\cdot x}\right)_{k\in\mathcal{I}}$ is a Riesz basis in $L^2_\Lambda(D)$.

In order to give conditions ensuring that a domain is associated with a regular sequence Λ, we need some notation and a technical lemma. For every $\alpha > 0$ we denote by D_α, with $\alpha > 0$, the hypercube $D_\alpha = [-\alpha, \alpha]^n$.

Lemma 8.3.3. *Let $n \in \mathbb{N}$, $r > 0$, let χ_r be the characteristic function on the interval $[-r, r]$ and let $h_r = \frac{1}{4r^2}\chi_r * \chi_r$. Moreover, let $K_r \in L^1(\mathbb{R}^n)$ be defined by $K_r(x) = \prod_{m=1}^n h_r(x_m)$ and let $\widehat{K_r}$ be the Fourier transform of K_r. Then*

$$K_r(0) = \left(\frac{1}{2r}\right)^n, \tag{8.3.3}$$

$$K_r(x) = 0 \quad \text{if } x \notin D_{2r}, \tag{8.3.4}$$

$$\widehat{K}_r(0) = 1, \tag{8.3.5}$$

$$\widehat{K}_r(\xi) = \frac{1}{r^{2n}}\prod_{m=1}^n \frac{\sin^2(r\xi_m)}{\xi_m^2} \qquad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}. \tag{8.3.6}$$

Proof. By using the definition of h_r and some basic properties of the convolution, it follows that $h_r(0) = \frac{1}{2r}$ and $h_r(x) = 0$ if $|x| \geqslant 2r$. These facts and the definition of K_r clearly imply (8.3.3) and (8.3.4).

On the other hand, the Fourier transform of h_r is clearly given by

$$\widehat{h_r}(0) = 1 \quad \text{and} \quad \widehat{h_r}(\xi) = \frac{\sin^2(r\xi)}{r^2\xi^2} \qquad \forall\, \xi \neq 0\,.$$

These facts and the formula

$$\widehat{K_r}(\xi) = \prod_{m=1}^{n} \widehat{h_r}(\xi_m)$$

clearly imply (8.3.5) and (8.3.6). □

Remark 8.3.4. From (8.3.4) it easily follows that

$$K_r(x) = 0 \text{ if } |x| \geqslant 2r\sqrt{n}\,. \tag{8.3.7}$$

Proposition 8.3.5. *Let $(\mu_m)_{m\in\mathcal{I}}$ be a sequence of vectors in $\mathbb{R}^n$ satisfying*

$$\inf_{\substack{m,l\in\mathcal{I}\\ m\neq l}} |\mu_m - \mu_l| \geqslant \sqrt{n}\,. \tag{8.3.8}$$

Then there exists $\beta > 0$ such that the ball centered at the origin and of radius β is a domain associated with (μ_m).

Proof. Let $(a_m)_{m\in\mathcal{I}}$ be an l^2 sequence having a finite number of non-vanishing terms and set

$$f(x) = \sum_{m\in\mathcal{I}} a_m e^{i\mu_m\cdot x}\,.$$

Let $(K_r)_{r>0}$ be the functions introduced in Lemma 8.3.3. For every $r > 0$ we have

$$\int_{\mathbb{R}^n} K_r(x)|f(x)|^2\,\mathrm{d}x = \sum_{m\in\mathcal{I}} |a_m|^2 + \sum_{\substack{m,l\in\mathcal{I}\\ m\neq l}} a_m\overline{a_l}\widehat{K}_r(\mu_l - \mu_m)\,. \tag{8.3.9}$$

The last term on the right-hand side of the above relation satisfies

$$\begin{aligned}
&\left|\sum_{\substack{m,l\in\mathcal{I}\\ m\neq l}} a_m\overline{a_l}\widehat{K}_r(\mu_m - \mu_l)\right| \\
&\quad\leqslant \frac{1}{2}\left(\sum_{m\in\mathcal{I}} |a_m|^2 \sum_{l\neq m} |\widehat{K}_r(\mu_m - \mu_l)| + \sum_{l\in\mathcal{I}} |a_l|^2 \sum_{m\neq l} |\widehat{K}_r(\mu_m - \mu_l)|\right) \\
&\quad= \sum_{m\in\mathcal{I}} |a_m|^2 \sum_{l\neq m} |\widehat{K}_r(\mu_m - \mu_l)|\,.
\end{aligned} \tag{8.3.10}$$

From (8.3.8) it follows that for every $p \in \mathbb{Z}_+$ the number of terms of the sequence (μ_m) in $D_{p+1} \setminus D_p$ is bounded by $c_1 p^{n-1}$, where c_1 is a universal constant, and that $\mu_k - \mu_l \notin D_1$ if $k \neq l$. From these facts and the estimate (following from (8.3.6))

$$\widehat{K_r}(\xi) \leqslant \frac{1}{r^{2n} p^{2n}} \qquad \forall\, \xi \in D_{p+1} \setminus D_p\,,$$

it follows that for every fixed $m \in \mathcal{I}$ we have

$$\begin{aligned}\sum_{l \neq m} |\widehat{K}_r(\mu_m - \mu_l)| &= \sum_{p=1}^{\infty} \sum_{\mu_m - \mu_l \in D_{p+1} \setminus D_p} |\widehat{K}_r(\mu_m - \mu_l)| \\ &\leqslant c_1 \sum_{p=1}^{\infty} \frac{p^{n-1}}{r^{2n} p^{2n}} = \frac{c_1}{r^{2n}} \sum_{p=1}^{\infty} \frac{1}{p^{n+1}}\,.\end{aligned}$$

It follows that

$$\lim_{r \to \infty} \sum_{l \neq m} |\widehat{K}_r(\mu_m - \mu_l)| = 0\,,$$

so that, by using (8.3.10), it follows that for r_0 large enough we have

$$\left| \sum_{\substack{m,l \in \mathcal{I} \\ m \neq l}} a_m \overline{a_l} \widehat{K}_{r_0}(\mu_m - \mu_l) \right| \leqslant \frac{1}{2} \sum_{m \in \mathcal{I}} |a_m|^2\,. \tag{8.3.11}$$

Using (8.3.9) and (8.3.11) we obtain

$$\frac{1}{2} \sum_{m \in \mathcal{I}} |a_m|^2 \leqslant \int_{\mathbb{R}^n} K_{r_0}(x) |f(x)|^2 \,\mathrm{d}x\,.$$

The above estimate, combined with (8.3.3), (8.3.4) and with the fact, easy to check, that $K_r(x)$ is maximum for $x = 0$, implies that

$$\int_{D_{2r_0}} |f(x)|^2 \,\mathrm{d}x \geqslant \frac{1}{2^{n+1} r_0^n} \sum_{m \in \mathcal{I}} |a_m|^2\,. \tag{8.3.12}$$

Moreover, it is easy to check that $K_{r_0}(x) \geqslant \left(\frac{1}{r_0}\right)^n$ for $x \in D_{r_0}$, so that (8.3.9) and (8.3.11) yield that

$$\int_{D_{r_0}} |f(x)|^2 \,\mathrm{d}x \leqslant \frac{3 r_0^n}{2} \sum_{m \in \mathcal{I}} |a_m|^2\,.$$

By a change of variables, we see that the last inequality still holds if we replace D_{r_0} by any domain obtained from D_{r_0} by a translation. Since D_{2r_0} can be covered by three such hypercubes, it follows that

$$\int_{D_{2r_0}} |f(x)|^2 \,\mathrm{d}x \leqslant \frac{9 r_0^n}{2} \sum_{m \in \mathcal{I}} |a_m|^2\,.$$

The above estimate and (8.3.12) imply that D_{2r_0} is a domain associated with the sequence (μ_m). It follows that if $\beta > \frac{n\sqrt{2r_0}}{2}$, then the ball centered at the origin and of radius β is a domain associated with the sequence (μ_m). $\square$

Corollary 8.3.6. *Let $\Lambda = (\lambda_m)_{m\in\mathcal{I}}$ be a sequence satisfying (8.3.1). Then there exists an $\alpha > 0$ such that every ball in $\mathbb{R}^n$ of radius $\frac{\alpha}{\gamma}$ is a domain associated with Λ.*

Proof. Let $(\mu_m)_{m\in\mathcal{I}}$ be the sequence defined by

$$\mu_m = \frac{\sqrt{n}}{\gamma}\lambda_m \qquad \forall\, m \in \mathcal{I}.$$

The sequence (μ_m) satisfies (8.3.8) so that, by Proposition 8.3.5, there exist constants $\beta,\ \delta_1,\ \delta_2 > 0$ such that for every $(a_m)_{m\in\mathcal{I}}$ with a finite number of non-vanishing terms we have

$$\delta_2 \sum_{m\in\mathcal{I}} |a_m|^2 \leqslant \int_{|x|<\beta} \left| \sum_{m\in\mathcal{I}} a_m e^{i\mu_m\cdot x} \right|^2 \mathrm{d}x \leqslant \delta_1 \sum_{m\in\mathcal{I}} |a_m|^2 .$$

Since

$$\int_{|x|<\beta} \left| \sum_{m\in\mathcal{I}} a_m e^{i\mu_m\cdot x} \right|^2 \mathrm{d}x = \left(\frac{\gamma}{\sqrt{n}}\right)^n \int_{|x|<\frac{\beta\sqrt{n}}{\gamma}} \left| \sum_{m\in\mathcal{I}} a_m e^{i\lambda_m\cdot x} \right|^2 \mathrm{d}x ,$$

it follows that every ball in $\mathbb{R}^n$ of radius $\frac{\beta\sqrt{n}}{\gamma}$ is a domain associated with Λ. $\square$

Proposition 8.3.7. *Given an open bounded set D, a regular sequence Λ in $\mathbb{R}^n$ and a sequence $(a_l) \in l^2(\mathcal{I},\mathbb{C})$, the series $\sum_{m\in\mathcal{I}} a_m e^{i\lambda_m\cdot x}$ converges in $L^2(D)$. Let $F_\Lambda : l^2 \to L^2(D)$ be the linear map associating with a sequence $(a_m)_{m\in\mathcal{I}}$ the function f defined by*

$$f(x) = \sum_{m\in\mathcal{I}} a_m e^{i\lambda_m\cdot x} \qquad \forall\, x \in \mathbb{R}^n . \tag{8.3.13}$$

Then $F_\Lambda \in \mathcal{L}(l^2, L^2(D))$, $\|F_\Lambda\|_{\mathcal{L}(l^2,L^2(D))}$ depends only on γ and on D, and the adjoint of F_Λ is given by

$$F_\Lambda^*(\varphi) = (\widehat{\varphi}(\lambda_m))_{m\in\mathcal{I}} \qquad \forall\, \varphi \in L^2_\Lambda(D), \tag{8.3.14}$$

the Fourier transform $\widehat{\varphi}$ being computed after the extension of φ by zero outside D.

Proof. By compactness, $\operatorname{clos} D$ can be covered by a finite number of balls of radius α, where α is the constant from Corollary 8.3.6. It follows that there exists a constant δ_2, depending only on γ and on D, such that

$$\int_D \left| \sum_{m\in\mathcal{I}} a_m e^{i\lambda_m\cdot x} \right|^2 \mathrm{d}x \leqslant \delta_2 \sum_{m\in\mathcal{I}} |a_m|^2$$

for every sequence $(a_m)_{m\in\mathcal{I}}$ having a finite number of non-vanishing terms. From the above relation it follows that the series $\sum_{m\in\mathcal{I}} a_m e^{i\lambda_m\cdot x}$ converges in $L^2(D)$ to some function f and that

$$\int_D |f(x)|^2\,\mathrm{d}x \leqslant \delta_2 \sum_{m\in\mathcal{I}} |a_m|^2,$$

so that F_Λ is well defined. Moreover, F_Λ is bounded and its norm depends only on γ and on D.

Let $\varphi \in L^2(D)$ and also denote by φ its extension to $\mathbb{R}^n$ obtained by setting $\varphi \equiv 0$ outside D. Then

$$\begin{aligned}\langle F_\Lambda a, \varphi\rangle_{L^2(D)} &= \sum_{m\in\mathcal{I}} a_m \int_D e^{i\lambda_m\cdot x}\overline{\varphi(x)}\,\mathrm{d}x \\ &= \sum_{m\in\mathcal{I}} a_m \overline{\int_D e^{-i\lambda_m\cdot x}\varphi(x)\,\mathrm{d}x} = \sum_{m\in\mathcal{I}} a_m \overline{\widehat{\varphi}(\lambda_m)},\end{aligned}$$

which implies (8.3.14). □

Proposition 8.3.8. *The open set D is a domain associated with Λ if and only if for every l^2 sequence $(b_k)_{k\in\mathcal{I}}$ there exists a function $G \in L^2(\mathbb{R}^n)$ such that* $\operatorname{supp} G \subset D$ *and $\widehat{G}(\lambda_m) = b_m$ for every $m \in \mathcal{I}$.*

Proof. Assume that D is a domain associated with Λ and let $(\phi_k)_{k\in\mathcal{I}}$ be a Riesz basis in $L^2_\Lambda(D)$ which is biorthogonal to $(e^{i\lambda_k\cdot x})_{k\in\mathcal{I}}$ (see Definition 2.5.1 and the comments following it). According to Proposition 2.5.3, the series $\sum_{k\in\mathcal{I}} b_k\phi_k$ converges in $L^2_\Lambda(D)$. Denote

$$G = \sum_{k\in\mathcal{I}} b_k\phi_k\,,$$

and extend G by zero outside D. Then $G \in L^2(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ and for every $k \in \mathcal{I}$ we have

$$\widehat{G}(\lambda_m) = \sum_{k\in\mathcal{I}} b_k \int_{\mathbb{R}^n} \phi_k(x)e^{-i\lambda_m\cdot x}\,\mathrm{d}x.$$

By using the fact that $(\phi_k)_{k\in\mathcal{I}}$ is biorthogonal to $(e^{i\lambda_k\cdot x})_{k\in\mathcal{I}}$, we get that $\widehat{G}(\lambda_m) = b_m$ for every $m \in \mathcal{I}$ so that we have shown one of the claimed implications.

Conversely, assume that for every sequence $(b_k)_{k\in\mathcal{I}}$ there exists a function $G \in L^2(\mathbb{R}^n)$ such that $\operatorname{supp} G \subset D$ and $\widehat{G}(\lambda_m) = b_m$ for every $m \in \mathcal{I}$. This means that the map $F_\Lambda^* \in \mathcal{L}(L^2_\Lambda(D), l^2)$, which is the adjoint of the operator F_Λ from Proposition 8.3.7, is onto. This implies, according to Propositions 12.1.3 and 2.8.1, that F_Λ is bounded from below; i.e., (8.3.2) holds for some $\delta_1 > 0$. □

Proposition 8.3.9. *Assume that $(G_m)_{m\in\mathcal{I}}$ is a sequence in $L^2(\mathbb{R}^n)$ such that*

- $\operatorname{supp} G_m \subset D$ *for every $m \in \mathcal{I}$,*
- *There exists $M > 0$ such that $\left\|\widehat{G_m}\right\|_{L^\infty} \leqslant M$ for every $m \in \mathcal{I}$,*

- *for every* $l,\ m \in \mathcal{I}$ *we have* $\widehat{G_l}(\lambda_m) = \delta_{lm}$ *(the Kronecker symbol).*

Then any open set D' *such that* $\operatorname{clos} D \subset D'$ *is a domain associated with* Λ.

Proof. First we choose $\varepsilon \in (0, \gamma/2)$ small enough in order to have $\operatorname{clos} D + B(0, 2\varepsilon) \subset D'$ (here $B(0, r)$ denotes the open ball of radius r with center 0). Let $(K_r)_{r>0}$ be the functions introduced in Lemma 8.3.3. For $m \in \mathcal{I}$ we define $\rho_m(x) = e^{-i\lambda_m \cdot x} K_\varepsilon(x)$ so that $\operatorname{supp} \rho_m \subset B(0, 2\varepsilon)$ for every $m \in \mathcal{I}$.

Let $(b_m)_{m\in\mathcal{I}}$ be a sequence containing only a finite number of non-vanishing terms and define

$$G = \sum_{m\in\mathcal{I}} b_m G_m * \rho_m .$$

We clearly have that $\operatorname{supp} G \subset D'$ and

$$\widehat{G}(\xi) = \sum_{m\in\mathcal{I}} b_m \widehat{G_m}(\xi) \widehat{K_\varepsilon}(\xi - \lambda_m) \qquad \forall\, \xi \in \mathbb{R}^n, \tag{8.3.15}$$

so that

$$\widehat{G}(\lambda_l) = b_l \qquad \forall\, l \in \mathcal{I}. \tag{8.3.16}$$

On the other hand, by using Parseval's theorem and (8.3.15),

$$\int_{D'} |G(x)|^2 \,\mathrm{d}x \leqslant \frac{M^2}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\sum_{m\in\mathcal{I}} |b_m| \widehat{K_\varepsilon}(\xi - \lambda_m) \right)^2 \mathrm{d}\xi . \tag{8.3.17}$$

By using again Parseval's theorem and the fact that K_ε is even, we obtain that

$$\begin{aligned}
\int_{\mathbb{R}^n} \left| \sum_{m\in\mathcal{I}} b_m \widehat{K_\varepsilon}(\xi - \lambda_m) \right|^2 \mathrm{d}\xi &= (2\pi)^n \int_{\mathbb{R}^n} \left| \sum_{m\in\mathcal{I}} K_\varepsilon(x) |b_m| e^{-i\lambda_m \cdot x} \right|^2 \mathrm{d}x \\
&= (2\pi)^n \int_{\mathbb{R}^n} K_\varepsilon^2(x) \left| \sum_{m\in\mathcal{I}} |b_m| e^{-i\lambda_m \cdot x} \right|^2 \mathrm{d}x \\
&\leqslant (2\pi)^n \|K_\varepsilon\|_{L^\infty}^2 \int_{B(0,2\varepsilon)} \left| \sum_{m\in\mathcal{I}} |b_m| e^{i\lambda_m \cdot x} \right|^2 \mathrm{d}x .
\end{aligned}$$

The above relation, combined with (8.3.17) and with Proposition 8.3.7, yields that there exists a constant $\widetilde{M}$, independent of the finite sequence (b_k), such that

$$\int_{D'} |G(x)|^2 \,\mathrm{d}x \leqslant \widetilde{M}^2 \sum_{m\in\mathcal{I}} |b_m|^2 . \tag{8.3.18}$$

Thus, there exists $\widetilde{M} > 0$ such that for every finite sequence $(b_m)_{m\in\mathcal{I}}$ there exists a function $G \in L^2(D')$ satisfying (8.3.16) and (8.3.18). An easy approximation argument yields that for every $(b_k) \in l^2(\mathcal{I})$ there exists a function $G \in L^2(D')$ satisfying (8.3.16). The conclusion follows now from Proposition 8.3.8. $\square$

8.4 The results of Kahane and Beurling

In this section we give some extensions of Theorem 8.1.1 (Ingham's theorem) which have been obtained by Kahane and by Beurling. The results obtained in this section will be used in Section 8.5 to derive exact observability results for the Schrödinger and the Euler–Bernoulli equations in a rectangular domain. First we need some more results on domains associated with a regular sequence.

Proposition 8.4.1. *Let D be a domain associated with the sequence $\Lambda = (\lambda_m)_{m\in\mathcal{I}}$, let $\mu \in \mathbb{R}^n$ be such that*

$$\inf_{m\in\mathcal{I}} |\mu - \lambda_m| = d > 0 .$$

Let $D' \subset \mathbb{R}^n$ be an open bounded set such that $\overline{D} \subset D'$. Then the function $x \mapsto e^{i\mu\cdot x}$ does not belong to $L^2_\Lambda(D')$ and the distance in $L^2(D')$ from this function to $L^2_\Lambda(D')$ is larger than a constant depending only on Λ, D' and d.

Proof. Let $\mathcal{I} \subset \mathbb{Z}$, let $(a_m)_{m\in\mathcal{I}}$ be an l^2 sequence and let $f \in L^2_{loc}(\mathbb{R}^n)$ be the function defined by

$$f(x) = \sum_{m\in\mathcal{I}} a_m e^{i\lambda_m\cdot x} . \tag{8.4.1}$$

Consider the function $q \in L^2_{\rm loc}(\mathbb{R}^n)$ defined by

$$q(x) = e_\mu(x) - f(x), \tag{8.4.2}$$

where

$$e_\mu(x) = e^{i\mu\cdot x} \qquad \forall\, x \in \mathbb{R}^n . \tag{8.4.3}$$

Let $\alpha > 0$ be such that $D + B_\alpha \subset D'$, where B_α stands for the ball in $\mathbb{R}^n$ centered at the origin and of radius α. We denote by V_α the Lebesgue measure (the volume) of B_α and we set

$$r(x) = q(x) - \frac{1}{V_\alpha}\int_{B_\alpha} e^{-i\mu\cdot y} q(x+y)\,{\rm d}y . \tag{8.4.4}$$

A simple calculation shows that

$$r(x) = \sum_{m\in\mathcal{I}} b_m e^{i\lambda_m\cdot x} , \tag{8.4.5}$$

where

$$b_m = a_m \left(\frac{1}{V_\alpha}\int_{B_\alpha} e^{i(\lambda_m-\mu)\cdot x}\,{\rm d}x - 1\right) \qquad \forall\, m \in \mathcal{I} .$$

It is easy to check that there exists $c_1 = c_1(\alpha, d) > 0$ such that

$$\left|\frac{1}{V_\alpha}\int_{B_\alpha} e^{i(\lambda_m-\mu)\cdot x}\,{\rm d}x - 1\right| \geqslant c_1 \qquad \forall\, m \in \mathcal{I},$$

so that, using (8.4.5) combined with the fact that D is a sequence associated with Λ, we get that there exists $c_2 = c_2(\Lambda, D, D', d) > 0$ such that

$$\int_D |r(x)|^2 \, \mathrm{d}x \geqslant c_2 \sum_{m \in \mathcal{I}} |a_m|^2 . \tag{8.4.6}$$

On the other hand, from (8.4.4) it follows, by applying the Cauchy–Schwarz inequality, that

$$\begin{aligned}
\int_D |r(x)|^2 \, \mathrm{d}x &\leqslant 2 \int_D |q(x)|^2 \, \mathrm{d}x + \frac{2}{V_\alpha^2} \int_D \left| \int_{B_\alpha} e^{-i\mu \cdot y} q(x+y) \, \mathrm{d}y \right|^2 \mathrm{d}x \\
&\leqslant 2 \int_D |q(x)|^2 \, \mathrm{d}x + \frac{2}{V_\alpha} \int_D \int_{B_\alpha} |q(x+y)|^2 \, \mathrm{d}y \, \mathrm{d}x \\
&= 2 \int_D |q(x)|^2 \, \mathrm{d}x + \frac{2}{V_\alpha} \int_{B_\alpha} \int_D |q(x+y)|^2 \, \mathrm{d}x \, \mathrm{d}y \\
&= 2 \int_D |q(x)|^2 \, \mathrm{d}x + \frac{2}{V_\alpha} \int_{B_\alpha} \int_{D'} |q(x)|^2 \, \mathrm{d}x \, \mathrm{d}y = 4 \int_{D'} |q(x)|^2 \, \mathrm{d}x .
\end{aligned}$$

The above inequality, combined with (8.4.6), implies that

$$\int_{D'} |q(x)|^2 \, \mathrm{d}x \geqslant \frac{c_2}{4} \sum_{m \in \mathcal{I}} |a_m|^2 . \tag{8.4.7}$$

On the other hand, according to Proposition 8.3.7, there exists $c_3 = c_3(D', \Lambda, d)$ such that

$$\sum_{m \in \mathcal{I}} |a_m|^2 \geqslant c_3 \int_{D'} \left| \sum_{m \in \mathcal{I}} a_m e^{i\lambda_m \cdot x} \right|^2 \mathrm{d}x .$$

By combining the above estimate with (8.4.1)–(8.4.3) and with (8.4.7), we obtain that

$$\|e_\mu - f\|^2_{L^2(D')} \geqslant c_4 \|f\|^2_{L^2(D')} , \tag{8.4.8}$$

where $c_4 = \frac{c_2 c_3}{4}$ depends only on Λ, d, D and D'. For the remaining part of this proof we distinguish between two cases.

Case 1. Assume that $\|f\|_{L^2(D')} \geqslant \frac{\mathrm{Vol}(D')}{2}$, where $\mathrm{Vol}(D')$ stands for the volume of D'. This assumption and (8.4.8) imply that

$$\|e_\mu - f\|_{L^2(D')} \geqslant \frac{\mathrm{Vol}(D') \sqrt{c_4}}{2} .$$

Case 2. Assume that $\|f\|_{L^2(D')} \, \mathrm{d}x \leqslant \frac{\mathrm{Vol}(D')}{2}$. Then

$$\|e_\mu - f\|_{L^2(D')} \geqslant \|e_\mu\|_{L^2(D')} - \|f\|_{L^2(D')} \geqslant \frac{\mathrm{Vol}(D')}{2} .$$

Consequently, if we denote $c_5 = \min\left(\frac{\mathrm{Vol}(D')\sqrt{c_4}}{2}, \frac{\mathrm{Vol}(D')}{2}\right)$, we have that c_5 depends only on D, D', Λ and d and

$$\int_{D'} \left| e^{i\mu\cdot x} - \sum_{m\in\mathcal{I}} a_m e^{i\lambda_m\cdot x}\right|^2_{L^2(D')} \mathrm{d}x \geqslant c_5^2 > 0 \qquad \forall\, (a_m) \in l^2(\mathcal{I}, \mathbb{C}). \qquad \square$$

Corollary 8.4.2. *With the assumptions of Proposition* 8.4.1, *there exists a function* $H_\mu \in L^2(D')$ *such that*

$$\widehat{H_\mu}(\mu) = 1, \quad \widehat{H_\mu}(\lambda_m) = 0 \qquad \forall\, m \in \mathcal{I}, \quad \|H_\mu\|_{L^2(D')} \leqslant M,$$

with $\|H_\mu\|_{L^2(D')}$ *depending only on* Λ, D' *and* d.

Proof. Let P_Λ denote the orthogonal projector from $L^2(D')$ onto $L^2_\Lambda(D')$ and let e_μ be the $L^2(D')$-function $x \mapsto e^{i\mu\cdot x}$. Then $\|e_\mu - P_\Lambda e_\mu\|_{L^2(D')}$ is the distance from e_μ to $L^2_\Lambda(D')$ so that, by Proposition 8.4.1, we have $\|e_\mu - P_\Lambda e_\mu\| \geqslant \beta > 0$, with β depending only on Λ, D' and d. Denote $G_\mu = e_\mu - P_\Lambda e_\mu$. A simple calculation shows that $\widehat{G}_\mu(\mu) = \beta^2$. This implies that the function $H_\mu = \frac{1}{\beta^2} G_\mu$ satisfies $\widehat{H}_\mu(\mu) = 1$. Moreover, the fact that $H_\mu \perp L^2_\Lambda(D')$ implies that

$$\widehat{H_\mu}(\lambda_m) = 0 \qquad \forall\, m \in \mathcal{I}.$$

Moreover,

$$\|H_\mu\|_{L^2(D')} = \frac{1}{\beta},$$

so that $\|H_\mu\|_{L^2(D')}$ depends only on Λ, D' and d. $\square$

Theorem 8.4.3. *Let* Λ_1, Λ_2 *be two regular sequences in* $\mathbb{R}^n$, *with* $n \in \mathbb{N}$. *Assume that* $D_1 \subset \mathbb{R}^n$ (*respectively*, $D_2 \subset \mathbb{R}^n$) *is a domain associated with* Λ_1 (*respectively with* Λ_2) *and that the sequence* $\Lambda = \Lambda_1 \cup \Lambda_2$ *is regular. Then any open set* $D \subset \mathbb{R}^n$ *containing the closure of* $D_1 + D_2$ *is a domain associated with* Λ.

Proof. We first denote the sequence $\Lambda_1 \cup \Lambda_2$ by $(\lambda_k)_{k\in\mathcal{I}}$ and we set

$$\inf_{\substack{m,l\in\mathcal{I}\\ m\neq l}} |\lambda_m - \lambda_l| = d > 0.$$

Let D', D'' be domains containing the closure of $D_1 + D_2$ such that the closure of D'' is contained in D'. According to Proposition 8.3.9, the claimed result is established if we prove that for every $\mu \in \Lambda$ there exists $G_\mu \in L^2(\mathbb{R}^n)$ such that $\operatorname{supp} G_\mu \subset D''$, the sequence $\left\|\widehat{G_\mu}\right\|_{L^\infty(\mathbb{R}^n)}$ is bounded by a constant depending only on d, D and D'', $\widehat{G_\mu}(\mu) = 1$ and $\widehat{G_\mu}(\lambda) = 0$ for every $\lambda \in \Lambda \setminus \{\mu\}$.

Without loss of generality, we can assume that μ is a term of the sequence Λ_1. Since D_1 is a domain associated with the sequence Λ_1, we can apply Proposition

8.3.8 to get the existence of a function $G_{\mu,1} \in L^2(D_1)$, depending only on Λ_1 and on D_1 such that $\widehat{G_{\mu,1}}(\mu) = 1$, and $\widehat{G_{\mu,1}}(\lambda) = 0$ for every $\lambda \in \Lambda_1 \setminus \{\mu\}$. Moreover, after extending $G_{\mu,1}$ by zero outside D_1 an by using the Cauchy–Schwarz inequality, we get that $\|\widehat{G_{\mu,1}}\|_{L^\infty(\mathbb{R}^n)}$ is bounded by a constant depending only on d and D_1.

On the other hand, according to Corollary 8.4.2, there exists a function $G_{\mu,2} \in L^2(D'')$ such that

$$\widehat{G_{\mu,2}}(\mu) = 1\,, \quad \widehat{G_{\mu,2}}(\lambda) = 0 \qquad \forall\, \lambda \in \Lambda_2\,, \quad \|G_{\mu,2}\|_{L^2(D'')} \leqslant M\,,$$

where M is a constant depending only on $\Lambda_2,\ D''$ and d. The last inequality implies, by applying the Cauchy–Schwarz inequality, that

$$\|\widehat{G_{\mu,2}}\|_{L^\infty(\mathbb{R}^n)} \leqslant \widetilde{M}\,,$$

where M is a constant depending only on $\Lambda_2,\ D''$ and d. We have thus constructed a function $G_\mu = G_{\mu,1} * G_{\mu,2}$ satisfying the required conditions, which ends up our proof. □

One of the applications of Proposition 8.4.3 is the following generalization of Ingham's theorem (Theorem 8.1.1), due to Beurling.

Proposition 8.4.4. *Let $\mathcal{I} \in \{\mathbb{Z}, \mathbb{N}\}$ and let $(\lambda_m)_{m\in\mathcal{I}}$ be a regular increasing sequence of real numbers. Assume that there exist $p \in \mathbb{N}$ and $\gamma > 0$ such that*

$$|\lambda_{m+p} - \lambda_m| \geqslant p\gamma \qquad \forall\, m \in \mathcal{I}. \tag{8.4.9}$$

Then every interval of length strictly larger than $\frac{2\pi}{\gamma}$ is a domain associated with the sequence Λ.

Proof. For $l \in \{0, \ldots, p-1\}$ we denote by $\Lambda^l = (\lambda^l_m)_{m\in\mathcal{I}}$ the sequence defined by

$$\lambda^l_m = \lambda_{mp+l} \qquad \forall\, m \in \mathcal{I}\,.$$

We clearly have

$$\lambda^l_{m+1} - \lambda^l_m \geqslant p\gamma \qquad \forall\ \in \mathcal{I},\ l \in \{0, \ldots, p-1\}\,.$$

By applying Theorem 8.1.1 it follows that, for every $l \in \{0, \ldots, p-1\}$, any interval of length strictly larger than $\frac{2\pi}{p\gamma}$ is a domain associated with the sequence Λ^l. By applying iteratively Proposition 8.4.3, it follows that any interval of length strictly larger than $\frac{2\pi}{\gamma}$ is a domain associated with the sequence Λ. □

Theorem 8.4.5. *Let Λ be a regular sequence in $\mathbb{R}^n$. For $d > 0$ denote by $\omega(d)$ the upper limit when $|b| \to \infty$ of the number of terms of Λ contained in the ball of center b and of radius d. If $\omega(d) = o(d)$ when $d \to \infty$, then every ball in $\mathbb{R}^n$ of strictly positive radius is a domain associated with Λ.*

Proof. For an arbitrary $d > 0$ we consider all the hypercubes in $\mathbb{R}^n$ with edges of length d and with summits having all the coordinates multiples of d. This family of hypercubes can be divided into 2^n subfamilies such that the distance between two hypercubes from the same family is larger than d. On the other hand, all but a finite number of these hypercubes contain at most $\omega(nd)$ points. Consequently, for every $d > 0$, the sequence Λ can be seen as the union of a finite sequence and of $\widetilde{\omega}(d) = 2^n\omega(nd)$ sequences Λ_j such that

$$\inf_{\substack{\lambda,\mu\in\Lambda_j \\ \lambda\neq\mu}} |\lambda - \mu| \geqslant d.$$

From Corollary 8.3.6 it follows that there exists $\alpha > 0$ such that, for every $j \in \{1, \ldots, \widetilde{\omega}(d)\}$, any ball of radius $> \frac{\alpha}{d}$ is a domain associated with the sequence Λ_j. By applying Theorem 8.4.3 it follows that any ball of radius $> \frac{\alpha\widetilde{\omega}(d)}{d}$ is a domain associated with Λ. Since $\frac{\alpha\widetilde{\omega}(d)}{d} = o(d)$ when $d \to \infty$ it follows that any ball of strictly positive radius is a domain associated with Λ. □

8.5 The Schrödinger and plate equations in a rectangular domain with distributed observation

We have seen in Section 7.5 that if Ω is a bounded domain with $\partial\Omega$ of class C^2 or if Ω is a rectangular domain, then the Schrödinger and the plate equations in Ω with distributed observation define an exactly observable system provided that the observation region satisfies a geometric condition. In this section we show that if Ω is a rectangular domain, then the above-mentioned systems are exactly observable for any observation region.

Notation. Let $a,\ b > 0$ and denote $\Omega = [0, a] \times [0, b]$. We use some of the notation in Section 7.5. More precisely, set $H = L^2(\Omega)$ and $\mathcal{D}(A_0) = H_1$ is the Sobolev space $\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$. The strictly positive operator $A_0 : \mathcal{D}(A_0) \to H$ is defined by $A_0\varphi = -\Delta\varphi$ for all $\varphi \in \mathcal{D}(A_0)$ and we denote $H_2 = \mathcal{D}(A_0^2)$. The inner product on H is denoted by $\langle\cdot,\cdot\rangle$ and the corresponding norm by $\|\cdot\|$. $\mathcal{O}$ is a non-empty open subset of Ω, and we introduce the observation operator $C_0 \in \mathcal{L}(H)$ by

$$C_0 g = g\chi_{\mathcal{O}} \qquad \forall\, g \in H, \tag{8.5.1}$$

where $\chi_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$.

We denote by $\mathcal{X}$ the Hilbert space $H_1 \times H$, with the scalar product

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_{\mathcal{X}} = \langle A_0 f_1, A_0 f_2\rangle + \langle g_1, g_2\rangle .$$

We define a dense subspace of $\mathcal{X}$ by $\mathcal{D}(\mathcal{A}) = H_2 \times H_1$ and the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ is defined by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0^2 & 0 \end{bmatrix}, \quad \text{i.e.,} \quad \mathcal{A}\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -A_0^2 f \end{bmatrix}, \tag{8.5.2}$$

which generates a unitary group on $\mathcal{X}$. We denote by $\mathcal{X}_1$ the space $\mathcal{D}(\mathcal{A})$ endowed with the graph norm and we introduce the observation operator $\mathcal{C} \in \mathcal{L}(\mathcal{X}_1, H)$ by

$$\mathcal{C} = \begin{bmatrix} 0 & C_0 \end{bmatrix}.$$

The main result of this section is the following.

Theorem 8.5.1. *With the above notation, the pairs (iA_0, C_0) and $(\mathcal{A}, \mathcal{C})$ are exactly observable in any time $\tau > 0$.*

Remark 8.5.2. For the Schrödinger equation, the result in Theorem 8.5.1 means that for every $\tau > 0$ there exists $k_\tau > 0$ such that the solution z of

$$\frac{\partial z}{\partial t}(x,t) = i\Delta z(x,t) \qquad \forall\, (x,t) \in \Omega \times [0,\infty),$$

with

$$z(x,t) = 0 \qquad \forall\, (x,t) \in \partial\Omega \times [0,\infty),$$

and $z(\cdot,0) = z_0 \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$, satisfies

$$\int_0^\tau \int_{\mathcal{O}} |z(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \geqslant k_\tau^2 \|z_0\|^2 \qquad \forall\, z_0 \in L^2(\Omega).$$

For the plate equation, the result in Theorem 8.5.1 means that for every $\tau > 0$ there exists $k_\tau > 0$ such that the solution w of the Euler–Bernoulli plate equation

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \Delta^2 w(x,t) = 0, \quad (x,t) \in \Omega \times [0,\infty),$$

with

$$w_{|\partial\Omega\times[0,\infty)} = \Delta w_{|\partial\Omega\times[0,\infty)} = 0,$$

and $w(\cdot,0) = w_0 \in \mathcal{D}(A_0^2)$, $\frac{\partial w}{\partial t}(\cdot,0) = w_1 \in \mathcal{D}(A_0)$, satisfies

$$\int_0^\tau \int_{\mathcal{O}} \left|\frac{\partial w}{\partial t}\right|^2 \mathrm{d}x\,\mathrm{d}t \geqslant k_\tau^2 \left(\|w_0\|^2_{\mathcal{H}^2(\Omega)} + \|w_1\|^2_{L^2(\Omega)}\right) \qquad \forall \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in \mathcal{D}(\mathcal{A}).$$

The main ingredient of the proof of Theorem 8.5.1 is the following proposition.

Proposition 8.5.3. *Let $r,\ s > 0$ and let $\Lambda \in l^2(\mathbb{Z}^2, \mathbb{R}^3)$ be defined by*

$$\lambda_{mn} = \begin{bmatrix} m\sqrt{r} \\ n\sqrt{s} \\ rm^2 + sn^2 \end{bmatrix} \qquad \forall\, m,n \in \mathbb{Z}. \tag{8.5.3}$$

Then any ball of strictly positive radius in $\mathbb{R}^3$ is a domain associated with Λ.

In order to prove Proposition 8.5.3 we need some notation and a lemma. For $R > 0$ and $\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right] \in \mathbb{Z}^2 \setminus \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ with $|k| < R$ and $|l| < R$, we denote

$$S_{R,k,l} = \left\{ \begin{bmatrix} m \\ n \end{bmatrix} \in \mathbb{Z}^2 \ \mid \ |2rkm + 2sln| < 3R^2 \right\},$$

and we introduce the subsequence $\Lambda_{R,k,l} = (\lambda_{mn})_{\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] \in S_{R,k,l}}$ of Λ.

Lemma 8.5.4. *With the above notation, any ball in $\mathbb{R}^3$ of strictly positive radius is a domain associated with $\Lambda_{R,k,l}$.*

Proof. Without loss of generality, we can assume that $k \neq 0$. Then the condition $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] \in S_{R,k,l}$ implies that there exists a constant $c > 0$ (depending on $r,\ s,\ R,\ l$ and k) such that

$$rm^2 + sn^2 = cn^2 + O(n) \qquad \forall \begin{bmatrix} m \\ n \end{bmatrix} \in S_{R,k,l}\,. \tag{8.5.4}$$

The above formula implies that the number of terms of $\Lambda_{R,k,l}$ contained in a ball of center $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$ and of radius $d > 0$ is bounded by the number of terms of the sequence $(rm^2 + sn^2)_{\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] \in S_{R,k,l}}$ in $(b_3 - d, b_3 + d)$. Relation (8.5.4) implies that, after possibly eliminating a finite number of terms, the sequence $\Lambda_{R,k,l}$ can be rewritten as a strictly increasing sequence $(\alpha_n)_{n \geqslant 1}$ satisfying

$$\alpha_{n+p} - \alpha_n \geqslant c(2np + p^2) + O(n)\,.$$

By choosing p large enough it follows that

$$\alpha_{n+p} - \alpha_n \geqslant np\,.$$

Consequently, the number of terms of $\Lambda_{R,k,l}$ contained in a ball of center b and of radius d is smaller than $c\left(\sqrt{|b_3| + d} - \sqrt{|b_3| - d}\right) + 1$, which tends to 1 when $b_3 \to \infty$. The conclusion follows now by applying Theorem 8.4.5. $\square$

Proof of Proposition 8.5.3. Let $\beta > 0$. It is easy to see that the assertion saying that any ball of strictly positive radius is a domain associated with Λ is equivalent to the assertion saying that any ball of strictly positive radius is a domain associated with $\beta\Lambda$. Therefore it suffices to tackle the case in which $r,\ s$ from (8.5.3) satisfy $r,\ s \in (0, 1]$.

Let $\varepsilon > 0$, $R > \max(1, 2\alpha/\varepsilon)$, where α is the constant in Corollary 8.3.6, and let $\mathcal{I}_R$ be the union of all the strips $S_{R,k,l}$ with $k^2 + l^2 \neq 0$, $|k| \leqslant R$ and $|l| \leqslant R$ (there are at most $(2R + 1)^2$ such strips). Denote $\Lambda_1 = (\lambda_{mn})_{\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] \in \mathcal{I}_R}$. Then

$$\Lambda_1 = \bigcup_{\substack{k,l \in [-R,R] \\ k^2 + l^2 \neq 0}} \Lambda_{R,k,l}\,,$$

so that, by combining Theorem 8.4.3 and Lemma 8.5.4 it follows that any ball in $\mathbb{R}^3$ of strictly positive radius is a domain associated with Λ_1.

Let $\mathcal{J}_R = \mathbb{Z}^2 \setminus \mathcal{I}_R$ and let $\Lambda_2 = (\lambda_{mn})_{\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] \in \mathcal{J}_R}$, so that $\Lambda = \Lambda_1 \cup \Lambda_2$. If we admit that

$$\inf_{\substack{\lambda,\mu \in \Lambda_2 \\ \lambda \neq \mu}} |\lambda - \mu| \geqslant R, \tag{8.5.5}$$

then, by Corollary 8.3.6, we have that any ball of radius $\varepsilon/2$ is a domain associated with Λ_2 so that, by applying Theorem 8.4.3, we obtain that any ball of radius ε is a domain associated with Λ.

We still have to show (8.5.5). Let $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} m' \\ n' \end{smallmatrix}\right] \in \mathcal{J}_R$ with $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} m' \\ n' \end{smallmatrix}\right]$. If $|m - m'| \geqslant R$ or $|n - n'| \geqslant R$, then (8.5.5) clearly holds. If $|m - m'| < R$ and $|n - n'| < R$, then there exist $k,\ l \in [-R, R] \cap \mathbb{Z}$ with $k^2 + l^2 \neq 0$ such that

$$m' = m + k \quad n' = n + l .$$

Then, by using the facts that $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] \notin \mathcal{I}_R$, $r,\ s \in (0, 1]$ and $R > 1$, it follows that

$$\begin{aligned} \left| rm^2 + sn^2 - rm'^2 - sn'^2 \right| &= |2rmk + 2snl + rm^2 + sn^2| \\ &\geqslant |2rmk + 2snl| - |rk^2 + sl^2| > 3R^2 - rR^2 - sR^2 \geqslant R^2 \geqslant R, \end{aligned}$$

which implies (8.5.5). □

Proof of Theorem 8.5.1. We have seen in Example 3.6.5 that the eigenvalues of A_0 are

$$\mu_{mn} = rm^2 + sn^2 \qquad \forall\, m, n \in \mathbb{N},$$

where $r = \frac{\pi^2}{a^2}$ and $s = \frac{\pi^2}{b^2}$ and that a corresponding orthonormal basis formed of eigenvectors of A_0 is given by

$$\varphi_{mn}(x, y) = \frac{2}{\sqrt{ab}} \sin(\sqrt{r}\, mx) \sin(\sqrt{s}\, ny) \qquad \forall\, m, n \in \mathbb{N}.$$

The above facts imply that the semigroup $\mathbb{T}$ generated by iA_0 satisfies

$$\mathbb{T}_t z = \sum_{m,n \in \mathbb{N}} z_{mn} e^{i(rm^2 + sn^2)t} \varphi_{mn} \qquad \forall\, z \in \mathcal{D}(A_0),$$

where we have denoted

$$z_{mn} = \langle z, \varphi_{mn} \rangle \qquad \forall\, m, n \in \mathbb{N}.$$

Let $\tau > 0$. From the definition (8.5.1) of C_0 it follows that

$$\begin{aligned} \int_0^\tau \|C\mathbb{T}_t z\|^2 \mathrm{d}t &= \int_0^\tau \int_{\mathcal{O}} \left| \sum_{m,n \in \mathbb{N}} z_{mn} e^{i(rm^2 + sn^2)t} \varphi_{mn}(x, y) \right|^2 \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t \qquad (8.5.6) \\ &= \frac{4}{ab} \int_0^\tau \int_{\mathcal{O}} \left| \sum_{m,n \in \mathbb{N}} z_{mn} e^{i(rm^2 + sn^2)t} \sin(\sqrt{r}\, mx) \sin(\sqrt{s}\, ny) \right|^2 \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t . \end{aligned}$$

We now extend $(z_{mn})_{m,n\in\mathbb{N}}$ to a sequence denoted $(z_{mn})_{m,n\in\mathbb{Z}^*}$ by setting

$$z_{-m,n} = -z_{mn}, \; z_{m,-n} = -z_{mn}, z_{-m,-n} = z_{mn} \qquad \forall\, m,n \in \mathbb{N}.$$

With the above relation notation, formula (8.5.6) can be easily put in the form

$$\begin{aligned}\int_0^\tau \|C\mathbb{T}_t z\|^2 \,\mathrm{d}t &= \frac{1}{iab}\int_0^\tau \int_{\mathcal{O}} \left| \sum_{m,n\in\mathbb{Z}^*} z_{mn} e^{i(rm^2+sn^2)t} e^{i(\sqrt{r}\,mx+\sqrt{s}\,ny)} \right|^2 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &= \frac{1}{iab}\int_0^\tau \int_{\mathcal{O}} \left| \sum_{m,n\in\mathbb{Z}^*} z_{mn} e^{i\lambda_{mn}\cdot \left[\begin{smallmatrix} x\\ y\\ t\end{smallmatrix}\right]} \right|^2 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t,\end{aligned}$$

where $(\lambda_{mn})_{m,n\in\mathbb{Z}}$ is the sequence of vectors defined in (8.5.3). By applying Proposition 8.5.3 it follows that there exists a constant $c > 0$ (depending only on $\mathcal{O}$ and on τ) such that

$$\int_0^\tau \|C\mathbb{T}_t z\|^2 \,\mathrm{d}t \geqslant c^2 \sum_{m,n\in\mathbb{Z}} |z_{mn}|^2,$$

so that the pair (iA_0, C_0) is exactly observable in any time $\tau > 0$.

On the other hand, by using the fact that $(rm^2 + sn^2)^2 > rs\, m^2 n^2$ for all $m, n \in \mathbb{N}$, it follows that $\sum_{m,n\in\mathbb{N}} \mu_{mn}^{-2} < \infty$. This fact and the exact observability in any time of (iA_0, C_0) imply, by using Proposition 6.8.2, that the pair $(\mathcal{A}, \mathcal{C})$ is exactly observable in any time $\tau > 0$. □

8.6 Remarks and bibliographical notes on Chapter 8

General remarks. The fact that a bounded interval J is a domain associated with the real sequence $(\lambda_n)_{n\in\mathbb{N}}$ (in the sense of Definition 8.3.1) is equivalent to the fact that, for any sequence $(c_n)_{n\in\mathbb{N}}$, the moment problem

$$\int_J f(t) e^{-i\lambda_n t}\,\mathrm{d}t = c_n \qquad \forall\, n \in \mathbb{N} \tag{8.6.1}$$

admits at least one solution $f \in L^2(J)$. We refer the reader to [240, p. 151] for the proof of this equivalence. Therefore the inequalities of Ingham and of Beurling from Theorem 8.1.1 and Proposition 8.4.4 can be interpreted as giving conditions for the sequence (λ_n) guaranteeing the solvability of the moment problem (8.6.1). Consequently, the exact observability of the systems considered in this chapter can be reduced to moment problems of the form (8.6.1). This equivalence has been used in the pioneering papers of Fattorini and Russell [63, 62] and of Russell [199, 197] for systems governed by hyperbolic or by parabolic PDEs in one space dimension. The method of moments has then been developed and systematically applied to systems governed by PDEs in the book of Avdonin and Ivanov [9]. The direct use of Ingham-type inequalities in exact observability problems has been initiated by Haraux in [92, 94]. The book of Komornik and Loreti [131] gives the state of the art on this method.

An interesting subject which is not tackled in this book consists in giving precise estimates of the constants involved in Ingham–Beurling-type inequalities in function of the distribution of the frequencies and of the length of the interval. We refer the reader to Seidman [206], Seidman, Avdonin and Ivanov [207], Miller [170] and Tenenbaum and Tucsnak [218] for results in this direction.

Section 8.1. Our proof of Ingham's theorem is essentially the same as one of the original proofs in Ingham [108]. Note that [108] contains two other proofs (based on different choices of the kernel k) which are also very interesting.

Section 8.2. The results here are essentially contained in [197]. The multiplier method used in the proof of Lemma 8.2.1 is inspired from Lagnese [136].

Sections 8.3 and 8.4. The presentation follows closely Kahane [126]. The proofs of Propositions 8.4.1 and 8.4.4 are borrowed from [131]. Note that, based on ideas of the original proof in Beurling [18], the recent paper of Tenenbaum and Tucsnak [219] provides more information on the constants involved in Proposition 8.4.4.

Section 8.5. The presentation follows closely Jaffard [123]. The main result has been generalized to several space dimensions in Komornik [129]. The corresponding boundary observability problem is more delicate. We refer the reader to Ramdani et al. [186] and to [219] for results in this direction. Note that the exact observability for the Schrödinger equation with an arbitrary observation region fails if the considered domain is a disk in $\mathbb{R}^2$ (see Chen et al. [34]). For more complicated examples of exact observability for the Schrödinger equation without the geometric optics condition we refer the reader to Burq and Zworski [27].

Chapter 9

Observability for Parabolic Equations

9.1 Preliminary results

In this section and the following one, we shall use the notation from Section 3.4: H is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. The operator $A_0 : \mathcal{D}(A_0) \to H$ is assumed to be strictly positive. The space $\mathcal{D}(A_0)$ endowed with the norm $\|z\|_1 = \|A_0 z\|$ is denoted by H_1 and $H_{\frac{1}{2}}$ is the completion of $\mathcal{D}(A_0)$ with respect to the norm

$$\|w\|_{\frac{1}{2}} = \sqrt{\langle A_0 w, w \rangle},$$

so that $H_{\frac{1}{2}}$ coincides with $\mathcal{D}(A_0^{\frac{1}{2}})$ with the norm $\|w\|_{\frac{1}{2}} = \|A_0^{\frac{1}{2}} w\|$. We have seen in Proposition 3.8.5 that $-A_0$ generates an exponentially stable semigroup $\mathbb{S}$ on H.

We assume that A_0^{-1} is compact so that, according to Proposition 3.2.12, there exists an orthonormal basis $(\varphi_k)_{k \in \mathbb{N}}$ in H consisting of eigenvectors of A_0. We denote by $(\lambda_k)_{k \in \mathbb{N}}$ the corresponding sequence of strictly positive eigenvalues of A_0. We know from Proposition 3.2.12 that $\lim_{k \to \infty} \lambda_k = \infty$.

Let Y be a Hilbert space and assume that $C_0 \in \mathcal{L}(H_{\frac{1}{2}}, Y)$. Recall from Proposition 5.1.3 that C_0 is an admissible observation operator for $\mathbb{S}$. In this section we give some preliminary results concerning the observability properties of the pair $(-A_0, C_0)$.

Proposition 9.1.1. *Assume that $C_0 \in \mathcal{L}(H_{\frac{1}{2}}, Y)$ is compact. Then the pair $(-A_0, C_0)$ is not exactly observable.*

Proof. We denote by Ψ the extended output map of $(-A_0, C)$; see (4.3.6). Since $\mathbb{S}$ is exponentially stable, according to Remark 4.3.5, we have $\Psi \in \mathcal{L}(H, L^2([0,\infty);Y))$. We compute

$$\|\Psi \varphi_n\|^2 = \int_0^\infty \|e^{-\lambda_n t} C_0 \varphi_n\|^2 \, \mathrm{d}t = \frac{1}{2\lambda_n} \|C_0 \varphi_n\|^2 .$$

Define $\widetilde{C}_0 = C_0 A_0^{-\frac{1}{2}}$, so that $\widetilde{C}_0 \in \mathcal{L}(H, Y)$ is compact. Then our earlier computation shows that

$$\|\Psi\varphi_n\| = \frac{1}{\sqrt{2\lambda_n}} \left\|\widetilde{C}_0(\sqrt{\lambda_n}\varphi_n)\right\| = \frac{1}{\sqrt{2}}\|\widetilde{C}_0\varphi_n\|.$$

The sequence (φ_n) is weakly convergent to zero in H. Since $\widetilde{C}_0$ is compact, it follows that $\widetilde{C}_0\varphi_n \to 0$ in Y; see Proposition 12.2.5 in Appendix I. We have shown that $\Psi\varphi_n \to 0$, which implies our claim. □

Remark 9.1.2. Since the embedding $H_{\frac{1}{2}} \subset H$ is compact, the above result holds, in particular, for every $C_0 \in \mathcal{L}(H, Y)$.

Example 9.1.3. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with C^2 boundary and put $H = \mathcal{H}_0^1(\Omega)$. Let $-A_0$ be the Dirichlet Laplacian on Ω, as defined in Section 3.6, but restricted such that it is a densely defined strictly positive operator on H. Then using Theorem 3.6.2 we can show that $H_{\frac{1}{2}} = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$. Let $Y = L^2(\partial\Omega)$ and let $C_0 \in \mathcal{L}(H_{\frac{1}{2}}, Y)$ be the Neumann trace operator: $C_0 f = \frac{\partial f}{\partial\nu}$. According to Corollary 13.6.8, C_0 is compact. Thus, according to the last proposition, $(-A_0, C_0)$ is not exactly observable. In terms of PDEs this means that if z is the solution of the heat equation

$$\begin{aligned}
\frac{\partial z}{\partial t}(x,t) &= \Delta z(x,t), & x &\in \Omega, & t &\geqslant 0,\\
z(x,t) &= 0, & x &\in \partial\Omega, & t &\geqslant 0,\\
z(x,0) &= z_0(x), & x &\in \Omega,
\end{aligned}$$

where $z_0 \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$, then, denoting by $\|\cdot\|$ the norm on $\mathcal{H}_0^1(\Omega)$,

$$\inf_{\|z_0\|=1} \int_0^\tau \int_{\partial\Omega} \left|\frac{\partial z}{\partial\nu}\right|^2 \mathrm{d}\sigma\,\mathrm{d}t = 0 \qquad \forall\, \tau > 0.$$

The last proposition may explain why exact observability rarely holds for semigroups generated by negative operators. For this reason, we shall concentrate on final state observability, as defined in Section 6.1.

We give a quite technical sufficient condition for final state observability. This condition uses the concept of a biorthogonal sequence, as defined Section 2.5.

Lemma 9.1.4. *Let $\tau > 0$ and assume that there exists a family $(G_n)_{n\in\mathbb{N}}$ which is biorthogonal, in $L^2([0,\tau];Y)$, to the family $\left(e^{-\lambda_k t}C_0\varphi_k\right)_{k\in\mathbb{N}}$. Moreover, assume that*

$$\sum_{n\in\mathbb{N}} e^{-2\tau\lambda_n}\|G_n\|^2_{L^2([0,\tau],Y)} = M^2 < \infty. \tag{9.1.1}$$

Then the pair $(-A_0, C_0)$ is final state observable in time τ.

Proof. For $z_0 \in H$ and $t \in [0,\tau]$ we set $H(t) = \sum_{n\in\mathbb{N}} e^{-2\lambda_n \tau}\langle z_0, \varphi_n\rangle G_n(t)$. Then, by using the Cauchy–Schwarz inequality in $L^2([0,\tau], Y)$, it follows that

$$\begin{aligned}\int_0^\tau \|H(t)\|_Y^2\,\mathrm{d}t &\leqslant \sum_{k,n\in\mathbb{N}} \langle z_0, \varphi_n\rangle \overline{\langle z_0, \varphi_k\rangle} e^{-2\tau(\lambda_n+\lambda_k)} \|G_n\|_{L^2([0,\tau],Y)} \|G_k\|_{L^2([0,\tau],Y)} \\ &= \left| \sum_{n\in\mathbb{N}} \langle z_0, \varphi_n\rangle \|G_n\|_{L^2([0,\tau],Y)} e^{-2\tau\lambda_n} \right|^2 .\end{aligned}$$

Using the Cauchy–Schwarz inequality in l^2 and (9.1.1), this becomes

$$\begin{aligned}\int_0^\tau \|H(t)\|_Y^2\,\mathrm{d}t &\leqslant \left(\sum_{n\in\mathbb{N}} e^{-2\tau\lambda_n} |\langle z_0, \varphi_n\rangle|^2\right)\left(\sum_{n\in\mathbb{N}} e^{-2\tau\lambda_n} \|G_n\|^2_{L^2([0,\tau],Y)}\right) \\ &= M^2 \left(\sum_{n\in\mathbb{N}} e^{-2\tau\lambda_n} |\langle z_0, \varphi_n\rangle|^2\right) . \qquad (9.1.2)\end{aligned}$$

On the other hand, due to the assumed biorthogonality,

$$\begin{aligned}\int_0^\tau \langle C_0 \mathbb{S}_t z_0, H(t)\rangle_Y\,\mathrm{d}t &= \sum_{k,n\in\mathbb{N}} \langle z_0, \varphi_k\rangle \overline{\langle z_0, \varphi_n\rangle} e^{-2\lambda_n\tau} \int_0^\tau \langle e^{-\lambda_k t} C_0 \varphi_k, G_n(t)\rangle_Y\,\mathrm{d}t \\ &= \sum_{n\in\mathbb{N}} |\langle z_0, \varphi_n\rangle|^2 e^{-2\lambda_n\tau}.\end{aligned}$$

Using the above formula together with (9.1.2) and the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}\|\mathbb{S}_\tau z_0\|^2 = \sum_{n\in\mathbb{N}} |\langle z_0, \varphi_n\rangle|^2 e^{-2\lambda_n\tau} &= \int_0^\tau \langle C_0 \mathbb{S}_t z_0, H(t)\rangle_Y\,\mathrm{d}t \\ &\leqslant M\ \|C_0 \mathbb{S}_t z_0\|_{L^2([0,\tau],Y)} \sqrt{\sum_{n\in\mathbb{N}} |\langle z_0, \varphi_n\rangle|^2 e^{-2\lambda_n\tau}} \\ &= M\ \|C_0 \mathbb{S}_t z_0\|_{L^2([0,\tau],Y)} \|\mathbb{S}_\tau z_0\| ,\end{aligned}$$

which implies the conclusion of the lemma. $\square$

9.2 From $\ddot{w} = -A_0 w$ to $\dot{z} = -A_0 z$

We continue to use the notation introduced in the previous section.

Our aim is to show that if a system is described by the second-order equation $\ddot{w} = -A_0 w$ and by $y = C_0 \dot{w}$ (y being the output signal) and if this system is exactly observable, then the system described by the first-order equation $\dot{z} = -A_0 z$, with $y = C_0 z$, is final state observable. Such results imply the final state observability of systems governed by the heat or related parabolic equations, in arbitrarily small time, by using results available for systems governed by the wave equation.

We shall also use some notation from Section 6.7. More precisely, we set $X = H_{\frac{1}{2}} \times H$, which is a Hilbert space with the scalar product

$$\left\langle \begin{bmatrix} w_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ v_2 \end{bmatrix} \right\rangle_X = \langle A_0^{\frac{1}{2}} w_1, A_0^{\frac{1}{2}} w_2 \rangle + \langle v_1, v_2 \rangle,$$

we define a dense subspace of X by $\mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}$ and we consider the skew-adjoint operator $A : \mathcal{D}(A) \to X$ defined by

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}. \tag{9.2.1}$$

We denote by $\mathbb{T}$ the unitary group generated by A on X and we let $C \in \mathcal{L}(H_1 \times H_{\frac{1}{2}}, Y)$ be defined by

$$C = \begin{bmatrix} 0 & C_0 \end{bmatrix}. \tag{9.2.2}$$

For $k \in \mathbb{N}$ we set $\mu_k = \sqrt{\lambda_k}$, $\varphi_{-k} = -\varphi_k$ and $\mu_{-k} = -\mu_k$. With the above assumptions and notation, we know from Proposition 3.7.7 that A is diagonalizable, with the eigenvalues $(i\mu_k)_{k\in\mathbb{Z}^*}$ corresponding to the orthonormal basis of eigenvectors

$$\phi_k = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{i\mu_k}\varphi_k \\ \varphi_k \end{bmatrix} \qquad \forall\, k \in \mathbb{Z}^*. \tag{9.2.3}$$

In order to show that the exact observability of (A, C) implies the final state observability of $(-A_0, C_0)$, we give a necessary condition for the exact observability of (A, C), which will be combined with the sufficient condition for the final state observability of the pair $(-A_0, C_0)$ given in Lemma 9.1.4.

Lemma 9.2.1. *Assume that the pair (A, C) is exactly observable in time τ_0. Then there exists a bounded sequence $(F_n)_{n\in\mathbb{Z}^*}$ in $L^2([0, \tau_0]; Y)$ such that $(F_n)_{n\in\mathbb{Z}^*}$ is biorthogonal, in $L^2([0, \tau_0]; Y)$, to the sequence $\left(e^{i\mu_k t} C_0 \varphi_k\right)_{k\in\mathbb{Z}^*}$.*

Proof. Let $\Psi_{\tau_0} \in \mathcal{L}(X, L^2([0, \infty); Y))$ be the output operator associated with (A, C), which has been introduced in (4.3.1). By definition, the exact observability in time τ_0 of (A, C) means that there exists $m > 0$ such that $\|\Psi_{\tau_0} z_0\| \geqslant m\|z_0\|$ for every $z_0 \in X$. It is easy to check that

$$(\Psi_{\tau_0} z_0)(t) = \sum_{n\in\mathbb{Z}^*} \langle z_0, \phi_n \rangle e^{i\mu_n t} C\phi_n \qquad \forall\, z_0 \in \mathcal{D}(A), \quad \forall t \in [0, \tau_0].$$

The above formula and the exact observability of (A, C) implies that the sequence $\left(e^{i\mu_k t} C\phi_k\right)_{k\in\mathbb{Z}^*}$ satisfies the left inequality in (2.5.5). The right inequality in (2.5.5) holds due to the admissibility of C. According to Proposition 2.5.3, $\left(e^{i\mu_k t} C\phi_k\right)_{k\in\mathbb{Z}^*}$ is a Riesz basis in its closed linear span in $L^2([0, \tau_0]; Y)$. By Definition 2.5.1 the family $\left(e^{i\mu_k t} C\phi_k\right)_{k\in\mathbb{Z}^*}$ admits a bounded biorthogonal family $(\tilde{F}_n)_{n\in\mathbb{Z}^*}$ in $L^2([0, \tau_0]; Y)$. Finally, by using (9.2.3) and (9.2.2) it follows that the sequence $(F_n)_{n\in\mathbb{Z}^*}$ defined by $F_n = \frac{1}{\sqrt{2}}\tilde{F}_n$ has the required properties. $\square$

Theorem 9.2.2. *Assume that the pair (A, C) is exactly observable. Moreover, assume that the sequence of eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ of A_0 satisfies*

$$\sum_{m\geqslant 1} e^{-\beta\lambda_m} < \infty \qquad \forall\, \beta > 0\,. \tag{9.2.4}$$

Then the pair $(-A_0, C_0)$ is final state observable in any time $\tau > 0$.

In order to prove the above theorem we need a technical result asserting the existence of an appropriate entire function with fast decay on the real line. This will be a multiple of the Fourier transform of the $\mathbb{C}^\infty$ function defined by

$$\sigma_\nu(t) = \begin{cases} e^{-\frac{\nu}{1-t^2}} & \text{if } \ |t| < 1, \\ 0 & \text{if } \ |t| \geqslant 1, \end{cases} \tag{9.2.5}$$

where ν is a positive constant. By elementary considerations, for every $\eta \in (0, 1)$ we have

$$\int_{-1}^{1} \sigma_\nu(t)\mathrm{d}t \geqslant 2\eta e^{-\frac{\nu}{1-\eta^2}}\,.$$

Selecting $\eta = \frac{1}{\sqrt{\nu+1}}$ implies the left inequality in

$$\frac{2e^{-\nu-1}}{\sqrt{\nu+1}} \leqslant \int_{-1}^{1} \sigma_\nu(t)\mathrm{d}t \leqslant 2e^{-\nu}\,, \tag{9.2.6}$$

while the right inequality can be obtained by elementary considerations.

The following result furnishes the required fast decay property.

Lemma 9.2.3. *Let $\beta > 0, \delta > 0$, and set $\nu = (\pi + \delta)^2/\beta$. The function σ_ν being defined as in (9.2.5), put $C_\nu = \left(\int_{-1}^{1} \sigma_\nu(t)\mathrm{d}t\right)^{-1}$ and denote by $H_{\beta,\delta}$ the entire function defined by*

$$H_{\beta,\delta}(z) = C_\nu \int_{-1}^{1} \sigma_\nu(t) e^{-i\beta t z}\,\mathrm{d}t\,. \tag{9.2.7}$$

Then we have

$$H_{\beta,\delta}(0) = 1\,, \tag{9.2.8}$$

$$H_{\beta,\delta}(ix) \geqslant \frac{e^{\beta|x|/(2\sqrt{\nu+1})}}{11\sqrt{\nu+1}} \quad (x \in \mathbb{R})\,, \tag{9.2.9}$$

$$|H_{\beta,\delta}(z)| \leqslant e^{\beta|y|} \quad (z = x + iy, \ \ x, y \in \mathbb{R})\,, \tag{9.2.10}$$

$$|H_{\beta,\delta}(x)| \leqslant C\sqrt{\nu+1}\, e^{3\nu/4-(\pi+\delta/2)\sqrt{|x|}} \qquad (x \in \mathbb{R})\,, \tag{9.2.11}$$

for some constant $C > 0$ depending only on δ.

Proof. Conditions (9.2.8) and (9.2.10) follow from the definition of C_ν.

To show (9.2.9), we may assume $x \geqslant 0$. We first note that, from (9.2.6), we have

$$\frac{1}{2}e^\nu \leqslant C_\nu \leqslant \frac{3}{2}\sqrt{\nu+1}\,e^\nu . \tag{9.2.12}$$

Then, since $\sigma_\nu(t) \geqslant e^{-\nu-1}$ for $\frac{1}{2}\eta \leqslant t \leqslant \eta$ with $\eta := 1/\sqrt{\nu+1}$, we have (as required)

$$H_{\beta,\delta}(ix) \geqslant \frac{1}{2}C_\nu \eta e^{-\nu-1+\beta x\eta/2} \geqslant \frac{1}{11}\eta e^{\beta\eta x/2}.$$

Thus, it only remains to establish condition (9.2.11). Since $H_{\beta,\delta}$ is even, we consider only the case $x > 0$. Since all the derivatives of σ_ν vanish for $x = -1$ and $x = 1$, after several integrations by parts we get

$$|H_{\beta,\delta}(x)| \leqslant \frac{C_\nu \|\sigma_\nu^{(j)}\|_{L^\infty(\mathbb{R})}}{(\beta x)^j} \qquad (x > 0, \quad j \in \mathbb{N}). \tag{9.2.13}$$

For $t \in (-1, 1)$ we set $\varrho = 1 - t$ and $z = t + \varrho e^{i\vartheta}$, with $\vartheta \in (-\pi, \pi]$. We have

$$\operatorname{Re}\frac{2}{1-z^2} = \operatorname{Re}\frac{1}{1-z} + \operatorname{Re}\frac{1}{1+z} = \frac{1}{2\varrho} + \frac{1-\varrho(\sin\vartheta/2)^2}{2-2\varrho(2-\varrho)(\sin\vartheta/2)^2}.$$

Since the last term is an increasing function of $(\sin\vartheta/2)^2$, we obtain

$$\operatorname{Re}\frac{2}{1-z^2} \geqslant \frac{1}{2\varrho} + \frac{1}{2} \qquad (|z-t| = \varrho).$$

Therefore

$$|\sigma_\nu(z)| \leqslant e^{-\nu/4\varrho-\nu/4} \qquad (|z-t| = \varrho). \tag{9.2.14}$$

Applying Cauchy's integral formula, we obtain that

$$|\sigma_\nu^{(j)}(t)| \leqslant e^{-\nu/4}\sup_{\varrho>0}\frac{j!e^{-\nu/4\varrho}}{\varrho^j} \qquad (j \in \mathbb{N}, \quad t \in [-1, 1]),$$

which, in view of the elementary inequality $j! > j^j e^{-j}$ $(j \geqslant 1)$, yields

$$|\sigma_\nu^{(j)}(t)| \leqslant e^{-\nu/4}\frac{\left(2^j j!\right)^2}{\nu^j} \qquad (j \in \mathbb{N}, \quad t \in [-1, 1]). \tag{9.2.15}$$

From this, (9.2.12), (9.2.13) and the fact that $H_{\beta,\delta}$ is even, we get that

$$|H_{\beta,\delta}(x)| \leqslant \tfrac{3}{2}\sqrt{\nu+1}\,e^{3\nu/4}\frac{\left(2^j j!\right)^2}{(\beta\nu x)^j} \qquad (x > 0, \ j \in \mathbb{N}).$$

Selecting $j = 0$ when $0 \leqslant x \leqslant 1$ and $j = \left\lfloor \frac{1}{2}\sqrt{\beta\nu x}\right\rfloor$ otherwise, we readily check that (9.2.11) holds as required (recall that $\lfloor x \rfloor$ stands for the integer part of the real number x). Indeed, we deduce from the above that there exist positive constants C_0, C_1, C_2 and C (depending only on δ) such that for every $x > 1$, we have

$$\begin{aligned}\frac{|H_{\beta,\delta}(x)|}{\sqrt{\nu+1}\, e^{3\nu/4}} &\leqslant C_0 \frac{\left(2^j j!\right)^2}{(2j)^{2j}} \leqslant C_1 e^{-2j} j \\ &\leqslant C_2 e^{-(\pi+\delta)\sqrt{x}} \sqrt{x} \leqslant C e^{-(\pi+\delta/2)\sqrt{x}}.\end{aligned}$$

This concludes the proof. $\square$

We are now in a position to prove the main result in this section.

Proof of Theorem 9.2.2. Let $(F_n)_{n\in\mathbb{Z}^*}$ be the sequence constructed in Lemma 9.2.1. For $m \in \mathbb{N}$ we consider the function $\widetilde{\Upsilon}_m$ defined by

$$\widetilde{\Upsilon}_m(z) = \int_0^{\tau_0} e^{-itz} F_m(t)\,\mathrm{d}t \qquad \forall\, z \in \mathbb{C}.$$

It can be seen easily that $\widetilde{\Upsilon}_m$ is of exponential type at most τ_0. More precisely, for every $m \in \mathbb{Z}^*$,

$$\|\widetilde{\Upsilon}_m(z)\|_Y \leqslant M\sqrt{\tau_0} e^{\tau_0|z|} \qquad \forall\, z \in \mathbb{C}, \tag{9.2.16}$$

where $M = \sup_{n\in\mathbb{Z}^*} \|F_n\|_{L^2([0,\tau_0],Y)}$. Moreover, by using the fact that the families $(F_n)_{n\in\mathbb{Z}^*}$ and $\left(e^{i\mu_k t} C_0\varphi_k\right)_{k\in\mathbb{Z}^*}$ are biorthogonal in $L^2([0,\tau_0],Y)$, we have that

$$\begin{aligned}\left\langle C_0\varphi_k, \widetilde{\Upsilon}_m(\mu_k)\right\rangle_Y &= \int_0^{\tau_0} \left\langle C_0\varphi_k, e^{-i\mu_k t} F_m(t)\right\rangle_Y \mathrm{d}t \\ &= \int_0^{\tau_0} \left\langle e^{i\mu_k t} C_0\varphi_k, F_m(t)\right\rangle_Y \mathrm{d}t = \delta_{km} \qquad \forall\, k,\ m \in \mathbb{N},\end{aligned} \tag{9.2.17}$$

$$\begin{aligned}\left\langle C_0\varphi_k, \widetilde{\Upsilon}_m(-\mu_k)\right\rangle_Y &= \int_0^{\tau_0} \left\langle C_0\varphi_k, e^{i\mu_k t} F_m(t)\right\rangle_Y \mathrm{d}t \\ &= \int_0^{\tau_0} \left\langle e^{-i\mu_k t} C_0\varphi_k, F_m(t)\right\rangle_Y \mathrm{d}t \\ &= \int_0^{\tau_0} \left\langle e^{i\mu_{-k} t} C_0\varphi_{-k}, F_m(t)\right\rangle_Y \mathrm{d}t = 0 \qquad \forall\, k,\ m \in \mathbb{N}.\end{aligned} \tag{9.2.18}$$

For each $m \in \mathbb{N}$ the function $z \mapsto \widetilde{\Upsilon}_m(z) + \widetilde{\Upsilon}_m(-z)$ is even. Therefore, there exists a family of entire functions $(\Upsilon_m)_{m\in\mathbb{N}}$ such that, for every $m \in \mathbb{N}$,

$$\Upsilon_m(-iz^2) = \widetilde{\Upsilon}_m(z) + \widetilde{\Upsilon}_m(-z) \qquad \forall\, z \in \mathbb{C}.$$

The above relation, combined with (9.2.16), (9.2.17) and (9.2.18), implies that

$$\|\Upsilon_m(z)\|_Y \leqslant 2M\sqrt{\tau_0} e^{\tau_0\sqrt{|z|}} \qquad \forall\, z \in \mathbb{C}, \tag{9.2.19}$$

$$\langle C_0\varphi_k, \Upsilon_m(-i\lambda_k)\rangle_Y = \delta_{km} \qquad \forall\, k,\ m \in \mathbb{N}. \tag{9.2.20}$$

For $\delta > \max(0, 2(\tau_0 - \pi))$ and $\beta \in (0, \frac{\tau}{2})$, we consider the function $H_{\beta,\delta}$ introduced in Lemma 9.2.3 and we define the family of functions $(Q_m)_{m\in\mathbb{N}}$ by

$$Q_m(z) = \frac{H_\beta(z/2)}{H_\beta(i\lambda_m/2)}\,\Upsilon_m(z) \qquad \forall\, m \in \mathbb{N}. \tag{9.2.21}$$

Relations (9.2.20) and (9.2.21) imply that

$$\langle C_0\varphi_k, Q_m(-i\lambda_k)\rangle_Y = \delta_{km} \qquad \forall\, k,\ m \in \mathbb{N}. \tag{9.2.22}$$

On the other hand, by using (9.2.9), (9.2.10) and (9.2.19), it follows that the function Q_m is, for each $m \in \mathbb{N}$, exponential of type $\frac{\tau}{2}$ and by combining (9.2.11) and (9.2.19), it follows that

$$Q_m \in L^1(\mathbb{R}; Y) \cap L^2(\mathbb{R}; Y) \qquad \forall\, m \in \mathbb{N}.$$

Moreover, by using (9.2.4), (9.2.9), (9.2.10) and (9.2.11), it is easy to check that

$$\sum_{m\in\mathbb{N}} \|Q_m\|^2_{L^2(\mathbb{R};Y)} < \infty. \tag{9.2.23}$$

By the Paley–Wiener theorem on entire functions (Theorem 12.4.3 from Appendix I), Q_m is, for each $m \in \mathbb{N}$, the Fourier transform of a function $g_m \in L^2(\mathbb{R})$ with $\operatorname{supp} g_m \subset \left[-\frac{\tau}{2}, \frac{\tau}{2}\right]$; i.e.,

$$Q_m(z) = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} g_m(t)e^{-itz}\,\mathrm{d}t \qquad (z \in \mathbb{C}).$$

Now we consider the family of functions $(G_m)_{m\in\mathbb{N}}$ defined by

$$G_m(t) = e^{\frac{\lambda_m \tau}{2}} g_m\left(t - \frac{\tau}{2}\right) \qquad (t \in \mathbb{R}). \tag{9.2.24}$$

Then

$$\begin{aligned}
\int_0^\tau \left\langle e^{-\lambda_k t} C_0\varphi_k, G_m(t)\right\rangle_Y \mathrm{d}t &= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \left\langle e^{-\lambda_k (s+\frac{\tau}{2})} C_0\varphi_k, G_m(s + \frac{\tau}{2})\right\rangle_Y \mathrm{d}s \\
&= e^{(\lambda_m - \lambda_k)\frac{\tau}{2}} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \left\langle e^{-\lambda_k s} C_0\varphi_k, g_m(s)\right\rangle_Y \mathrm{d}s \\
&= e^{(\lambda_m - \lambda_k)\frac{\tau}{2}} \left\langle C_0\varphi_k, \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-\lambda_k s} g_m(s)\,\mathrm{d}s \right\rangle_Y \\
&= e^{(\lambda_m - \lambda_k)\frac{\tau}{2}} \langle C_0\varphi_k, Q_m(-i\lambda_k)\rangle_Y\ .
\end{aligned}$$

The above formula and (9.2.22) imply that the family $(G_m)_{m\in\mathbb{N}}$ is biorthogonal, in $L^2([0,\tau], Y)$, to the family $\left(e^{-\lambda_k t} C_0\varphi_k\right)_{k\in\mathbb{N}}$. Moreover, by combining (9.2.23) and (9.2.24), it follows that condition (9.1.1) holds. By applying Lemma 9.1.4 we get the conclusion of the theorem. ☐

Example 9.2.4. Let $H = L^2[0,\pi]$ and let A_0 be the positive operator from Example 3.4.12, where we have seen that $H_{\frac{1}{2}} = \mathcal{H}^1_R(0,\pi)$. Let C_0 be the observation operator defined by

$$C_0 f = f(0) \qquad \forall\, f \in H_{\frac{1}{2}}.$$

Since $\mathcal{H}^1(0,\pi)$ is continuously embedded in $C[0,\pi]$, C_0 is well defined and it is in $\mathcal{L}(H_{\frac{1}{2}},\mathbb{C})$. Let $X = \mathcal{H}^1_R(0,\pi) \times L^2[0,\pi]$ and let $A : \mathcal{D}(A) \to X$ be defined by

$$\mathcal{D}(A) = \left\{ f \in \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_R(0,\pi) \;\middle|\; \frac{\mathrm{d}f}{\mathrm{d}x}(0) = 0 \right\} \times \mathcal{H}^1_R(0,\pi), \quad A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}.$$

We have seen in Proposition 6.2.5 that the pair (A, C), with $C = \begin{bmatrix} 0 & C_0 \end{bmatrix}$, is exactly observable. According to Theorem 9.2.2, it follows that the pair $(-A_0, C_0)$ (with state space H) is final state observable in any time $\tau > 0$.

9.3 Final state observability with geometric conditions

In this section we apply the results from the previous section and from Chapter 7 to several systems governed by parabolic PDEs. We use the notation H, A_0, H_1, H_1, X, X_1, A from the previous section, but H and A_0 will be chosen in several manners in order to tackle the variety of examples considered. We denote by Ω an open bounded set in $\mathbb{R}^n$ which either has a C^2 boundary or it is rectangular.

First we consider a heat equation with locally distributed observation.

Proposition 9.3.1. *Let $H = L^2(\Omega)$ and let $-A_0$ be the Dirichlet Laplacian on Ω, introduced in Section 3.6. Let $\mathcal{O} \subset \Omega$ be an open set satisfying the assumptions in Theorem 7.4.1, let $Y = L^2(\mathcal{O})$ and let $C_0 \in \mathcal{L}(H, Y)$ be defined by*

$$C_0 f = f|_{\mathcal{O}}.$$

Then the pair $(-A_0, C_0)$ is final state observable in any time $\tau > 0$.

Proof. We know from Theorem 7.4.1 that (A, C) with $C = \begin{bmatrix} 0 & C_0 \end{bmatrix}$ is exactly observable. Moreover, from Proposition 3.6.9 it follows that the eigenvalues (λ_k) of A_0 satisfy (9.2.4). Therefore, the conclusion follows by applying Theorem 9.2.2. □

Remark 9.3.2. In terms of PDEs, the above proposition says that if z is the solution of the heat equation

$$\frac{\partial z}{\partial t}(x,t) = \Delta z(x,t), \quad x \in \Omega, \quad t \geqslant 0, \tag{9.3.1}$$

$$z(x,t) = 0, \qquad x \in \partial\Omega, \quad t \geqslant 0, \tag{9.3.2}$$

$$z(\cdot,0) = z_0(x), \qquad x \in \Omega, \tag{9.3.3}$$

where $z_0 \in \mathcal{H}^2(\Omega) \cap \mathcal{H}^1_0(\Omega)$, then

$$\inf_{\|z(\tau)\|_{L^2(\Omega)}=1} \int_0^\tau \int_{\mathcal{O}} |z(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t > 0 \qquad \forall\, \tau > 0.$$

Our next example concerns the heat equation with boundary observation.

Proposition 9.3.3. *Let $H = \mathcal{H}_0^1(\Omega)$ and let $-A_0$ be the Dirichlet Laplacian on Ω, but restricted such that it is a densely defined strictly positive operator on H. Let $\Gamma \subset \partial\Omega$ be an open set satisfying the assumptions in Theorem 7.2.4, let $Y = L^2(\Gamma)$ and let $C_1 \in \mathcal{L}(H_{\frac{1}{2}}, Y)$ be defined by*

$$C_1 f = \frac{\partial f}{\partial \nu}|_\Gamma .$$

Then the pair $(-A_0, C_1)$ is final state observable in any time $\tau > 0$.

Proof. We consider the operator A, defined in (9.2.1), so that it is a densely defined skew-adjoint operator on $H_{\frac{1}{2}} \times H$. Consider the initial and boundary value problem

$$\begin{aligned} \frac{\partial^2 \eta}{\partial t^2} - \Delta \eta &= 0 \quad \text{in} \quad \Omega \times (0, \infty), \\ \eta &= 0 \qquad \text{on} \quad \partial\Omega \times (0,\infty), \\ \eta(x,0) = f(x), \quad \frac{\partial \eta}{\partial t}(x,0) &= g(x) \quad \text{for} \quad x \in \Omega, \end{aligned}$$

where $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{D}(A^2)$. According to Theorem 7.2.4 and Remark 7.2.6, for $\tau > 0$ large enough, there exists a constant $k_\tau > 0$ such that

$$\int_0^\tau \int_\Gamma \left|\frac{\partial \dot\eta}{\partial \nu}\right|^2 \mathrm{d}\sigma \, \mathrm{d}t \geqslant k_\tau^2 \left(\|\Delta f\|^2 + \|\nabla g\|^2\right) \qquad \forall \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{D}(A^2),$$

where $\|\cdot\|$ stands for the norm in $L^2(\Omega)$. The above estimate means that the pair (A, C), with $C = \begin{bmatrix} 0 & C_1 \end{bmatrix}$, state space $H_{\frac{1}{2}} \times H$ and output space $Y = L^2(\Gamma)$, is exactly observable. The conclusion follows now by applying Theorem 9.2.2. $\square$

Remark 9.3.4. In terms of PDEs, the above proposition says that if z is the solution of the heat equation (9.3.1)–(9.3.3), with $z_0 \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$, then

$$\inf_{\|z(\tau)\|_{\mathcal{H}_0^1(\Omega)}=1} \int_0^\tau \int_\Gamma \left|\frac{\partial z}{\partial \nu}(x,t)\right|^2 \mathrm{d}\sigma \, \mathrm{d}t > 0 \qquad \forall \, \tau > 0 .$$

Remark 9.3.5. Recall from the comments on Section 7.2 in Section 7.7 that the assumptions in Theorems 7.4.1 and 7.2.4 can be replaced by the weaker geometric optics condition of Bardos, Lebeau and Rauch [15]. Therefore, the conclusions in propositions 9.3.1 and 9.3.3 still hold if the assumptions on $\mathcal{O}$ and Γ are replaced by the geometric optics condition.

The example in the proposition below concerns a one-dimensional heat equation with variable coefficients and boundary observation.

Proposition 9.3.6. *Denote $J = (0,1)$ and let $a, b : J \to \mathbb{R}$ be two functions such that $a \in C^2(J)$, $b \in \mathcal{H}^1(J)$ and a is bounded from below (i.e., there exists $m > 0$ such that $a(x) \geqslant m > 0$ for all $x \in J$). We denote by H the space $\mathcal{H}_0^1(J)$ and we consider the Sturm–Liouville operator $A_0 : \mathcal{D}(A_0) \to H$ which has been introduced in Section 8.2, but restricted such that it is a densely defined self-adjoint operator on H. Let $Y = \mathbb{C}$ and $C_1 \in \mathcal{L}(H_{\frac{1}{2}}, Y)$ be defined by*

$$C_1 z = \frac{\mathrm{d}z}{\mathrm{d}x}(0) \qquad \forall\, z \in H_{\frac{1}{2}} .$$

Then the pair $(-A_0, C_1)$ is final state observable in any time $\tau > 0$.

Proof. In this proof A, defined in (9.2.1), is restricted such that it is a densely defined strictly positive operator on $H_{\frac{1}{2}} \times H$.

Consider the initial and boundary value problem

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2}(x,t) = \dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial w}{\partial x}(x,t)\right) - b(x)w(x,t), & x \in J,\ t \geqslant 0, \\ \quad w(0,t) = 0, \quad w(\pi,t) = 0, & t \in [0,\infty), \\ \quad w(x,0) = f(x), \quad \dfrac{\partial w}{\partial t}(x,0) = g(x), & x \in J, \end{cases}$$

where $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{D}(A^2)$. According to Remark 8.2.3, for $\tau > 0$ large enough, there exists $k_\tau > 0$ such that

$$\int_0^\tau \left|\frac{\partial \dot{w}}{\partial x}(0,t)\right|^2 \mathrm{d}t \ \geqslant\ k_\tau^2 \left(\|f\|^2_{\mathcal{H}^2(\Omega)} + \|g\|^2_{\mathcal{H}^1(\Omega)}\right) \qquad \forall\ \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{D}(A^2). \quad (9.3.4)$$

The above estimate means that the pair (A, C), with $C = \begin{bmatrix} 0 & C_1 \end{bmatrix}$, state space $H_{\frac{1}{2}} \times H$ and output space $Y = \mathbb{C}$, is exactly observable. Since, by Proposition 3.5.5, the eigenvalues (λ_k) of A_0 satisfy (9.2.4), the conclusion follows now by applying Theorem 9.2.2. □

Remark 9.3.7. In terms of PDEs, the above proposition says that if z is the solution of the variable coefficients heat equation

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) = \dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial z}{\partial x}(x,t)\right) - b(x)z(x,t), & x \in J,\ t \geqslant 0, \\ \quad z(0,t) = 0, \quad w(1,t) = 0, & t \in [0,\infty), \\ \quad z(x,0) = f(x), & x \in J, \end{cases}$$

with $f \in \mathcal{H}^2(J) \cap \mathcal{H}_0^1(J)$, then

$$\inf_{\|z(\tau)\|_{\mathcal{H}_0^1(J)}=1} \int_0^\tau \left|\frac{\partial z}{\partial x}(0,t)\right|^2 \mathrm{d}t > 0 \qquad \forall\, \tau > 0 .$$

We consider a system corresponding to the linearized Cahn–Hilliard equation.

Proposition 9.3.8. *Let $H = L^2(\Omega)$ and let $-A_0$ be the Dirichlet Laplacian on Ω, introduced in Section* 3.6. *Let $\mathcal{O} \subset \Omega$ be an open set, let $Y = H$ and let $C_0 \in \mathcal{L}(H)$ be defined by*

$$C_0 f = f\chi_{\mathcal{O}},$$

where $\chi_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$. Assume that one of the following conditions holds:

1. *$\mathcal{O}$ satisfies the assumptions in Theorem* 7.4.1*.*
2. *Ω is a rectangle in $\mathbb{R}^2$.*

Then the pair $(-A_0^2, C_0)$ is final state observable in any time $\tau > 0$.

Proof. Consider the space $\mathcal{X}$ and the operator $\mathcal{A}$ introduced in Section 7.5, i.e., $\mathcal{X} = H_1 \times H$, $\mathcal{D}(\mathcal{A}) = H_2 \times H_1$ and $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ defined by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0^2 & 0 \end{bmatrix}.$$

Let $\mathcal{C} \in \mathcal{L}(\mathcal{X}_1, Y)$ be defined by $\mathcal{C} = \begin{bmatrix} 0 & C_0 \end{bmatrix}$. Then the pair $(\mathcal{A}, \mathcal{C})$ is exactly observable in any time $\tau > 0$. Indeed, this has been shown in Proposition 7.5.7 if $\mathcal{O}$ satisfies the assumptions in Theorem 7.4.1 and in Theorem 8.5.1 if Ω is a rectangle in $\mathbb{R}^2$. We can now conclude by applying Theorem 9.2.2. □

Remark 9.3.9. In terms of PDEs, the above proposition says that if z is the solution of the linearized Cahn–Hilliard equation

$$\begin{aligned} \frac{\partial z}{\partial t}(x,t) + \Delta^2 z(x,t) &= 0, \quad x \in \Omega, \quad t \geqslant 0, \\ z(x,t) = \Delta(x,t) &= 0, \quad x \in \partial\Omega, \quad t \geqslant 0, \\ z(\cdot,0) &= z_0(x), \quad x \in \Omega, \end{aligned}$$

where $z_0 \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$, then

$$\inf_{\|z(\tau)\|_{L^2(\Omega)}=1} \int_0^\tau \int_{\mathcal{O}} |z(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t > 0 \qquad \forall\, \tau > 0.$$

9.4 A global Carleman estimate for the heat operator

The aim of this section is to provide a proof of a quite technical result, called the global Carleman estimate for the heat equation. This estimate will be the main tool in the proof of the final state observability for arbitrary observation regions, which will be proved in the next section.

Throughout this section, $\Omega \subset \mathbb{R}^n$ is an open bounded and connected set with boundary $\partial\Omega$ of class C^4 or Ω is a rectangular domain, and $T > 0$. The main result of this section is the following global Carleman estimate for the heat operator $\frac{\partial}{\partial t} - \Delta$.

Theorem 9.4.1. *Let $\mathcal{O}$ be an open non-empty subset of Ω. Then there exist a positive function $\alpha \in C^4(\text{clos}\ \Omega)$ and the constants $C_0 > 0$, $s_0 > 0$, depending only on Ω, $\mathcal{O}$ and T such that for all*

$$\varphi \in C\left([0,T]; \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right) \tag{9.4.1}$$

and all $s \geqslant s_0$, we have

$$\begin{aligned}\int_0^T \int_\Omega e^{\frac{-2s\alpha(x)}{t(T-t)}} \left[\frac{s}{t(T-t)}|\nabla\varphi|^2 + \frac{s^3}{t^3(T-t)^3}|\varphi|^2\right] \mathrm{d}x\,\mathrm{d}t \\ \leqslant C_0 \left[\int_0^T \int_\Omega e^{\frac{-2s\alpha(x)}{t(T-t)}} \left|\frac{\partial\varphi}{\partial t} - \Delta\varphi\right|^2 \mathrm{d}x\,\mathrm{d}t + s^3 \int_0^T \int_{\mathcal{O}} \frac{e^{-\frac{2s\alpha}{t(T-t)}}}{t^3(T-t)^3}|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t\right].\end{aligned} \tag{9.4.2}$$

Remark 9.4.2. The condition that $\partial\Omega$ is of class C^4 can be weakened to $\partial\Omega$ of class C^2; see the comments in Section 9.6.

The function α in the above theorem is constructed by using the theorem below.

Theorem 9.4.3. *Let $\mathcal{O}$ be an open subset of Ω and assume that $\partial\Omega$ is of class C^m, with $m \geqslant 2$. Then there exists a function $\eta_0 \in C^m(\text{clos}\ \Omega)$ such that*

- $\eta_0(x) > 0$ *for all* $x \in \Omega$,
- $\eta_0(x) = 0$ *for all* $x \in \partial\Omega$,
- $|\nabla\eta_0(x)| > 0$ *for all* $x \in \text{clos}\ (\Omega \setminus \mathcal{O})$.

If Ω is a rectangular domain, then there exists a function $\eta_0 \in C^\infty(\text{clos}\ \Omega)$ satisfying the above three conditions.

The proof of the above lemma is obvious in the case of a rectangular domain. For an arbitrary Ω with boundary of class C^m, $m \geqslant 2$, the proof is more complicated, and it is given in Chapter 14.

We introduce now some notation. We set $\eta(x) = \eta_0(x) + K_0$ and we define the function

$$\alpha(x) = e^{\lambda K_1} - e^{\lambda\eta(x)} \qquad \forall\, x \in \text{clos}\ \Omega, \tag{9.4.3}$$

where

$$K_0 = 4 \max_{x\in\text{clos}\ \Omega} \eta_0(x), \qquad K_1 = 6 \max_{x\in\text{clos}\ \Omega} \eta_0(x), \tag{9.4.4}$$

and λ is a constant which will be specified later. Moreover, for every $x \in \text{clos}\ \Omega$ and every $t \in (0,T)$ we set

$$\beta(x,t) = \frac{\alpha(x)}{t(T-t)}, \ \rho(x,t) = e^{\beta(x,t)}. \tag{9.4.5}$$

Several useful properties of the function β are summarized in the following lemma.

Lemma 9.4.4. *Assume that K_0 and K_1 are given by* (9.4.4). *Then*

$$\left|\frac{\partial\beta}{\partial t}(x,t)\right| \leqslant \frac{Te^{2\lambda\eta(x)}}{t^2(T-t)^2} \qquad \forall (x,t)\in\Omega\times(0,T), \tag{9.4.6}$$

$$\left|\frac{\partial^2\beta}{\partial t^2}(x,t)\right| \leqslant \frac{2T^2\lambda^2e^{2\lambda\eta(x)}}{t^3(T-t)^3} \qquad \forall (x,t)\in\Omega\times(0,T). \tag{9.4.7}$$

Moreover, there exists $C_1>0$ (depending on Ω and on $\mathcal{O}$) such that, for every $x\in\operatorname{clos}\Omega$, $\lambda\geqslant 1$ and every $t\in(0,T)$, we have

$$|\nabla\beta(x,t)| \leqslant C_1\frac{\lambda e^{\lambda\eta(x)}}{t(T-t)} \qquad \forall (x,t)\in\Omega\times(0,T), \tag{9.4.8}$$

$$|\Delta\beta(x,t)| \leqslant C_1\frac{\lambda^2 e^{\lambda\eta(x)}}{t(T-t)} \qquad \forall (x,t)\in\Omega\times(0,T), \tag{9.4.9}$$

$$\left|\nabla\left(\frac{\partial\beta}{\partial t}(x,t)\right)\cdot\nabla\beta(x,t)\right| \leqslant \frac{C_1T\lambda^2e^{3\lambda\eta(x)}}{t^3(T-t)^3} \qquad \forall (x,t)\in\Omega\times(0,T), \tag{9.4.10}$$

$$\left|\frac{\partial\beta}{\partial t}(x,t)(\Delta\beta)(x,t)\right| \leqslant \frac{C_1T\lambda^2e^{3\lambda\eta(x)}}{t^3(T-t)^3} \qquad \forall (x,t)\in\Omega\times(0,T). \tag{9.4.11}$$

Proof. We first remark that, according to (9.4.4) and to the fact that $K_1 > \max_{x\in\operatorname{clos}\Omega}\eta$, we have

$$2K_1 = 3K_0 \leqslant 3\eta(x) \qquad \forall\, x\in\operatorname{clos}\Omega. \tag{9.4.12}$$

The estimate (9.4.6) follows from

$$\frac{\partial\beta}{\partial t} = \frac{2t-T}{t^2(T-t)^2}\left(e^{\lambda K_1}-e^{\lambda\eta}\right) \tag{9.4.13}$$

and from the fact that $2\eta(x)\geqslant K_1$ for every $x\in\Omega$ (this follows from (9.4.12)).

In order to prove (9.4.7) we note that

$$\left|\frac{\partial^2\beta}{\partial t^2}(x,t)\right| = \frac{2\left|T^2-3Tt+3t^2\right|}{t^3(T-t)^3}\left(e^{\lambda K_1}-e^{\lambda\eta(x)}\right),$$

which, combined with (9.4.4) and (9.4.12), implies (9.4.7).

Inequality (9.4.8) follows from

$$\nabla\beta = -\frac{\lambda e^{\lambda\eta}}{t(T-t)}\nabla\eta. \tag{9.4.14}$$

From (9.4.14) it follows that

$$\Delta\beta = -\frac{\lambda e^{\lambda\eta}}{t(T-t)}\Delta\eta - \frac{\lambda^2e^{\lambda\eta}}{t(T-t)}|\nabla\eta|^2, \tag{9.4.15}$$

which yields (9.4.9). Moreover, by using (9.4.13) or (9.4.14) we obtain that

$$\left|\nabla\left(\frac{\partial\beta}{\partial t}\right)\cdot\nabla\beta\right| = \frac{|T-2t|\lambda^2 e^{2\lambda\eta}}{t^3(T-t)^3}|\nabla\eta|^2,$$

which implies (9.4.10).

Finally, inequality (9.4.11) is an obvious consequence of (9.4.6) and (9.4.9). □

We define the functions

$$f_s = \rho^{-s}\left(\frac{\partial\varphi}{\partial t} - \Delta\varphi\right) \tag{9.4.16}$$

and

$$\psi = \rho^{-s}\varphi, \tag{9.4.17}$$

with $s > 0$ and ρ defined in (9.4.5).

The main ingredient of the proof of Theorem 9.4.1 is the following lemma.

Lemma 9.4.5. *With the above notation, there exist the constants s_0, $\lambda_0 > 0$, $K > 0$, depending only on Ω, $\mathcal{O}$ and T, such that the inequality*

$$\int_0^T\int_\Omega\left(\frac{t(T-t)}{s}\left(\left|\frac{\partial\psi}{\partial t}\right|^2 + |\Delta\psi|^2\right) + \frac{s}{t(T-t)}|\nabla\psi|^2 + \frac{s^3}{t^3(T-t)^3}|\psi|^2\right)\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant K\int_0^T\left(\|f_s\|^2_{L^2(\Omega)} + \int_{\mathcal{O}}\frac{s^3}{t^3(T-t)^3}|\psi|^2\,\mathrm{d}x\right)\mathrm{d}t \tag{9.4.18}$$

holds for every φ satisfying (9.4.1) *and for every $s \geqslant s_0$ and $\lambda \geqslant \lambda_0$.*

Proof. The proof is divided into four steps.

First step. It can be easily checked that

$$\frac{\partial}{\partial t}(e^{s\beta}) = s\frac{\partial\beta}{\partial t}e^{s\beta}, \tag{9.4.19}$$

$$\nabla(e^{s\beta}) = se^{s\beta}\nabla\beta,\quad \Delta(e^{s\beta}) = se^{s\beta}\Delta\beta + s^2e^{s\beta}|\nabla\beta|^2. \tag{9.4.20}$$

Notice that

$$\lim_{t\to 0+}\psi(x,t) = \lim_{t\to T-}\psi(x,t)$$
$$= \lim_{t\to 0+}\frac{\partial\psi}{\partial t}(x,t) = \lim_{t\to T-}\frac{\partial\psi}{\partial t}(x,t) = 0 \qquad \forall\, x\in\Omega. \tag{9.4.21}$$

By (9.4.19), (9.4.20) and the fact, following from (9.4.16) and (9.4.17), that

$$\rho^{-s}\left[\frac{\partial}{\partial t}(\rho^s\psi) - \Delta(\rho^s\psi)\right] = f_s,$$

we obtain that

$$M_1\psi + M_2\psi = g_s\,, \tag{9.4.22}$$

where we have denoted

$$M_1\psi = \frac{\partial\psi}{\partial t} - 2s\nabla\beta\cdot\nabla\psi, \tag{9.4.23}$$

$$M_2\psi = s\frac{\partial\beta}{\partial t}\psi - \Delta\psi - s^2|\nabla\beta|^2\psi, \tag{9.4.24}$$

and

$$g_s = f_s + s(\Delta\beta)\psi. \tag{9.4.25}$$

These relations imply (using the notation $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ for the norm and the inner product in $L^2(\Omega)$) that

$$\int_0^T \left(\|M_1\psi\|^2 + \|M_2\psi\|^2 + 2\langle M_1\psi, M_2\psi\rangle\right) \mathrm{d}t = \int_0^T \|g_s\|^2\,\mathrm{d}t. \tag{9.4.26}$$

Second step. We estimate the crossed term $2\langle M_1\psi, M_2\psi\rangle_{L^2(\Omega\times(0,T))}$ in (9.4.26). Relations (9.4.23) and (9.4.24) imply that

$$2\langle M_1\psi, M_2\psi\rangle_{L^2(\Omega\times(0,T))} = I_1 + I_2 + I_3\,, \tag{9.4.27}$$

where

$$I_1 = 2\int_0^T\int_\Omega \left(s\frac{\partial\beta}{\partial t}\psi - \Delta\psi - s^2|\nabla\beta|^2\psi\right)\frac{\partial\psi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t, \tag{9.4.28}$$

$$I_2 = 4s\int_0^T\int_\Omega (\nabla\beta\cdot\nabla\psi)\,\Delta\psi\,\mathrm{d}x\,\mathrm{d}t \tag{9.4.29}$$

and

$$I_3 = 4s\int_0^T\int_\Omega \left(s^2|\nabla\beta|^2\psi - s\frac{\partial\beta}{\partial t}\psi\right)(\nabla\beta\cdot\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t. \tag{9.4.30}$$

Integrating by parts with respect to x in (9.4.28) and using the fact that $\psi = 0$ on $\partial\Omega\times(0,T)$, we obtain that

$$I_1 = \int_0^T\int_\Omega \left[\frac{\partial}{\partial t}\left(|\nabla\psi|^2\right) - \left(s^2|\nabla\beta|^2 - s\frac{\partial\beta}{\partial t}\right)\frac{\partial}{\partial t}\left(|\psi|^2\right)\right]\mathrm{d}x\,\mathrm{d}t.$$

By integrating the above relation by parts with respect to t and by using (9.4.21), we get

$$I_1 = \int_0^T\int_\Omega \left\{2s^2\left[\nabla\left(\frac{\partial\beta}{\partial t}\right)\right]\cdot\nabla\beta - s\frac{\partial^2\beta}{\partial t^2}\right\}|\psi|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{9.4.31}$$

Integrating by parts in (9.4.29) we obtain that

$$I_2 = 4s\int_0^T\int_{\partial\Omega}(\nabla\beta\cdot\nabla\psi)\frac{\partial\psi}{\partial\nu}\,\mathrm{d}\sigma\,\mathrm{d}t - 4s\int_0^T\int_\Omega\nabla(\nabla\beta\cdot\nabla\psi)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t. \quad (9.4.32)$$

Since β and ψ are, for each $t\in(0,T)$, constant with respect to $x\in\partial\Omega$, the first term on the right-hand side of the above relation can be written as

$$4s\int_0^T\int_{\partial\Omega}(\nabla\beta\cdot\nabla\psi)\frac{\partial\psi}{\partial\nu}\,\mathrm{d}\sigma\,\mathrm{d}t = 4s\int_0^T\int_{\partial\Omega}\frac{\partial\beta}{\partial\nu}\left|\frac{\partial\psi}{\partial\nu}\right|^2\mathrm{d}\sigma\,\mathrm{d}t. \quad (9.4.33)$$

The last term on the right-hand side of (9.4.32) can be written as

$$\begin{aligned}4s\int_0^T\int_\Omega\nabla(\nabla\beta\cdot\nabla\psi)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t &= 4s\sum_{i,j=1}^n\int_0^T\int_\Omega\frac{\partial^2\beta}{\partial x_i\partial x_j}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t\\ &+4s\sum_{i,j=1}^n\int_0^T\int_\Omega\frac{\partial\beta}{\partial x_i}\frac{\partial^2\psi}{\partial x_j\partial x_i}\frac{\partial\psi}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

By integrating by parts with respect to x in the last term on the right-hand side, the above relation becomes

$$\begin{aligned}4s\int_0^T\int_\Omega\nabla(\nabla\beta\cdot\nabla\psi)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t &= 4s\sum_{i,j=1}^n\int_0^T\int_\Omega\frac{\partial^2\beta}{\partial x_i\partial x_j}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t\\ &+2s\int_0^T\int_{\partial\Omega}\frac{\partial\beta}{\partial\nu}\left|\frac{\partial\psi}{\partial\nu}\right|^2\mathrm{d}\sigma\,\mathrm{d}t - 2s\int_0^T\int_\Omega(\Delta\beta)|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The above relation, combined with (9.4.32) and (9.4.33), gives

$$\begin{aligned}I_2 = 2s\int_0^T\int_{\partial\Omega}\frac{\partial\beta}{\partial\nu}\left|\frac{\partial\psi}{\partial\nu}\right|^2\mathrm{d}\sigma\,\mathrm{d}t - 4s\int_0^T\int_\Omega\sum_{i,j=1}^n\frac{\partial^2\beta}{\partial x_i\partial x_j}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t\\ + 2s\int_0^T\int_\Omega(\Delta\beta)|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t. \quad (9.4.34)\end{aligned}$$

In order to transform I_3 (which was defined in (9.4.30)), we notice that by integrating by parts we get

$$\begin{aligned}4s^3\int_0^T\int_\Omega|\nabla\beta|^2\psi(\nabla\beta\cdot\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t &= -2s^3\int_0^T\int_\Omega|\nabla\beta|^2|\psi|^2\Delta\beta\,\mathrm{d}x\,\mathrm{d}t\\ &- 4s^3\sum_{i,j=1}^n\int_0^T\int_\Omega\frac{\partial^2\beta}{\partial x_i\partial x_j}\frac{\partial\beta}{\partial x_i}\frac{\partial\beta}{\partial x_j}|\psi|^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}4s^2\int_0^T\int_\Omega\frac{\partial\beta}{\partial t}\psi(\nabla\beta\cdot\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t &= -2s^2\int_0^T\int_\Omega\nabla\left(\frac{\partial\beta}{\partial t}\right)\cdot(|\psi|^2\nabla\beta)\,\mathrm{d}x\,\mathrm{d}t\\ &- 2s^2\int_0^T\int_\Omega\frac{\partial\beta}{\partial t}(\Delta\beta)|\psi|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The above two formulas and (9.4.30) imply that

$$I_3 = -2s^3\int_0^T\int_\Omega |\nabla\beta|^2|\psi|^2\Delta\beta\,\mathrm{d}x\,\mathrm{d}t - 4s^3\sum_{i,j=1}^n\int_0^T\int_\Omega \frac{\partial^2\beta}{\partial x_i\partial x_j}\frac{\partial\beta}{\partial x_i}\frac{\partial\beta}{\partial x_j}|\psi|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$+2s^2\int_0^T\int_\Omega \nabla\left(\frac{\partial\beta}{\partial t}\right)\cdot(|\psi|^2\nabla\beta)\,\mathrm{d}x\,\mathrm{d}t + 2s^2\int_0^T\int_\Omega \frac{\partial\beta}{\partial t}(\Delta\beta)|\psi|^2\,\mathrm{d}x\,\mathrm{d}t. \quad (9.4.35)$$

Relations (9.4.27), (9.4.31), (9.4.34) and (9.4.35) imply that

$$2\langle M_1\psi, M_2\psi\rangle_{L^2(\Omega\times(0,T))} = J_1 + J_2 + J_3 - 4s\sum_{i,j=1}^n\int_0^T\int_\Omega \frac{\partial^2\beta}{\partial x_i\partial x_j}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t$$

$$+2\int_0^T\int_\Omega \left(s(\Delta\beta)|\nabla\psi|^2 - s^3(\Delta\beta)|\nabla\beta|^2|\psi|^2\right)\mathrm{d}x\,\mathrm{d}t, \quad (9.4.36)$$

where

$$J_1 = -4s^3\sum_{i,j=1}^n\int_0^T\int_\Omega \frac{\partial^2\beta}{\partial x_i\partial x_j}\frac{\partial\beta}{\partial x_i}\frac{\partial\beta}{\partial x_j}|\psi|^2\,\mathrm{d}x\,\mathrm{d}t, \quad (9.4.37)$$

$$J_2 = 2s\int_0^T\int_{\partial\Omega} \frac{\partial\beta}{\partial\nu}\left|\frac{\partial\psi}{\partial\nu}\right|^2\mathrm{d}\sigma\,\mathrm{d}t, \quad (9.4.38)$$

$$J_3 = 2s^2\int_0^T\int_\Omega \left\{2\left[\nabla\left(\frac{\partial\beta}{\partial t}\right)\right]\cdot\nabla\beta + \frac{\partial\beta}{\partial t}\Delta\beta\right\}|\psi|^2\,\mathrm{d}x\,\mathrm{d}t - s\int_0^T\int_\Omega \frac{\partial^2\beta}{\partial t^2}|\psi|^2\,\mathrm{d}x\,\mathrm{d}t. \quad (9.4.39)$$

By setting

$$c_0 = 2\min_{x\in\mathrm{clos}\,(\Omega\setminus\mathcal{O})}\sum_{i,j=1}^n\left|\frac{\partial\eta}{\partial x_i}\frac{\partial\eta}{\partial x_j}\right|^2, \quad (9.4.40)$$

and using the fact that

$$t^3(T-t)^3\sum_{i,j=1}^n\frac{\partial^2\beta}{\partial x_i\partial x_j}\frac{\partial\beta}{\partial x_i}\frac{\partial\beta}{\partial x_j}$$

$$= -\lambda^3e^{3\lambda\eta}\sum_{i,j}\frac{\partial^2\eta}{\partial x_i\partial x_j}\frac{\partial\eta}{\partial x_i}\frac{\partial\eta}{\partial x_j} - \lambda^4e^{3\lambda\eta}\sum_{i,j=1}^n\left|\frac{\partial\eta}{\partial x_i}\frac{\partial\eta}{\partial x_j}\right|^2,$$

together with (9.4.37), we obtain that

$$J_1 = 4s^3\lambda^3\left[\sum_{i,j=1}^n\int_0^T\int_\Omega\left(\frac{\partial^2\eta}{\partial x_i\partial x_j}\frac{\partial\eta}{\partial x_i}\frac{\partial\eta}{\partial x_j} + \frac{\lambda c_0}{2}\right)\frac{e^{3\lambda\eta}|\psi|^2}{t^3(T-t)^3}\,\mathrm{d}x\,\mathrm{d}t\right.$$

$$\left.-\int_0^T\frac{\lambda c_0}{2}\int_\Omega\frac{e^{3\lambda\eta}|\psi|^2}{t^3(T-t)^3}\,\mathrm{d}x\,\mathrm{d}t\right] + 4s^3\lambda^4\sum_{i,j=1}^n\int_0^T\int_\Omega\left|\frac{\partial\eta}{\partial x_i}\frac{\partial\eta}{\partial x_j}\right|^2\frac{e^{3\lambda\eta}|\psi|^2}{t^3(T-t)^3}\,\mathrm{d}x\,\mathrm{d}t.$$

The above relation implies, if we assume that λ satisfies

$$\lambda \geqslant \frac{4}{c_0} \max_{x\in \operatorname{clos}\,\Omega} \sum_{i,j=1}^{n} \left| \frac{\partial^2 \eta}{\partial x_i \partial x_j} \frac{\partial \eta}{\partial x_i} \frac{\partial \eta}{\partial x_j} \right| , \tag{9.4.41}$$

that

$$\begin{aligned} J_1 \geqslant{} & c_0 s^3 \lambda^4 \int_0^T \int_\Omega \frac{e^{3\lambda\eta(x)}|\psi|^2}{t^3(T-t)^3} \,\mathrm{d}x\,\mathrm{d}t \\ & + 4s^3\lambda^4 \int_0^T \int_\Omega \left[\sum_{i,j=1}^{n} \left| \frac{\partial \eta}{\partial x_i} \frac{\partial \eta}{\partial x_j} \right|^2 - \frac{c_0}{2} \right] \frac{e^{3\lambda\eta}|\psi|^2}{t^3(T-t)^3} \,\mathrm{d}x\,\mathrm{d}t . \end{aligned} \tag{9.4.42}$$

By using (9.4.40) it follows that, for every $t \in (0,T)$, we have

$$\begin{aligned} & \int_\Omega \left[\sum_{i,j=1}^{n} \left| \frac{\partial \eta}{\partial x_i} \frac{\partial \eta}{\partial x_j} \right|^2 - \frac{c_0}{2} \right] e^{3\lambda\eta} |\psi|^2 \,\mathrm{d}x \\ & \quad \geqslant \int_{\Omega\setminus\mathcal{O}} \left[\sum_{i,j=1}^{n} \left| \frac{\partial \eta}{\partial x_i} \frac{\partial \eta}{\partial x_j} \right|^2 - \frac{c_0}{2} \right] e^{3\lambda\eta} |\psi|^2 \,\mathrm{d}x - \frac{c_0}{2} \int_{\mathcal{O}} e^{3\lambda\eta}|\psi|^2 \,\mathrm{d}x \\ & \quad \geqslant -\frac{c_0}{2} \int_{\mathcal{O}} e^{3\lambda\eta}|\psi|^2 \,\mathrm{d}x . \end{aligned}$$

The above inequality and (9.4.42) yield that

$$J_1 \geqslant c_0 s^3 \lambda^4 \int_0^T \int_\Omega \frac{e^{3\lambda\eta(x)}|\psi|^2}{t^3(T-t)^3} \,\mathrm{d}x\,\mathrm{d}t - 2c_0 s^3\lambda^4 \int_0^T \int_{\mathcal{O}} \frac{e^{3\lambda\eta(x)}|\psi|^2}{t^3(T-t)^3} \,\mathrm{d}x\,\mathrm{d}t . \tag{9.4.43}$$

On the other hand, the facts that $\eta = 0$ on $\partial\Omega$ and $\eta > 0$ in Ω imply that

$$\frac{\partial \alpha}{\partial \nu}(x) = -\lambda \frac{\partial \eta}{\partial \nu}(x) e^{\lambda\eta} \geqslant 0 \qquad \forall\, x \in \partial\Omega ,$$

so that, by using (9.4.38), it follows that

$$J_2 \geqslant 0 . \tag{9.4.44}$$

The definition (9.4.39) of J_3, together with (9.4.10)–(9.4.7), implies that there exists a constant $C_2 > 0$ (depending only on Ω, $\mathcal{O}$ and T) such that for every $\lambda,\ s \geqslant 1$ we have

$$|J_3| \leqslant C_2 s^2 \lambda^2 \int_0^T \int_\Omega \frac{e^{3\lambda\eta(x)}}{t^3(T-t)^3} |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t . \tag{9.4.45}$$

Relation (9.4.36), combined with (9.4.41), (9.4.43), (9.4.44) and (9.4.45), implies that there exists a constant $C_3 > 0$, depending only on Ω, $\mathcal{O}$ and T, such that

$$\begin{aligned}
&2\langle M_1\psi, M_2\psi\rangle_{L^2(\Omega\times(0,T))} \\
&\geqslant C_3 s^3\lambda^4 \int_0^T\int_\Omega \frac{e^{3\lambda\eta(x)}|\psi|^2}{t^3(T-t)^3}\,\mathrm{d}x\,\mathrm{d}t - 2c_0 s^3\lambda^4 \int_0^T\int_{\mathcal{O}} \frac{e^{3\lambda\eta(x)}|\psi|^2}{t^3(T-t)^3}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - 4s\sum_{i,j=1}^n \int_0^T\int_\Omega \frac{\partial^2\beta}{\partial x_i\partial x_j}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2\int_0^T\int_\Omega \left(s(\Delta\beta)|\nabla\psi|^2 - s^3(\Delta\beta)|\nabla\beta|^2|\psi|^2\right)\mathrm{d}x\,\mathrm{d}t, \qquad (9.4.46)
\end{aligned}$$

provided that

$$s \geqslant 1, \quad \lambda \geqslant \max\left\{1, \frac{4}{c_0}\max_{x\in\operatorname{clos}\Omega}\sum_{i,j=1}^n \left|\frac{\partial^2\eta}{\partial x_i\partial x_j}\frac{\partial\eta}{\partial x_i}\frac{\partial\eta}{\partial x_j}\right|, \frac{2C_2}{c_0}\right\}. \qquad (9.4.47)$$

(Recall that c_0 has been defined in (9.4.40).) From the estimate (9.4.46), combined with (9.4.26), it follows that

$$\begin{aligned}
&\int_0^T \left(\|M_1\psi\|^2_{L^2(\Omega)} + \|M_2\psi\|^2_{L^2(\Omega)}\right)\mathrm{d}t + C_3 s^3\lambda^4\int_0^T\int_\Omega \frac{e^{3\lambda\eta(x)}|\psi|^2}{t^3(T-t)^3}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \leqslant \int_0^T \|g_s\|^2\,\mathrm{d}t + 2c_0 s^3\lambda^4\int_0^T\int_{\mathcal{O}} \frac{e^{3\lambda\eta(x)}|\psi|^2}{t^3(T-t)^3}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad + 4s\sum_{i,j=1}^n\int_0^T\int_\Omega \frac{\partial^2\beta}{\partial x_i\partial x_j}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j} - 2J_4, \qquad (9.4.48)
\end{aligned}$$

where

$$J_4 = \int_0^T\int_\Omega \left(s(\Delta\beta)|\nabla\psi|^2 - s^3(\Delta\beta)|\nabla\beta|^2|\psi|^2\right)\mathrm{d}x\,\mathrm{d}t. \qquad (9.4.49)$$

On the other hand, (9.4.25) implies that

$$\int_0^T \|g_s\|^2\,\mathrm{d}t \leqslant 2\int_0^T \|f_s\|^2\,\mathrm{d}t + 2s^2\int_0^T \|(\Delta\beta)\psi\|^2\,\mathrm{d}t.$$

By using (9.4.9) it follows that

$$\int_0^T \|g_s\|^2\,\mathrm{d}t \leqslant 2\int_0^T \|f_s\|^2\,\mathrm{d}t + 2C_1^2 s^2\lambda^4\int_0^T\int_\Omega \frac{e^{2\lambda\eta}|\psi|^2}{t^2(T-t)^2}\,\mathrm{d}x\,\mathrm{d}t.$$

If we take, in the above inequality,

$$s \geqslant \max\left\{1, \frac{4C_1^2T^2}{C_3}\right\} \qquad (9.4.50)$$

and we use the elementary fact that

$$\frac{1}{t^2(T-t)^2} \leqslant \frac{T^2}{t^3(T-t)^3} \qquad \forall\, t \in (0,T),$$

we obtain

$$\int_0^T \|g_s\|^2 \,\mathrm{d}t \leqslant 2\int_0^T \|f_s\|^2 \,\mathrm{d}t + \frac{C_3 s^3\lambda^4}{2}\int_0^T\int_\Omega \frac{e^{3\lambda\eta}|\psi|^2}{t^3(T-t)^3}\,\mathrm{d}x\,\mathrm{d}t.$$

The above estimate and (9.4.48) yield that

$$\begin{aligned}\int_0^T \left(\|M_1\psi\|^2_{L^2(\Omega)} + \|M_2\psi\|^2_{L^2(\Omega)}\right)\,\mathrm{d}t + \frac{C_3 s^3\lambda^4}{2}\int_0^T\int_\Omega \frac{e^{3\lambda\eta(x)}|\psi|^2}{t^3(T-t)^3}\,\mathrm{d}x\,\mathrm{d}t \\ \leqslant 2\int_0^T \|f_s\|^2\,\mathrm{d}t + 2c_0 s^3\lambda^4\int_0^T\int_{\mathcal{O}} \frac{e^{3\lambda\eta(x)}|\psi|^2}{t^3(T-t)^3}\,\mathrm{d}x\,\mathrm{d}t \\ + 4s\sum_{i,j=1}^n \int_0^T\int_\Omega \frac{\partial^2\beta}{\partial x_i\partial x_j}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j} - 2J_4, \end{aligned} \tag{9.4.51}$$

provided that s and λ satisfy (9.4.47) and (9.4.50).

Third step. We estimate J_4, defined in (9.4.49). First we notice that

$$J_4 = \int_0^T\int_\Omega \left\{s(\Delta\beta)|\nabla\psi|^2 - s(\Delta\beta)\psi\left(s^2|\nabla\beta|^2\psi\right)\right\}\,\mathrm{d}x\,\mathrm{d}t.$$

The above relation, combined with (9.4.22)–(9.4.25), implies that

$$\begin{aligned} J_4 &= \int_0^T\int_\Omega s(\Delta\beta)|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t - s\int_0^T\int_\Omega(\Delta\beta)\psi\left(s\frac{\partial\beta}{\partial t}\psi - M_2\psi - \Delta\psi\right)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T\int_\Omega s(\Delta\beta)|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t - s\int_0^T\int_\Omega(\Delta\beta)\psi\left(s\frac{\partial\beta}{\partial t}\psi + M_1\psi - g_s - \Delta\psi\right)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T\int_\Omega s(\Delta\beta)|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + s\int_0^T\int_\Omega(\Delta\beta)\psi\left(f_s - M_1\psi - s\frac{\partial\beta}{\partial t}\psi + \Delta\psi + s(\Delta\beta)\psi\right)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

By using the fact that $\operatorname{div}(\psi\nabla\psi) = |\nabla\psi|^2 + \psi\Delta\psi$ in the above formula, we obtain

$$\begin{aligned} J_4 &= s\int_0^T\int_\Omega(\Delta\beta)\operatorname{div}(\psi\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + s\int_0^T\int_\Omega(\Delta\beta)\psi\left(f_s - M_1\psi - s\frac{\partial\beta}{\partial t}\psi + s(\Delta\beta)\psi\right)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

A double integration by parts with respect to x in the first term on the right-hand side of the above relation implies that

$$J_4 = \frac{s}{2}\int_0^T\int_\Omega (\Delta^2\beta)|\psi|^2\,\mathrm{d}x\,\mathrm{d}t - s^2\int_0^T\int_\Omega (\Delta\beta)\frac{\partial\beta}{\partial t}|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \qquad (9.4.52)$$
$$+ s\int_0^T\int_\Omega (\Delta\beta)\psi\left(f_s - M_1\psi + s(\Delta\beta)\psi\right)\mathrm{d}x\,\mathrm{d}t.$$

On the other hand, by using the elementary inequalities

$$\begin{aligned} ab &\leqslant \frac{a^2}{4} + 2b^2 && \forall\, a,b\in\mathbb{R}, \\ |a+b|^2 &\leqslant 2(a^2+b^2) && \forall\, a,b\in\mathbb{R}, \end{aligned}$$

we have

$$\begin{aligned} &\left| s\int_0^T\int_\Omega (\Delta\beta)\psi\left(f_s - M_1\psi + s(\Delta\beta)\psi\right)\mathrm{d}x\,\mathrm{d}t\right| \\ &\quad\leqslant \int_0^T\int_\Omega |f_s - M_1\psi|\cdot|s(\Delta\beta)\psi|\,\mathrm{d}x\,\mathrm{d}t + s^2\int_0^T\int_\Omega (\Delta\beta)^2|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leqslant \frac{1}{4}\int_0^T\int_\Omega |f_s - M_1\psi|^2\,\mathrm{d}x\,\mathrm{d}t + 3s^2\int_0^T\int_\Omega (\Delta\beta)^2|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leqslant \frac{1}{2}\int_0^T\left(\|M_1\psi\|^2_{L^2(\Omega)} + \|f_s\|^2_{L^2(\Omega)}\right)\mathrm{d}t + 3s^2\int_0^T\int_\Omega (\Delta\beta)^2|\psi|^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

From the above estimate and (9.4.52), it follows that

$$|J_4| \leqslant \frac{s}{2}\int_0^T\int_\Omega (\Delta^2\beta)|\psi|^2\,\mathrm{d}x\,\mathrm{d}t - s^2\int_0^T\int_\Omega (\Delta\beta)\frac{\partial\beta}{\partial t}|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \qquad (9.4.53)$$
$$+ \frac{1}{2}\int_0^T\left(\|M_1\psi\|^2_{L^2(\Omega)} + \|f_s\|^2_{L^2(\Omega)}\right)\mathrm{d}t + 3s^2\int_0^T\int_\Omega (\Delta\beta)^2|\psi|^2\,\mathrm{d}x\,\mathrm{d}t.$$

On the other hand, by using (9.4.5) it is easy to check that there exists a constant $C(\Omega,\mathcal{O})>0$ such that

$$|\Delta^2\beta| = \frac{1}{t(T-t)}\left|\Delta^2(e^{\lambda\eta})\right| \leqslant \frac{C(\Omega,\mathcal{O})}{t(T-t)}\lambda^4 e^{3\lambda\eta}.$$

The above formula, combined with (9.4.11), (9.4.53) and with the fact that

$$\frac{1}{t(T-t)} \leqslant \frac{T^4}{t^3(T-t)^3} \qquad \forall\, t\in(0,t), \qquad (9.4.54)$$

yields the existence of a constant $C_4 > 0$ (depending only on Ω, $\mathcal{O}$ and T) such that, for every $s, \lambda \geqslant 1$ we have

$$|J_4| \leqslant \frac{1}{2}\int_0^T \left(\|M_1\psi\|^2_{L^2(\Omega)} + \|f_s\|^2_{L^2(\Omega)}\right) \mathrm{d}t + C_4 s^2\lambda^4 \int_0^T \int_\Omega \frac{e^{3\lambda\eta}}{t^3(T-t)^3}|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Fourth step. From the above formula and (9.4.51), we obtain that there exists $C_5 > 0$ (depending only on Ω, $\mathcal{O}$ and T) such that, for every s and λ satisfying (9.4.47) and (9.4.50), with $s \geqslant \frac{8C_4}{C_3}$, we have

$$\begin{aligned}\int_0^T \left(\|M_1\psi\|^2_{L^2(\Omega)} + \|M_2\psi\|^2_{L^2(\Omega)}\right) \mathrm{d}t + C_5 s^3\lambda^4 \int_0^T \int_\Omega \frac{e^{3\lambda\eta(x)}|\psi|^2}{t^3(T-t)^3} \,\mathrm{d}x\,\mathrm{d}t \\ \leqslant 5\int_0^T \|f_s\|^2_{L^2(\Omega)} \,\mathrm{d}t + 4c_0 s^3\lambda^4 \int_0^T \int_{\mathcal{O}} \frac{e^{3\lambda\eta(x)}|\psi|^2}{t^3(T-t)^3} \,\mathrm{d}x\,\mathrm{d}t \\ + 8s\sum_{i,j=1}^n \int_0^T \int_\Omega \frac{\partial^2\beta}{\partial x_i\partial x_j}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}.\end{aligned} \tag{9.4.55}$$

Now we transform the above estimate into an inequality involving the terms containing $\Delta\psi$, $\nabla\psi$ and $\frac{\partial\psi}{\partial t}$, which occur on the left-hand side of (9.4.18). We begin with the term containing $\Delta\psi$ by noticing that

$$\begin{aligned}&\frac{1}{s}\int_0^T \int_\Omega t(T-t)e^{-\lambda\eta}|\Delta\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \frac{1}{s}\int_0^T \int_\Omega t(T-t)e^{-\lambda\eta}\left(M_2\psi - s^2|\nabla\beta|^2\psi - s\frac{\partial\beta}{\partial t}\psi\right)^2 \mathrm{d}x\,\mathrm{d}t \\ &\quad \leqslant \frac{3}{s}\int_0^T \int_\Omega t(T-t)e^{-\lambda\eta}|M_2\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + 3s^3\int_0^T \int_\Omega t(T-t)e^{-\lambda\eta}|\nabla\beta|^4|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + 3s\int_0^T \int_\Omega t(T-t)e^{-\lambda\eta}\left|\frac{\partial\beta}{\partial t}\right|^2 |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The above estimate, combined with (9.4.8) and (9.4.6), yields that

$$\begin{aligned}&\frac{1}{s}\int_0^T \int_\Omega t(T-t)e^{-\lambda\eta}|\Delta\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqslant \frac{3T^2}{s}\int_0^T \|M_2\psi\|^2 \,\mathrm{d}t + C(\Omega, \mathcal{O}, T)s^3\lambda^4 \int_0^T \int_\Omega \frac{e^{3\lambda\eta}}{t^3(T-t)^3}|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{9.4.56}$$

Now we give an upper bound for the term containing $\nabla\psi$ on the left-hand side of (9.4.18). Integrating by parts, we obtain

$$\begin{aligned}&2s\lambda^2\int_0^T \int_\Omega \frac{e^{\lambda\eta}}{t(T-t)}|\nabla\psi|^2 \,\mathrm{d}x\,\mathrm{d}t = 2s\lambda^2\int_0^T \int_\Omega \frac{e^{\lambda\eta}}{t(T-t)}(\nabla\psi)\cdot(\nabla\psi) \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = 2s\lambda^2\int_0^T \int_\Omega \frac{e^{\lambda\eta}}{t(T-t)}(-\Delta\psi)\psi \,\mathrm{d}x\,\mathrm{d}t - 2s\lambda^3\int_0^T \int_\Omega \frac{e^{\lambda\eta}}{t(T-t)}(\nabla\eta\cdot\nabla\psi)\psi \,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= 2s\lambda^2 \int_0^T \int_\Omega \frac{e^{\lambda\eta}}{t(T-t)}(-\Delta\psi)\psi \,\mathrm{d}x\,\mathrm{d}t - s\lambda^3 \int_0^T \int_\Omega \frac{e^{\lambda\eta}}{t(T-t)}(\nabla\eta\cdot\nabla|\psi|^2)\,\mathrm{d}x\,\mathrm{d}t \\
&= 2s\lambda^2 \int_0^T \int_\Omega \frac{e^{\lambda\eta}}{t(T-t)}(-\Delta\psi)\psi \,\mathrm{d}x\,\mathrm{d}t + s\lambda^3 \int_0^T \int_\Omega \frac{e^{\lambda\eta}}{t(T-t)}(\Delta\eta)|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + s\lambda^4 \int_0^T \int_\Omega \frac{e^{\lambda\eta}}{t(T-t)}|\nabla\eta|^2|\psi|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

The above estimate, combined with (9.4.54) and with other elementary inequalities, yields that, for every $s \geqslant 1$,

$$
\begin{aligned}
&2s\lambda^2 \int_0^T \int_\Omega \frac{e^{\lambda\eta}}{t(T-t)}|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = -2\int_0^T \int_\Omega \left(\frac{\sqrt{t(T-t)}}{\sqrt{s}}e^{-\frac{\lambda\eta}{2}}\Delta\psi\right)\left(\frac{s^{\frac{3}{2}}\lambda^2 e^{3\frac{\lambda\eta}{2}}}{t^{\frac{3}{2}}(T-t)^{\frac{3}{2}}}\psi\right)\mathrm{d}x\,\mathrm{d}t \\
&\qquad + s\lambda^3 \int_0^T \int_\Omega \frac{e^{\lambda\eta}}{t(T-t)}(\Delta\eta)|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + s\lambda^4 \int_0^T \int_\Omega \frac{e^{\lambda\eta}}{t(T-t)}|\nabla\eta|^2|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \frac{1}{s}\int_0^T \int_\Omega t(T-t)e^{-\lambda\eta}|\Delta\psi|^2\,\mathrm{d}x\,\mathrm{d}t + C(\Omega,\mathcal{O})s^3\lambda^4 \int_0^T \int_\Omega \frac{e^{3\lambda\eta}}{t^3(T-t)^3}|\psi|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

From the above and (9.4.56), it follows that if $s \geqslant 3T^2$, then

$$
\begin{aligned}
&2s\lambda^2 \int_0^T \int_\Omega t^{-1}(T-t)^{-1}e^{\lambda\eta}|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \int_0^T \|M_2\psi\|^2\,\mathrm{d}t + s^3\lambda^4 \int_0^T \int_\Omega \frac{e^{3\lambda\eta}}{t^3(T-t)^3}|\psi|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{9.4.57}
$$

Now we move to the term containing $\frac{\partial\psi}{\partial t}$. By using (9.4.23), (9.4.8) and some elementary inequalities, it follows that

$$
\begin{aligned}
\frac{1}{s}\int_0^T \int_\Omega t(T-t)\left|\frac{\partial\psi}{\partial t}\right|^2\mathrm{d}x\,\mathrm{d}t &= \frac{1}{s}\int_0^T \int_\Omega t(T-t)\left|M_1\psi + 2s\nabla\beta\cdot\nabla\psi\right|^2\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \frac{2}{s}\int_0^T t(T-t)\|M_1\psi\|^2\,\mathrm{d}t + 4s\int_0^T \int_\Omega t(T-t)|\nabla\beta|^2|\nabla\psi|^2 \\
&\leqslant \frac{2T^2}{s}\int_0^T \|M_1\psi\|^2 + 4s\lambda^2 C_1 \int_0^T \int_\Omega \frac{e^{2\lambda\eta}}{t(T-t)}|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

From the above it follows that if

$$
s \geqslant 3T^2, \tag{9.4.58}
$$

then

$$\frac{1}{s}\int_0^T\int_\Omega t(T-t)\left|\frac{\partial\psi}{\partial t}\right|^2 \mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \int_0^T \|M_1\psi\|^2\,\mathrm{d}t + 4s\lambda^2 C_1\int_0^T\int_\Omega \frac{e^{2\lambda\eta}}{t(T-t)}|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Let us now fix λ satisfying (9.4.47). By combining (9.4.55), (9.4.56), (9.4.57) and the last inequality, it follows that there exists $C_6 > 0$ (depending only on Ω, $\mathcal{O}$ and T) such that the inequality

$$\begin{aligned}
&\frac{1}{s}\int_0^T\int_\Omega t(T-t)e^{-\lambda\eta}|\Delta\psi|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{s}\int_0^T\int_\Omega t(T-t)\left|\frac{\partial\psi}{\partial t}\right|^2\mathrm{d}x\,\mathrm{d}t \\
&\quad + s\lambda^2\int_0^T\int_\Omega \frac{e^{\lambda\eta}}{t(T-t)}|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^4\int_0^T\int_\Omega \frac{e^{3\lambda\eta(x)}|\psi|^2}{t^3(T-t)^3}\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant C_6\Bigg(\int_0^T \|f_s\|^2\,\mathrm{d}t + s^3\lambda^4\int_0^T\int_{\mathcal{O}} \frac{e^{3\lambda\eta(x)}|\psi|^2}{t^3(T-t)^3}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + s\sum_{i,j=1}^n\int_0^T\int_\Omega \frac{\partial^2\beta}{\partial x_i\partial x_j}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t\Bigg)
\end{aligned}\tag{9.4.59}$$

holds for every s satisfying (9.4.50), (9.4.58) together with $s \geqslant \frac{8C_4}{C_3}$. In order to eliminate the last term on the right-hand side of (9.4.59), we note that

$$\begin{aligned}
\sum_{i,j=1}^n \frac{\partial^2\beta}{\partial x_i\partial x_j}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j} &= -e^{\lambda\eta}t^{-1}(T-t)^{-1}\left(\lambda^2\frac{\partial\eta}{\partial x_i}\frac{\partial\eta}{\partial x_j} + \lambda\frac{\partial^2\eta}{\partial x_i\partial x_j}\right)\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j} \\
&= -\frac{\lambda^2 e^{\lambda\eta}}{t(T-t)}\left(\nabla\eta\cdot\nabla\psi\right)^2 - \frac{\lambda e^{\lambda\eta}}{t(T-t)}\left(\frac{\partial^2\eta}{\partial x_i\partial x_j}\right)\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j} \\
&\leqslant C(\Omega,\mathcal{O})\frac{\lambda e^{\lambda\eta}}{t(T-t)}|\nabla\psi|^2.
\end{aligned}$$

From the above estimate and (9.4.59) we get the desired conclusion. □

We are now in a position to prove the main result in this section.

Proof of Theorem 9.4.1. We fix λ satisfying (9.4.47) and we consider an arbitrary s satisfying (9.4.50), (9.4.58) together with $s \geqslant \frac{8C_4}{C_3}$.

By using (9.4.17) and the fact that $\rho = e^{\beta}$, it follows that $\psi = e^{-s\beta}\varphi$, so that

$$\nabla\psi = e^{-s\beta}(\nabla\varphi - s\varphi\nabla\beta).$$

From the above formula and the elementary inequality

$$|a - sb|^2 \geqslant \frac{a^2}{2} - s^2b^2 \qquad \forall\, a,\, b\in\mathbb{R},$$

it follows that

$$\begin{aligned} s\int_0^T \frac{1}{t(T-t)}|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t &= s\int_0^T \frac{e^{-2s\beta}}{t(T-t)}|\nabla\varphi - s\varphi\nabla\beta|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\geqslant \frac{s}{2}\int_0^T\int_\Omega \frac{e^{-2s\beta}}{t(T-t)}|\nabla\varphi|^2 - s^3\int_0^T \frac{e^{-2s\beta}}{t(T-t)}|\nabla\beta|^2|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\geqslant \frac{s}{2}\int_0^T \frac{e^{-2s\beta}}{t(T-t)}|\nabla\varphi|^2\mathrm{d}x\mathrm{d}t - K_1 s^3\int_0^T\int_\Omega \frac{e^{-2s\beta}}{t^3(T-t)^3}|\varphi|^2\mathrm{d}x\mathrm{d}t, \end{aligned}$$

where $K_1 > 0$ depends only on Ω and $\mathcal{O}$. The last formula implies that for every $\varepsilon \in (0, \frac{1}{2K_1})$ we have

$$\begin{aligned} &\varepsilon s\int_0^T \frac{1}{t(T-t)}|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t + s^3\int_0^T\int_\Omega \frac{1}{t^3(T-t)^3}|\psi|^2\mathrm{d}x\mathrm{d}t \\ &\quad = \varepsilon s\int_0^T \frac{1}{t(T-t)}|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t + s^3\int_0^T\int_\Omega \frac{e^{-2s\beta}}{t^3(T-t)^3}|\varphi|^2\mathrm{d}x\mathrm{d}t \\ &\quad \geqslant \frac{\varepsilon s}{2}\int_0^T \frac{e^{-2s\beta}}{t(T-t)}|\nabla\varphi|^2\mathrm{d}x\mathrm{d}t + \frac{s^3}{2}\int_0^T\int_\Omega \frac{e^{-2s\beta}}{t^3(T-t)^3}|\varphi|^2\mathrm{d}x\mathrm{d}t. \end{aligned}$$

The above estimate, combined with (9.4.18), yields the conclusion. □

9.5 Final state observability without geometric conditions

In this section $\Omega \subset \mathbb{R}^n$ is an open set with boundary of class C^2, $X = L^2(\Omega)$, $b \in L^\infty(\Omega; \mathbb{R}^n)$, $c \in L^\infty(\Omega, \mathbb{R})$ and A is the operator of domain $\mathcal{D}(A) = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$, defined by

$$Af = \Delta f + b\cdot\nabla f + cf \qquad \forall\, f \in \mathcal{D}(A),$$

where $\cdot$ stands for the inner product in $\mathbb{R}^n$. We know from Example 5.4.4 that A generates a semigroup $\mathbb{T}$ on X that corresponds to the convection-diffusion equation on Ω, with homogeneous Dirichlet boundary conditions.

Let $\mathcal{O}$ be a non-empty open subset of Ω and let $C_0 \in \mathcal{L}(X)$ be defined by

$$C_0 f = f\chi_{\mathcal{O}},$$

where $\chi_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$. The norm on X is denoted by $\|\cdot\|$.

In this section we show that the geometric assumptions on the observation region which have been used in the previous section are not necessary for the final state observability of a convection-diffusion equation with distributed observation.

Theorem 9.5.1. *The pair (A, C_0) is final state observable in any time $\tau > 0$. In terms of PDEs, this means that for every $\tau > 0$ there exists a constant $k_\tau > 0$ such that, for every $z_0 \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$, the solution of*

$$\frac{\partial z}{\partial t}(x,t) = \Delta z(x,t) + b(x) \cdot \nabla z(x,t) + c(x) z(x,t), \quad x \in \Omega, \ t \geqslant 0, \tag{9.5.1}$$

$$z(x,t) = 0, \quad x \in \partial\Omega, \ t \geqslant 0, \tag{9.5.2}$$

$$z(x,0) = z_0(x), \quad x \in \Omega, \tag{9.5.3}$$

satisfies

$$\int_0^\tau \int_{\mathcal{O}} |z(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \geqslant k_\tau^2 \int_\Omega |z(x,\tau)|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{9.5.4}$$

Proof. Let α be the function constructed in Section 9.4. According to Theorem 9.4.1, it follows that there exist s_0, $C_0 > 0$, depending only on Ω, $\mathcal{O}$ and τ, such that, for all $s \geqslant s_0$, we have

$$\int_0^\tau \int_\Omega e^{\frac{-2s\alpha(x)}{t(\tau-t)}} \left[s t^{-1}(\tau-t)^{-1} |\nabla z|^2 + \frac{s^3}{t^3(T-t)^3} |z|^2 \right] \mathrm{d}x\,\mathrm{d}t \tag{9.5.5}$$
$$\leqslant C_0 \left[s^3 \int_{\mathcal{O}\times(0,\tau)} e^{\frac{-2s\alpha(x)}{t(\tau-t)}} \frac{|z|^2}{t^3(\tau-t)^3} \,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \!\! \int_\Omega e^{\frac{-2s\alpha(x)}{t(\tau-t)}} \left(|cz|^2 + |b \cdot \nabla z|^2 \right) \mathrm{d}x\,\mathrm{d}t \right].$$

On the other hand, it is easy to see that

$$\int_0^\tau \int_\Omega e^{\frac{-2s\alpha(x)}{t(\tau-t)}} |cz|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \tau^6 \, \|c\|_{L^\infty(\Omega}^2 \int_0^\tau \int_\Omega e^{\frac{-2s\alpha(x)}{t(\tau-t)}} t^{-3}(\tau-t)^{-3} |z|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{9.5.6}$$

$$\int_0^\tau \int_\Omega e^{\frac{-2s\alpha(x)}{t(\tau-t)}} |b \cdot \nabla z|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \tau^2 \|b\|_{L^\infty(\Omega)}^2 \int_0^\tau \int_\Omega e^{\frac{-2s\alpha(x)}{t(\tau-t)}} t^{-1}(\tau-t)^{-1} |\nabla z|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{9.5.7}$$

Relations (9.5.5)–(9.5.7) imply that there exist $s_1, \ C_1 > 0$, depending only on Ω, $\mathcal{O}$, τ, $\|b\|_\infty$ and $\|c\|_\infty$, such that, for all $s \geqslant s_1$, we have

$$\int_0^\tau \int_\Omega e^{\frac{-2s\alpha(x)}{t(\tau-t)}} \frac{|\nabla z|^2}{t(\tau-t)} \,\mathrm{d}x\,\mathrm{d}t \leqslant C_1 s^2 \int_0^\tau \int_{\mathcal{O}} e^{\frac{-2s\alpha(x)}{t(\tau-t)}} \frac{|z|^2}{t^3(\tau-t)^3} \,\mathrm{d}x\,\mathrm{d}t. \tag{9.5.8}$$

It is easy to check that there exist two constants $C_2, C_3 > 0$, depending only on $\Omega, \ \mathcal{O}, \ \tau$, such that

$$\frac{e^{\frac{-2s\alpha(x)}{t(\tau-t)}}}{t^3(\tau-t)^3} \geqslant C_2 \qquad \forall \, (x,t) \in \Omega \times \left(\frac{\tau}{4}, \frac{3\tau}{4} \right),$$
$$\frac{e^{\frac{-2s\alpha(x)}{t(\tau-t)}}}{t^3(\tau-t)^3} \leqslant C_3 \qquad \forall \, (x,t) \in \Omega \times (0,\tau).$$

The above two relations and (9.5.8) imply that there exists $C_4 > 0$, depending only on Ω, $\mathcal{O}$, τ, $\|b\|_\infty$ and $\|c\|_\infty$, such that

$$\int_{\frac{\tau}{4}}^{\frac{3\tau}{4}} \int_\Omega |\nabla z|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C_4 \int_0^\tau \int_{\mathcal{O}} |z|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{9.5.9}$$

On the other hand, if M and ω are as in (2.1.4), then, for every $t \in \left(\frac{\tau}{4}, \frac{3\tau}{4}\right)$, we have

$$\|z(\tau)\| \leqslant M e^{\omega(\tau-t)} \|\mathbb{T}_t z_0\| \leqslant M e^{\frac{3\omega\tau}{4}} \|z(t)\|.$$

This fact, combined with (9.5.9) and with the Poincaré inequality, clearly implies the conclusion (9.5.4). $\square$

9.6 Remarks and bibliographical notes on Chapter 9

General remarks. The observability and the controllability of linear parabolic PDEs in one space dimension has been extensively studied about thirty years ago in a series of papers beginning with Fattorini and Russell [62]. The corresponding results in several space dimensions have been obtained much later, as described below. More recently, several researchers became interested in finding precise estimates of the final state observability constants when the observation time tends to zero. We did not tackle this challenging issue in this book, but we refer the reader to Zuazua [244], Miller [169] and to Tenenbaum and Tucsnak [218] for results on this topic.

Section 9.2. The fact that the exact observability of a system governed by the wave equation implies the final state observability of a corresponding system governed by the heat equation has been proved by Russell in [197, 198] (see also Seidman [205]). The abstract version given in this book is closer to the approach in Avdonin and Ivanov [9]. Lemma 9.2.3 is a key technical result which, in the form shown in this book, has been proved in [218]. For earlier versions of this result we refer the reader to Bombieri, Friedlander and Iwaniec [21] and to Jaffard and Micu [124]. A different proof for a generalization of Theorem 9.2.2 has been given recently in Miller [171], using the "control transmutation method". This generalization of Theorem 9.2.2 eliminates the assumption that A_0 is diagonalizable.

Section 9.3. The result in Proposition 9.3.6 has been obtained in Fattorini and Russell [62, 63] by tackling directly the one-dimensional parabolic equation. The observability for the system (9.3.4) has been used by Fernandez-Cara and Zuazua [67] to show that the result in Proposition 9.3.6 holds when a is less smooth, namely a function with bounded variation. By a different method, this result has been generalized recently in Alessandrini and Escauriaza [3] to the case $a \in L^\infty$.

Section 9.4. The type of estimates derived in this section have been introduced by Carleman in [29] in order to prove unique continuation results for linear elliptic

PDEs in two space dimensions. Their use for global estimates for the heat equation is due to Fursikov and Imanuvilov in [69]. Our approach follows Fernández-Cara and Zuazua [66]. For a proof of Theorem 9.4.1 under the assumption that $\partial\Omega$ is only of class C^2, we refer the reader to Fernández-Cara and Guerrero [64].

Section 9.5. The result in Theorem 9.5.1 has been obtained independently by Lebeau and Robbiano in [151] and by Fursikov and Imanuvilov in [69]. These works were the departure point of a series of papers devoted to the observability and controllability of other parabolic equations, linear or nonlinear, and in particular for the Navier–Stokes system (see, for instance, Barbu [14], Fabre [60], Fernández-Cara et al. [65]).

Chapter 10

Boundary Control Systems

Notation. We continue to use the notation listed at the beginning of Chapter 2. As in earlier chapters, if $\mathbb{T}$ is a strongly continuous semigroup on the Hilbert space X, with generator A, then the spaces X_1 and X_{-1} are as in Section 2.10 and the extension of A to X is still denoted by A.

10.1 What is a boundary control system?

In this section we introduce boundary control systems, in particular well-posed boundary control systems. Usually, boundary control systems are defined as systems having inputs and outputs. However, the novelty resides only in the equations linking the input to the state. For this reason, here we introduce a restricted version of the concept of boundary control system, which do not have outputs.

In Chapter 4 we have discussed infinite-dimensional control systems for which the evolution of the state z is described by the differential equation $\dot{z}(t) = Az(t) + Bu(t)$, where u is the input signal. Systems described by linear PDEs with non-homogeneous boundary conditions often appear in the following, quite different looking, form:

$$\dot{z}(t) = Lz(t), \qquad Gz(t) = u(t). \tag{10.1.1}$$

Often (but not necessarily) L is a differential operator and G is a boundary trace operator. It is not obvious what is meant by solutions of the above equations, and it is clear that some assumptions are needed in order to be able to translate these equations into the familiar form $\dot{z}(t) = Az(t) + Bu(t)$. In what follows, we assume that U, Z and X are complex Hilbert spaces such that

$$Z \subset X,$$

with continuous embedding. We shall call U the *input space*, Z the *solution space* and X the *state space*.

Definition 10.1.1. A *boundary control system* on U, Z and X is a pair of operators (L, G), where

$$L \in \mathcal{L}(Z, X), \qquad G \in \mathcal{L}(Z, U),$$

if there exists a $\beta \in \mathbb{C}$ such that the following properties hold:

(i) G is onto,

(ii) Ker G is dense in X,

(iii) $\beta I - L$ restricted to Ker G is onto,

(iv) Ker $(\beta I - L) \cap$ Ker $G = \{0\}$.

We think of the two operators in this definition as determining a system via (10.1.1). Broadly, our aim is to translate these equations into the familar form $\dot{z}(t) = Az(t) + Bu(t)$. The boundary control system will be called well-posed if A generates a semigroup on X and B is admissible for this semigroup.

With the assumptions of the last definition, we introduce the Hilbert space X_1 and the operator A by

$$X_1 = \text{Ker } G, \qquad A = L|_{X_1}. \tag{10.1.2}$$

Obviously, X_1 is a closed subspace of Z and $A \in \mathcal{L}(X_1, X)$. Condition (iii) means that $\beta I - A$ is onto. Condition (iv) means that Ker $(\beta I - A) = \{0\}$. Thus, (iii) and (iv) together are equivalent to the fact that $\beta \in \rho(A)$, so that

$$(\beta I - A)^{-1} \in \mathcal{L}(X).$$

In fact, $(\beta I - A)^{-1} \in \mathcal{L}(X, X_1)$, so that the norm on X_1 is equivalent to the norm

$$\|z\|_1 = \|(\beta I - A)z\|,$$

which has been discussed in detail in Section 2.10. As usual, we define the space X_{-1} as the completion of X with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1}z\|$. Then A has an extension, also denoted by A, such that $A \in \mathcal{L}(X, X_{-1})$, as explained in Section 2.10. Note that, so far, A has not been assumed to be a generator.

Proposition 10.1.2. *Let (L, G) be a boundary control system on U, Z and X. Let A and X_{-1} be as introduced earlier. Then there exists a unique operator $B \in \mathcal{L}(U, X_{-1})$ such that*

$$L = A + BG, \tag{10.1.3}$$

where A is regarded as an operator from X to X_{-1}. For every $\beta \in \rho(A)$ we have that $(\beta I - A)^{-1}B \in \mathcal{L}(U, Z)$ and

$$G(\beta I - A)^{-1}B = I, \tag{10.1.4}$$

so that in particular, B is bounded from below.

Proof. Since G is onto, it has at least one bounded right inverse $H \in \mathcal{L}(U, Z)$. We put

$$B = (L - A)H. \tag{10.1.5}$$

From $G(I - HG) = 0$ we see that the range of $I - HG$ is in $\operatorname{Ker} G = X_1$, so that $(L - A)(I - HG) = 0$. Thus we get that $BG = (L - A)HG = L - A$, as required in (10.1.3). It is easy to see that B is unique. To prove (10.1.4), first we rewrite (10.1.5) in the form

$$(\beta I - A)H - (\beta I - L)H = B.$$

If we apply $(\beta I - A)^{-1}$ to both sides, we get

$$H - (\beta I - A)^{-1}(\beta I - L)H = (\beta I - A)^{-1}B,$$

which shows that indeed $(\beta I - A)^{-1}B \in \mathcal{L}(U, Z)$. Therefore, we can apply G to both sides above and then the second term on the left-hand side disappears, because $X_1 = \operatorname{Ker} G$. Since $GH = I$, we obtain (10.1.4). □

When L, G, A and B are as in the above proposition, we say that A is the *generator* of the boundary control system (L, G) and B is the *control operator* of (L, G).

Remark 10.1.3. It follows from (10.1.4) that B is *strictly unbounded* with respect to X, meaning that $X \cap BU = \{0\}$. Another consequence of (10.1.4) is that

$$Z = X_1 + (\beta I - A)^{-1}BU.$$

Indeed, for each $z \in Z$, denoting $v = Gz$, we have $z = z_1 + (\beta I - A)^{-1}Bv$, where $z_1 \in X_1$ (because $Gz_1 = 0$). The converse inclusion is trivial. Thus, Z coincides with the space defined in (4.2.9). Moreover, by the closed-graph theorem, the norm of Z is equivalent to the norm defined after (4.2.9).

Remark 10.1.4. As a consequence of Proposition 10.1.2, (10.1.1) can be rewritten equivalently as

$$\dot{z}(t) = Az(t) + Bu(t), \quad \text{with} \quad \dot{z}(t) \in X. \tag{10.1.6}$$

This equivalence is meant in the algebraic sense, without making at this stage any assumptions about the existence or uniqueness of solutions for these equations (for example, we have not assumed that A generates a semigroup). Indeed, the transformation from (10.1.1) to (10.1.6) is obvious from (10.1.3). Conversely, if (10.1.6) holds, then applying $G(\beta I - A)^{-1}$ to both sides, we obtain with (10.1.4) that $Gz(t) = u(t)$. Now from (10.1.3) it follows that $\dot{z}(t) = Lz(t)$.

When transforming (10.1.1) into (10.1.6), the control operator B is determined in principle from (10.1.5). However, this way of determining B is not satisfactory for most examples, because it is awkward to work with the extended operator A and with the right inverse H. Thus, we need more practical ways to determine B. The following two remarks offer two ways to find B from L and G.

Remark 10.1.5. The following fact is an easy consequence of Proposition 10.1.2 (we use the notation of the proposition): For every $\mathrm{v} \in U$ and every $\beta \in \rho(A)$, the vector $z = (\beta I - A)^{-1} B\mathrm{v}$ is the unique solution of the "abstract elliptic problem"

$$Lz = \beta z \,, \qquad Gz = \mathrm{v} \,.$$

For many L and G, this problem has a well-known solution, and it is easier to describe $z \in X$ than to describe $B\mathrm{v} \in X_{-1}$, since X is usually a more "natural" space than X_{-1} (see the other sections of this chapter).

Remark 10.1.6. Often we need to express B^* in terms of L and G. Instead of finding the control operator B and then computing its adjoint, it is usually more convenient to use the following formula, which follows from (10.1.3):

$$\langle Lz, \psi \rangle = \langle z, A^* \psi \rangle + \langle Gz, B^* \psi \rangle \qquad \forall\, z \in Z \,,\ \psi \in \mathcal{D}(A^*) \,. \tag{10.1.7}$$

Sometimes the expression $\langle Lz, \psi \rangle - \langle z, A^* \psi \rangle$ can be written in a simple form using integration by parts, thus revealing the expression for B^*; see, for example, Propositions 10.2.1, 10.3.3, 10.4.1, 10.5.1 and 10.9.1 later in this chapter.

Definition 10.1.7. With the notation of Proposition 10.1.2, the boundary control system (L, G) is called *well-posed* if A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X and B is an admissible control operator for $\mathbb{T}$.

Proposition 10.1.8. *Let (L, G) be a boundary control system on U, Z and X, with A, B as in Proposition 10.1.2. We assume that A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X.*

Then for every $T > 0$, $z(0) \in Z$ and $u \in \mathcal{H}^2((0,T);U)$ which satisfy the compatibility condition $Gz(0) = u(0)$, equations (10.1.1) have a unique solution z and

$$z \in C([0,T];Z) \cap C^1([0,T];X) \,. \tag{10.1.8}$$

If (L, G) is well posed, then the same conclusion holds for every $T > 0$, $z(0) \in Z$ and $u \in \mathcal{H}^1((0,T);U)$ that satisfies $Gz(0) = u(0)$.

Proof. The identity $Gz(0) = u(0)$ is equivalent to $Az(0) + Bu(0) \in X$ (this follows from (10.1.4)). According to Remark 10.1.3, the space Z from the definition of a boundary control system coincides with Z defined in (4.2.9). According to Remark 10.1.4, (10.1.1) are equivalent to (10.1.6).

Now consider the case when (L, G) is well posed (i.e., B is admissible for $\mathbb{T}$). We know from Proposition 4.2.10 that (10.1.6) has the unique solution z defined by (4.2.7), where the operators Φ_t are defined by (4.2.1). Still by Proposition 4.2.10, z satisfies (10.1.8). If (L, G) is not assumed to be well posed, then we follow by the same reasoning, but with Proposition 4.2.11 instead of Proposition 4.2.10. $\square$

Example 10.1.9. We want to formulate the equations

$$\frac{\partial z(x,t)}{\partial t} = -\frac{\partial z(x,t)}{\partial x} \,, \qquad z(0,t) = u(t) \,,$$

as a boundary control system. Here, $x, t \geqslant 0$. Take $X = L^2[0,\infty)$, $Z = \mathcal{H}^1(0,\infty)$ and define the operators $L \in \mathcal{L}(Z,X)$ and $G \in \mathcal{L}(Z,\mathbb{C})$ by

$$Lz = -\frac{\mathrm{d}z}{\mathrm{d}x}, \qquad Gz = z(0).$$

Notice that $X_1 = \operatorname{Ker} G = \mathcal{H}^1_0(0,\infty)$ and $A = L|_{X_1}$ is the generator of the unilateral right shift semigroup on X, last encountered in Example 4.2.7. Now it is clear that all the conditions in Definition 10.1.1 are satisfied. To identify B, we follow the approach in Remark 10.1.6. First we notice that $A^*\psi = \psi'$ for all $\psi \in \mathcal{D}(A^*) = \mathcal{H}^1(0,\infty)$. Integrating by parts, we see that

$$\langle Lz, \psi\rangle - \langle z, A^*\psi\rangle = z(0)\overline{\psi}(0) \qquad \forall\, z,\psi \in \mathcal{H}^1(0,\infty).$$

Comparing this with (10.1.7), it follows that

$$B^*\psi = \psi(0), \quad \text{i.e.,} \quad B = \delta_0,$$

with δ_0 as defined in Example 4.2.7. Thus, our system is equivalent to the one from Example 4.2.7. In particular, this boundary control system is well posed.

Alternatively, we could solve the "abstract elliptic problem" from Remark 10.1.5 with $\beta = 1$ and $v = 1$:

$$-z'(x) = z(x), \qquad z(0) = 1,$$

which gives $z(x) = e^{-x}$. According to Remark 10.1.5, $Bv = (\beta I - A)z$. Using integration by parts, we can obtain from here that $B = \delta_0$ (we omit the computation). Overall, for this system, the approach in Remark 10.1.6 is more efficient.

The next proposition shows that certain perturbations of well-posed boundary control systems are again well-posed boundary control systems.

Proposition 10.1.10. *Let (L,G) be a well-posed boundary control system on U, Z and X, with generator A and control operator B_1. Let $B \in \mathcal{L}(Y,X)$ and let $C \in \mathcal{L}(X_1,Y)$ be an admissible observation operator for the semigroup $\mathbb{T}$ generated by A. Let C^e be an extension of C such that $C^e \in \mathcal{L}(Z,Y)$. Assume that there exist $\alpha \in \mathbb{R}$ and $M \geqslant 0$ such that $\mathbb{C}_\alpha \subset \rho(A)$ and*

$$\|C^e(sI-A)^{-1}B_1\|_{\mathcal{L}(U,Y)} \leqslant M \qquad \forall\, s \in \mathbb{C}_\alpha.$$

Then $(L + BC^e, G)$ is a well-posed boundary control system on U, Z and X. Its generator is $A + BC$ and its control operator is JB_1, where J is the extension of the identity operator introduced in Proposition 5.5.2.

Proof. According to Theorem 5.4.2, $A + BC$ is the generator of a strongly continuous semigroup $\mathbb{T}^{cl}$ on X. Let us show that $(L+BC^e, G)$ is a well-posed boundary control system. Conditions (i) and (ii) in Definition 10.1.1 are obviously satisfied, since U, Z and G have not changed. The restriction of $L+BC^e$ to Ker $G=\mathcal{D}(A)$ is $A+BC$, so that conditions (iii) and (iv) in Definition 10.1.1 are also satisfied. Clearly the generator $(L+BC^e,G)$ is $A+BC$. Let us determine the control operator of this boundary control system. For every $z\in Z$ we have, using (10.1.3),

$$(L+BC^e)z \,=\, Az+B_1Gz+BC^e z\,,$$

where A is regarded as an operator from X to X_{-1}. Applying J to both sides (so that the left-hand side and the term $BC^e z$ remain unchanged), we obtain

$$(L+BC^e)z \,=\, JAz+JB_1Gz+BC^e z\,.$$

Now using (5.5.2) we obtain

$$(L+BC^e)z \,=\, (A+BC)z+JB_1Gz\,,$$

where $A+BC$ is regarded as an operator from X to X_{-1}^{cl} (the space X_{-1}^{cl} is the analogue of X_{-1} for the operator $A+BC$, as in Proposition 5.5.2). Comparing the above formula with (10.1.3) (with $L+BC^e$ in place of L and $A+BC$ in place of A), we see that the control operator of $(L+BC^e,G)$ is JB_1.

It remains to show that $(L+BC^e,G)$ is well posed. The operators C^e and B_1 satisfy the assumptions in part (3) of Proposition 5.5.2. Therefore, according to this proposition, JB_1 is an admissible control operator for $\mathbb{T}^{cl}$. This means that our boundary control system is well posed. $\square$

We shall see an application of the last proposition in Section 10.8.

10.2 Two simple examples in one space dimension

Notation. Throughout this section we denote

$$\mathcal{H}_R^1(0,\pi) \,=\, \left\{\phi\in\mathcal{H}^1(0,\pi)\ \mid\ \phi(\pi)=0\right\},$$
$$H \,=\, L^2[0,\pi], \qquad U \,=\, \mathbb{C},$$
$$H_1 \,=\, \left\{ f\in\mathcal{H}^2(0,\pi)\cap\mathcal{H}_R^1(0,\pi)\ \left|\ \frac{\mathrm{d}f}{\mathrm{d}x}(0)=0\right.\right\},$$

and the operator $A_0:H_1\to H$ is defined by

$$A_0 f \,=\, -\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \qquad \forall\, f\in H_1\,.$$

We know from Example 3.4.12 that $A_0>0$ and that the Hilbert spaces $H_{\frac12}$ obtained from H and A_0, according to the definition in Section 3.4, is

$$H_{\frac12} \,=\, \mathcal{H}_R^1(0,\pi)\,.$$

Moreover, we set $U=\mathbb{C}$ and we define the operator $N:\mathbb{C}\to\mathcal{H}_R^1(0,\pi)$ by

$$(N\mathrm{v})(x) \,=\, \mathrm{v}(x-\pi) \qquad \forall\, x\in[0,\pi]\,. \tag{10.2.1}$$

10.2.1 A one-dimensional heat equation with Neumann boundary control

In this subsection we study a boundary control system modeling the heat propagation in a rod occupying the interval $[0, \pi]$. We want to control the temperature in the rod by means of a heat flux $u(t)$ acting at its left end. Normalizing the physical constants, the temperature z satisfies the initial and boundary value problem

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) = \dfrac{\partial^2 z}{\partial x^2}(x,t), & 0 < x < \pi, \quad t \geqslant 0, \\ \dfrac{\partial z}{\partial x}(0,t) = u(t), \quad z(\pi,t) = 0, & t \geqslant 0, \\ z(x,0) = z_0(x), & 0 < x < \pi. \end{cases} \tag{10.2.2}$$

Let $A = -A_0$. Since $A < 0$, it is the generator of an exponentially stable semigroup $\mathbb{T}$ on X and $\mathbb{T}_t \geqslant 0$ (see Proposition 3.8.5).

To formulate (10.2.2) as a boundary control system, we take the solution space $Z = \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_R(0,\pi)$ and the state space $X = H$. The operators $L \in \mathcal{L}(Z,X)$ and $G \in \mathcal{L}(Z,U)$ are defined by

$$Lf = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}, \qquad Gf = \frac{\mathrm{d}f}{\mathrm{d}x}(0) \qquad \forall\, f \in Z.$$

Proposition 10.2.1. *The pair (L,G) is a well-posed boundary control system on U, Z and X. The control operator and its adjoint are given by*

$$B\mathrm{v} = A_0 N \mathrm{v} \qquad \forall\, \mathrm{v} \in U, \tag{10.2.3}$$

$$B^*\psi = -\psi(0) \qquad \forall\, \psi \in \mathcal{D}(A^*). \tag{10.2.4}$$

Proof. We have $\operatorname{Ker} G = \mathcal{D}(A)$ and $A = L|_{\mathcal{D}(A)}$ is the generator of a semigroup on X. Consequently, all the conditions in Definition 10.1.1 are satisfied, which means that the pair (L,G) is indeed a boundary control system on U, Z and X.

In order to write a formula for B we use Remark 10.1.5. More precisely, for every $\mathrm{v} \in \mathbb{C}$, the abstract elliptic problem

$$Lf = 0, \qquad Gf = \mathrm{v}$$

is equivalent to

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 0 \quad \text{in} \quad [0,\pi], \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(0) = \mathrm{v}, \quad f(\pi) = 0.$$

It is easy to verify that the unique solution of the above boundary value problem is $f = N\mathrm{v}$. Using Remark 10.1.5 with $\beta = 0$, we obtain that $-A^{-1}B\mathrm{v} = N\mathrm{v}$. Applying $A_0 = -A$ to both sides, we obtain (10.2.3).

In order to check (10.2.4), we take $f \in Z$ and $\psi \in \mathcal{D}(A^*) = \mathcal{D}(A)$. Then, using integrations by parts, we obtain

$$\langle Lf, \psi\rangle - \langle f, A^*\psi\rangle = \langle Lf, \psi\rangle - \left\langle f, \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}\right\rangle = -\frac{\mathrm{d}f}{\mathrm{d}x}(0)\overline{\psi(0)}.$$

The above formula together with (10.1.7) imply (10.2.4).

Since $B^* \in \mathcal{L}(H_{\frac{1}{2}}, U)$, it follows from Proposition 5.1.3 that B^* is an admissible observation operator for the semigroup generated by $A^* = A$. From Theorem 4.4.3 it follows that B is an admissible control operator for the semigroup generated by A, so that we have indeed a well-posed boundary control system. $\square$

Remark 10.2.2. Using Remark 4.2.6, the above result can be stated as follows: For every $z_0 \in L^2[0,\pi]$ and every $u \in L^2_{\mathrm{loc}}[0,\infty)$ there exists a unique function $z \in C([0,\infty); L^2[0,\pi])$ that satisfies, for every $t \geqslant 0$ and every $\psi \in \mathcal{D}(A_0)$,

$$\int_0^\pi z(x,t)\overline{\psi(x)}\,\mathrm{d}x - \int_0^\pi z_0(x)\overline{\psi(x)}\,\mathrm{d}x = \int_0^t \left[\int_0^\pi z(x,\sigma)\frac{\mathrm{d}^2\overline{\psi}}{\mathrm{d}x^2}\,\mathrm{d}x - u(\sigma)\overline{\psi}(0)\right]\mathrm{d}\sigma.$$

In the PDE literature such formulas are used to define weak solutions for PDEs with boundary control. Using this terminology, Proposition 10.2.1 is an existence and uniqueness result for weak solutions of (10.2.2).

We show in Example 11.2.5 that this system is null-controllable in any time $\tau > 0$.

10.2.2 A string equation with Neumann boundary control

We consider the problem of controlling the vibrations of a string occupying the interval $[0,\pi]$ by means of a force $u(t)$ acting at its left end. If we assume that the string is fixed at its right end, then the transverse deflection w satisfies the following initial and boundary value problem:

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t), & 0 < x < \pi,\ t \geqslant 0, \\ w(\pi,t) = 0, \quad \dfrac{\partial w}{\partial x}(0,t) = u(t), & t \geqslant 0, \\ w(x,0) = f(x), \quad \dfrac{\partial w}{\partial t}(x,0) = g(x), & 0 < x < \pi. \end{cases} \tag{10.2.5}$$

To formulate (10.2.5) as a boundary control system, we take the solution space

$$Z = \left[\mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_R(0,\pi)\right] \times \mathcal{H}^1_R(0,\pi)$$

and the state space

$$X = \mathcal{H}^1_R(0,\pi) \times H.$$

We introduce the operator $A : \mathcal{D}(A) \to X$ by

$$\mathcal{D}(A) = \left\{ f \in \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_R(0,\pi) \;\middle|\; \frac{\mathrm{d}f}{\mathrm{d}x}(0) = 0 \right\} \times \mathcal{H}^1_R(0,\pi), \tag{10.2.6}$$

$$A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A). \tag{10.2.7}$$

Recall from Example 2.7.15 that A generates a group of isometries on X.

The operators $L \in \mathcal{L}(Z, X)$ and $G \in \mathcal{L}(Z, U)$ are defined by

$$L \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \end{bmatrix}, \qquad G \begin{bmatrix} f \\ g \end{bmatrix} = \frac{\mathrm{d}f}{\mathrm{d}x}(0) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in Z.$$

Proposition 10.2.3. *The pair (L, G) is a well-posed boundary control system on U, Z and X. The control operator and its adjoint are given by*

$$B\mathrm{v} = \begin{bmatrix} 0 \\ A_0 N \mathrm{v} \end{bmatrix} \qquad \forall\, \mathrm{v} \in U, \tag{10.2.8}$$

$$B^* \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = -\psi(0) \qquad \forall \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A^*), \tag{10.2.9}$$

where N has been defined by (10.2.1).

Proof. It is easy to see that Ker $G = \mathcal{D}(A)$ and $L|_{\mathcal{D}(A)} = A$, so that all the conditions in Definition 10.1.1 are satisfied. This means that the pair (L, G) is indeed a boundary control system on U, Z and X.

To determine B we use Remark 10.1.5. More precisely, for every $\mathrm{v} \in \mathbb{C}$, consider the abstract elliptic problem

$$L \begin{bmatrix} f \\ g \end{bmatrix} = 0, \qquad G \begin{bmatrix} f \\ g \end{bmatrix} = \mathrm{v}.$$

It is easy to check that the unique solution of the above equations is $g = 0$ and $f = N\mathrm{v}$. Using Remark 10.1.5 with $\beta = 0$, we obtain that $-A^{-1}B\mathrm{v} = N\mathrm{v}$. Applying $-A$ to both sides, we obtain (10.2.8).

In order to check (10.2.9), we take $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in Z$ and $\left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix}\right] \in \mathcal{D}(A^*) = \mathcal{D}(A)$. Then, using integrations by parts and the fact that A is skew-adjoint, we obtain

$$\left\langle L \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \right\rangle - \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, A^* \begin{bmatrix} f \\ g \end{bmatrix} \right\rangle = -\frac{\mathrm{d}f}{\mathrm{d}x}(0)\overline{\psi(0)}.$$

The above formula together with (10.1.7) imply (10.2.9).

In order to show that B is an admissible control operator for the semigroup $\mathbb{T}$ generated by A, we first notice that $B^* = C$, where C is the operator defined in (6.2.14). We have seen in Proposition 6.2.5 that C is an admissible observation operator for $\mathbb{T}$. Since $\mathbb{T}$ is invertible and $A^* = -A$, it follows that $B^* = C$ is an admissible observation operator for $\mathbb{T}^*$. From Theorem 4.4.3 it follows that B is an admissible control operator for the semigroup generated by A, so that we have indeed a well-posed boundary control system. □

Remark 10.2.4. Using Remark 4.2.6, the above result can be stated as follows: For every $f \in \mathcal{H}^1_R(0,\pi)$, every $g \in L^2[0,\pi]$ and every $u \in L^2_{\rm loc}[0,\infty)$, there exists a unique function

$$w \in C([0,\infty);\mathcal{H}^1_R[0,\pi]) \cap C^1([0,\infty);L^2[0,\pi]),$$

such that $w(0) = f$ and w satisfies, for every $t \geqslant 0$ and every $\psi \in \mathcal{H}^1_R(0,\pi)$,

$$\begin{aligned}\int_0^\pi \dot{w}(x,t)\overline{\psi(x)}\,{\rm d}x - \int_0^\pi g(x)\overline{\psi(x)}\,{\rm d}x \\ = -\int_0^t \left[\int_0^\pi \frac{\partial w}{\partial x}(x,\sigma)\frac{{\rm d}\overline{\psi}}{{\rm d}x}(x)\,{\rm d}x + u(\sigma)\overline{\psi}(0)\right]{\rm d}\sigma.\end{aligned}$$

Therefore, Proposition 10.2.3 can be interpreted as an existence and uniqueness result for weak solutions of (10.2.5).

We show in Example 11.2.6 that the system discussed in this subsection is exactly controllable in any time $\tau \geqslant 2\pi$.

10.3 A string equation with variable coefficients

Let $a \in \mathcal{H}^1((0,\pi);\mathbb{R})$ and $b \in L^\infty([0,\pi];\mathbb{R})$. Assume that there exists $m > 0$ with $a(x) \geqslant m$ for all $x \in [0,\pi]$ and that $b \geqslant 0$.

Consider the initial and boundary value problem

$$\frac{\partial^2 w}{\partial t^2}(x,t) = \frac{\partial}{\partial x}\left(a(x)\frac{\partial w}{\partial x}(x,t)\right) - b(x)w(x,t), \quad 0 < x < \pi,\ t > 0, \tag{10.3.1}$$

$$w(0,t) = u(t), \qquad w(\pi,t) = 0, \tag{10.3.2}$$

$$w(\cdot,0) = f, \qquad \frac{\partial w}{\partial t}(\cdot,0) = g. \tag{10.3.3}$$

These equations model the vibrations of a non-homogeneous elastic string which is fixed at the end $x = \pi$ and with a controlled displacement $w(0,t) = u(t)$.

Throughout this section, we denote

$$H = L^2[0,\pi], \qquad U = \mathbb{C},$$

and the operator $A_0 : H_1 \to H$ is defined by

$$H_1 = \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_0(0,\pi), \quad A_0 f = -\frac{\rm d}{{\rm d}x}\left(a\frac{{\rm d}f}{{\rm d}x}\right) + bf \quad \forall f \in H_1. \tag{10.3.4}$$

We know from Proposition 3.5.2 that $A_0 > 0$ and that the Hilbert spaces $H_{\frac{1}{2}}$ and $H_{-\frac{1}{2}}$ obtained from H and A_0, according to the definitions in Section 3.4, are

$$H_{\frac{1}{2}} = \mathcal{H}_0^1(0,\pi), \qquad H_{-\frac{1}{2}} = \mathcal{H}^{-1}(0,\pi).$$

From the same section we know that A_0 has a unique extension to a unitary operator from $H_{\frac{1}{2}}$ onto $H_{-\frac{1}{2}}$ and also from H onto H_{-1}. We denote these extensions also by A_0. The inner products in $H_{\frac{1}{2}}$, H and $H_{-\frac{1}{2}}$ will be denoted by $\langle\cdot,\cdot\rangle_{\frac{1}{2}}$, $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{-\frac{1}{2}}$, respectively. With H and A_0 as above we set

$$X = H \times H_{-\frac{1}{2}}, \qquad \mathcal{D}(A) = H_{\frac{1}{2}} \times H,$$

and $A : \mathcal{D}(A) \to X$ is defined by

$$A\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -A_0 f \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A). \tag{10.3.5}$$

Since A_0 is strictly positive, it follows from Proposition 3.7.6 that A is skew-adjoint and $0 \in \rho(A)$. As usual, X_1 is $\mathcal{D}(A)$ endowed with the graph norm.

To formulate (10.3.1)–(10.3.3) as a boundary control system, we take the input space $U = \mathbb{C}$ and we introduce the solution space Z by

$$Z = \mathcal{H}_R^1(0,\pi) \times L^2[0,\pi], \quad \text{where} \quad \mathcal{H}_R^1(0,\pi) = \{\psi \in \mathcal{H}^1(0,\pi) \mid \psi(\pi) = 0\}.$$

The state $z(t)$ of the boundary control system will correspond to $\begin{bmatrix} w(\cdot,t) \\ \dot{w}(\cdot,t) \end{bmatrix}$ from (10.3.1)–(10.3.3). The operators $L \in \mathcal{L}(Z,X)$ and $G \in \mathcal{L}(Z,U)$ are defined by

$$L\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}f}{\mathrm{d}x}\right) - bf \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in Z,$$

$$G\begin{bmatrix} f \\ g \end{bmatrix} = f(0) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in Z. \tag{10.3.6}$$

We need the following technical result.

Proposition 10.3.1. *For every* $\mathrm{v} \in U = \mathbb{C}$, *there exists a unique function* $D\mathrm{v} \in \mathcal{H}^2(0,\pi) \cap \mathcal{H}_R^1(0,\pi)$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}(D\mathrm{v})}{\mathrm{d}x}\right) - b(D\mathrm{v}) = 0 \quad \text{in} \quad [0,\pi], \tag{10.3.7}$$

$$D\mathrm{v}(0) = \mathrm{v}, \quad (D\mathrm{v})(\pi) = 0. \tag{10.3.8}$$

Clearly, D may be regarded as a bounded linear operator from U into H.

Proof. Let $\chi \in C^\infty[0,\pi]$ be such that $\chi(x) = 1$ for $x \in \left[0, \frac{\pi}{4}\right]$ and $\chi(x) = 0$ for $x \in \left[\frac{3\pi}{4}, \pi\right]$. We define the operator D by

$$(D\mathrm{v})(x) = \mathrm{v}A_0^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}\chi}{\mathrm{d}x}\right) - b\chi\right](x) + \mathrm{v}\chi(x). \qquad (10.3.9)$$

It is easy to check that the above formula defines a bounded linear map from U into H, that $D\mathrm{v} \in \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_R(0,\pi)$ and that it satisfies (10.3.8). Moreover, from (10.3.9), it follows that $D\mathrm{v} - \mathrm{v}\chi \in \mathcal{D}(A_0)$ and

$$A_0(D\mathrm{v} - \mathrm{v}\chi) = \mathrm{v}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}\chi}{\mathrm{d}x}\right) - b\chi\right],$$

which implies that $D\mathrm{v}$ also satisfies (10.3.7). The uniqueness of the operator D with the required properties follows easily from the fact that Ker $A_0 = \{0\}$. □

Remark 10.3.2. In the case $a = 1$ and $b = 0$, the map D introduced above is the one-dimensional counterpart of the Dirichlet map which will be studied in Section 10.6 and it is given explicitly by

$$(D\mathrm{v})(x) = \frac{\mathrm{v}}{\pi}(\pi - x) \qquad \forall\, x \in [0,\pi].$$

Proposition 10.3.3. *The pair (L, G) is a well-posed boundary control system on U, Z and X. Its control operator and its adjoint are given by*

$$B\mathrm{v} = \begin{bmatrix} 0 \\ A_0 D\mathrm{v}\end{bmatrix} \qquad \forall\, \mathrm{v} \in U, \qquad (10.3.10)$$

$$B^*\begin{bmatrix}\varphi \\ \psi\end{bmatrix} = a(0)\,\frac{\mathrm{d}}{\mathrm{d}x}\left(A_0^{-1}\psi\right)\Big|_{x=0} \qquad \forall \begin{bmatrix}\varphi \\ \psi\end{bmatrix} \in \mathcal{D}(A^*) = \mathcal{D}(A), \qquad (10.3.11)$$

where D is defined as in Proposition 10.3.1*.*

Proof. Notice that G is onto, Ker $G = X_1$ and $A = L|_{X_1}$ is the generator of a unitary group on X, so that all the conditions in Definition 10.1.1 are satisfied.

In order to write a formula for B we use Remark 10.1.5. More precisely, for every $\mathrm{v} \in \mathbb{C}$, the abstract elliptic problem

$$L\begin{bmatrix} f \\ g\end{bmatrix} = 0, \qquad G\begin{bmatrix} f \\ g\end{bmatrix} = \mathrm{v}$$

is equivalent to $g = 0$ and $f = D\mathrm{v}$. Using Remark 10.1.5 with $\beta = 0$, we obtain that $(-A)^{-1}B = \left[\begin{smallmatrix} D\mathrm{v} \\ 0\end{smallmatrix}\right]$. Applying A to both sides, we obtain (10.3.10).

In order to calculate B^* we take $\left[\begin{smallmatrix} f \\ g\end{smallmatrix}\right] \in Z$ and $\left[\begin{smallmatrix} \varphi \\ \psi\end{smallmatrix}\right] \in \mathcal{D}(A^*) = \mathcal{D}(A)$. Then

$$\left\langle L\begin{bmatrix} f \\ g\end{bmatrix}, \begin{bmatrix}\varphi \\ \psi\end{bmatrix}\right\rangle_X - \left\langle \begin{bmatrix} f \\ g\end{bmatrix}, A^*\begin{bmatrix}\varphi \\ \psi\end{bmatrix}\right\rangle_X = \left\langle L\begin{bmatrix} f \\ g\end{bmatrix}, \begin{bmatrix}\varphi \\ \psi\end{bmatrix}\right\rangle_X + \left\langle \begin{bmatrix} f \\ g\end{bmatrix}, A\begin{bmatrix}\varphi \\ \psi\end{bmatrix}\right\rangle_X$$

$$= \langle g, \varphi\rangle + \left\langle \frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}f}{\mathrm{d}x}\right) - bf, \psi\right\rangle_{-\frac{1}{2}} + \langle f, \psi\rangle - \langle g, A_0\varphi\rangle_{-\frac{1}{2}}. \qquad (10.3.12)$$

Assume for a moment that $f \in \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_R(0,\pi)$. Since $A_0^{\frac{1}{2}}$ is unitary from H onto $H_{-\frac{1}{2}}$ (see Section 3.4), it follows that

$$\begin{aligned}\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}f}{\mathrm{d}x}\right) - bf, \psi \right\rangle_{-\frac{1}{2}} &= \left\langle \frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}f}{\mathrm{d}x}\right) - bf, A_0^{-1}\psi \right\rangle \\ &= \int_0^\pi \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}f}{\mathrm{d}x}\right) - bf\right)\overline{A_0^{-1}\psi}\,\mathrm{d}x\,.\end{aligned}$$

Using integration by parts twice, the above relation becomes

$$\begin{aligned}&\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}f}{\mathrm{d}x}\right) - bf, \psi \right\rangle_{-\frac{1}{2}} \\ &= \int_0^\pi f\frac{\mathrm{d}}{\mathrm{d}x}\left(a\,\frac{\mathrm{d}\overline{A_0^{-1}\psi}}{\mathrm{d}x}\right)\mathrm{d}x + f(0)a(0)\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{A_0^{-1}\psi}\right)\Big|_{x=0} \\ &\quad - \int_0^\pi bf\,\overline{A_0^{-1}\psi}\,\mathrm{d}x \,=\, \langle f, (-A_0)A_0^{-1}\psi\rangle + f(0)a(0)\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{A_0^{-1}\psi}\right)\Big|_{x=0} \\ &= -\langle f,\psi\rangle + f(0)a(0)\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{A_0^{-1}\psi}\right)\Big|_{x=0}\,.\end{aligned}$$

Since $\mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_R(0,\pi)$ is dense in $\mathcal{H}^1_R(0,\pi)$, it follows that

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}f}{\mathrm{d}x}\right) - bf, \psi \right\rangle_{-\frac{1}{2}} \;=\; -\langle f,\psi\rangle + f(0)a(0)\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{A_0^{-1}\psi}\right)\Big|_{x=0} \tag{10.3.13}$$

for every $f \in \mathcal{H}^1_R(0,\pi)$ and $\psi \in L^2[0,\pi]$.

The inner product $\langle g, A_0\varphi\rangle_{-\frac{1}{2}}$ from (10.3.12) can be expressed, using that $A_0^{\frac{1}{2}}$ is unitary from $H_{\frac{1}{2}}$ to H, as

$$\langle g, A_0\varphi\rangle_{-\frac{1}{2}} \,=\, \langle g,\varphi\rangle \qquad \forall\, g \in L^2[0,\pi],\;\; \varphi \in \mathcal{H}^1_0(0,\pi)\,. \tag{10.3.14}$$

By combining (10.3.12), (10.3.13) and (10.3.14), it follows that

$$\left\langle L\begin{bmatrix} f \\ g\end{bmatrix}, \begin{bmatrix}\varphi \\ \psi\end{bmatrix}\right\rangle_X - \left\langle \begin{bmatrix} f \\ g\end{bmatrix}, A^*\begin{bmatrix}\varphi \\ \psi\end{bmatrix}\right\rangle_X \;=\; f(0)a(0)\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{A_0^{-1}\psi}\right)\Big|_{x=0}\,.$$

Comparing this with (10.1.7), we obtain (10.3.11).

To prove that the system is well posed, denote $X_2 = \mathcal{D}(A^2) = H_1 \times H_{\frac{1}{2}}$ with the graph norm (see Remark 2.10.5) and recall that X_1 is $\mathcal{D}(A)$ endowed with the graph norm. Notice that

$$B^*Az \,=\, -\,a(0)Cz \qquad \forall\, z \in X_2\,,$$

where C is the operator from Proposition 8.2.2. We know from this proposition that C is an admissible observation operator for the semigroup $\mathbb{T}$ generated by A, acting on X_1, and hence also for its inverse semigroup (whose generator is $-A$). Since $\mathbb{T}$ is unitary (on any of the spaces X, X_1), it follows that C is admissible for $\mathbb{T}^*$ acting on the space X_1. This implies that $B^* = CA^{-1}$ is an admissible observation operator for the semigroup $\mathbb{T}^*$ acting on X. From the duality result in Theorem 4.4.3 it follows that B is admissible for $\mathbb{T}$ acting on X. □

Remark 10.3.4. In (10.3.10) A_0 is the extension of the operator from (10.3.4) to an operator in $\mathcal{L}(H, H_{-1})$ and this extension cannot be expressed in a simple manner, other than by duality. More precisely,

$$\langle A_0 D\mathrm{v}, \psi\rangle_{H_{-1},H_1} = \langle D\mathrm{v}, A_0\psi\rangle \qquad \forall\, \psi \in H_1 .$$

In order to study the admissibility of the control operator B, it is more convenient to study the admissibility of the observation operator B^* for the semigroup generated by A^*. We refer the reader to Example 11.2.7 for a detailed discussion of this issue, together with a discussion of the controllability of this system.

10.4 An Euler–Bernoulli beam with torque control

In this section we study the initial and boundary value problem

$$\frac{\partial^2 w}{\partial t^2}(x,t) = -\frac{\partial^4 w}{\partial x^4}(x,t), \quad 0 < x < \pi,\ t > 0, \tag{10.4.1}$$

$$w(0,t) = 0, \qquad w(\pi,t) = 0, \tag{10.4.2}$$

$$\frac{\partial^2 w}{\partial x^2}(0,t) = u(t), \qquad \frac{\partial^2 w}{\partial x^2}(\pi,t) = 0, \tag{10.4.3}$$

$$w(\cdot,0) = f, \qquad \frac{\partial w}{\partial t}(\cdot,0) = g. \tag{10.4.4}$$

These equations model the vibrations of an Euler–Bernoulli beam which is hinged at the end $x = \pi$, whereas it is fixed at the end $x = 0$ and a bending torque $\frac{\partial^2 w}{\partial x^2}(0,t) = u(t)$ is applied at this end.

Throughout this section, we denote $H = L^2[0,\pi]$, $U = \mathbb{C}$ and the operator $A_0 : H_1 \to H$ is defined by

$$H_1 = \mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi), \qquad A_0 f = -\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \qquad \forall\, f \in H_1 . \tag{10.4.5}$$

We know from Proposition 3.5.1 that $A_0 > 0$ and that the Hilbert spaces $H_{\frac{1}{2}}$ and $H_{-\frac{1}{2}}$ obtained from H and A_0, according to the definitions in Section 3.4, are given by

$$H_{\frac{1}{2}} = \mathcal{H}_0^1(0,\pi), \qquad H_{-\frac{1}{2}} = \mathcal{H}^{-1}(0,\pi).$$

We know that A_0 has unique extensions to unitary operators from $H_{\frac{1}{2}}$ onto $H_{-\frac{1}{2}}$ and from H onto H_{-1}. These extensions are still denoted by A_0. The inner products in $H_{\frac{1}{2}}$, H and $H_{-\frac{1}{2}}$ will be denoted by $\langle\cdot,\cdot\rangle_{\frac{1}{2}}$, $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{-\frac{1}{2}}$. We denote $H_{\frac{3}{2}} = A_0^{-1}H_{\frac{1}{2}}$. It is not difficult to check that

$$H_{\frac{3}{2}} = \left\{ g \in \mathcal{H}^3(0,\pi) \cap \mathcal{H}_0^1(0,\pi) \;\middle|\; \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(0) = \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(\pi) = 0 \right\}.$$

With H and A_0 as above we set

$$\begin{aligned} X &= H_{\frac{1}{2}} \times H_{-\frac{1}{2}}, & \mathcal{D}(A) &= H_{\frac{3}{2}} \times H_{\frac{1}{2}}, \\ A\begin{bmatrix} f \\ g \end{bmatrix} &= \begin{bmatrix} g \\ -A_0^2 f \end{bmatrix} & \forall \begin{bmatrix} f \\ g \end{bmatrix} &\in \mathcal{D}(A). \end{aligned} \tag{10.4.6}$$

Since A_0^2 is strictly positive on $H_{-\frac{1}{2}}$, it follows from Proposition 3.7.6 that A is skew-adjoint. As usual, we denote $X_1 = \mathcal{D}(A)$, with the graph norm.

Let

$$W = \left\{ g \in \mathcal{H}^3(0,\pi) \cap \mathcal{H}_0^1(0,\pi) \;\middle|\; \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(\pi) = 0 \right\}.$$

To formulate (10.4.1)–(10.4.3) as a boundary control system, we introduce the solution space

$$Z = W \times H_{\frac{1}{2}}.$$

The operators $L \in \mathcal{L}(Z,X)$ and $G \in \mathcal{L}(Z,U)$ are defined by

$$\begin{aligned} L\begin{bmatrix} f \\ g \end{bmatrix} &= \begin{bmatrix} g \\ -\dfrac{\mathrm{d}^4 f}{\mathrm{d}x^4} \end{bmatrix} & \forall \begin{bmatrix} f \\ g \end{bmatrix} &\in Z, \\ G\begin{bmatrix} f \\ g \end{bmatrix} &= \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(0) & \forall \begin{bmatrix} f \\ g \end{bmatrix} &\in Z. \end{aligned} \tag{10.4.7}$$

We also define the operator $E : \mathbb{C} \to W$ by

$$(E\mathrm{v})(x) = \frac{\mathrm{v}}{6\pi}(\pi - x)^3 - \frac{\pi\mathrm{v}}{6}(\pi - x) \qquad \forall\, x \in [0,\pi]. \tag{10.4.8}$$

Proposition 10.4.1. *The pair (L, G) is a boundary control system on U, Z and X. The control operator and its adjoint are given by*

$$B\mathrm{v} = \begin{bmatrix} 0 \\ A_0^2 E\mathrm{v} \end{bmatrix} \qquad \forall\, \mathrm{v} \in U, \tag{10.4.9}$$

$$B^*\begin{bmatrix} \varphi \\ \psi \end{bmatrix} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(A_0^{-1}\psi\right)\Big|_{x=0} \qquad \forall \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A^*) = \mathcal{D}(A). \tag{10.4.10}$$

Proof. Notice that Ker $G = X_1$ and $A = L|_{X_1}$ is the generator of a unitary group on X, so that all the conditions in Definition 10.1.1 are satisfied. In order to write a formula for B we use Remark 10.1.5. More precisely, for every $\mathrm{v} \in \mathbb{C}$, the abstract elliptic problem

$$L\begin{bmatrix} f \\ g \end{bmatrix} = 0, \qquad G\begin{bmatrix} f \\ g \end{bmatrix} = \mathrm{v}$$

is equivalent to $g = 0$ and

$$\frac{\mathrm{d}^4 f}{\mathrm{d}x^4} = 0 \quad \text{in} \quad [0,\pi], \qquad f(0) = f(\pi) = 0, \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(0) = \mathrm{v}, \ \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(\pi) = 0.$$

It can easily be checked that the unique solution of the above boundary value problem is $f = E\mathrm{v}$, where the operator E has been defined in (10.4.8). Using Remark 10.1.5 with $\beta = 0$ we obtain $(-A)^{-1}B = \left[\begin{smallmatrix} E\mathrm{v} \\ 0 \end{smallmatrix}\right]$, which implies (10.4.9).

Let $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in Z$ and $\left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix}\right] \in \mathcal{D}(A^*) = \mathcal{D}(A)$. Then, using that $A^* = -A$, we obtain

$$\begin{aligned} &\left\langle L\begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \varphi \\ \psi \end{bmatrix}\right\rangle_X - \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, A^*\begin{bmatrix} \varphi \\ \psi \end{bmatrix}\right\rangle_X \\ &\quad = \langle g, \varphi\rangle_{\frac{1}{2}} - \left\langle \frac{\mathrm{d}^4 f}{\mathrm{d}x^4}, \psi\right\rangle_{-\frac{1}{2}} + \langle f, \psi\rangle_{\frac{1}{2}} - \langle g, A_0^2\varphi\rangle_{-\frac{1}{2}}. \end{aligned} \tag{10.4.11}$$

Assuming for a moment that $f \in \mathcal{H}^4(0,\pi) \cap Z$ and using the fact that $A_0^{\frac{1}{2}}$ is unitary from H onto $H_{-\frac{1}{2}}$, it follows that

$$\left\langle \frac{\mathrm{d}^4 f}{\mathrm{d}x^4}, \psi\right\rangle_{-\frac{1}{2}} = \left\langle \frac{\mathrm{d}^4 f}{\mathrm{d}x^4}, A_0^{-1}\psi\right\rangle = \int_0^\pi \frac{\mathrm{d}^4 f}{\mathrm{d}x^4}\, \overline{A_0^{-1}\psi}\, \mathrm{d}x.$$

Using integrations by parts, the above relation becomes

$$\begin{aligned} \left\langle \frac{\mathrm{d}^4 f}{\mathrm{d}x^4}, \psi\right\rangle_{-\frac{1}{2}} &= -\int_0^\pi \frac{\mathrm{d}^3 f}{\mathrm{d}x^3} \frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{A_0^{-1}\psi}\right) \mathrm{d}x \\ &= \int_0^\pi \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\overline{A_0^{-1}\psi}\right) \mathrm{d}x + \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(0)\, \frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{A_0^{-1}\psi}\right)\bigg|_{x=0}. \end{aligned}$$

Using the facts that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\overline{A_0^{-1}\psi}\right) = -\overline{\psi}, \quad f \in H_1 \quad \text{and} \quad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = -A_0 f,$$

together with the density of $\mathcal{H}^4(0,\pi) \cap W$ in W, it follows that

$$\left\langle \frac{\mathrm{d}^4 f}{\mathrm{d}x^4}, \psi\right\rangle_{-\frac{1}{2}} = \langle f, \psi\rangle_{\frac{1}{2}} + \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(0)\, \frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{A_0^{-1}\psi}\right)\bigg|_{x=0} \tag{10.4.12}$$

for every $f \in W$ and $\psi \in H_{\frac{1}{2}}$.

To evaluate the last term on the right-hand side of (10.4.11), we use the fact that $A_0^{\frac{1}{2}}$ is unitary from $H_{\frac{1}{2}}$ onto H and we obtain that

$$\langle g, A_0^2 \varphi \rangle_{-\frac{1}{2}} = -\langle g, \varphi \rangle_{\frac{1}{2}} \qquad \forall\, g \in H_{\frac{1}{2}}, \quad \forall\, \varphi \in H_{\frac{3}{2}}. \tag{10.4.13}$$

By combining (10.4.11), (10.4.12) and (10.4.13), it follows that

$$\left\langle L \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \right\rangle_X - \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, A^* \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \right\rangle_X = -\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(0)\, \frac{\mathrm{d}}{\mathrm{d}x} \left(\overline{A_0^{-1} \psi} \right) \Big|_{x=0}.$$

Comparing this with (10.1.7), it follows that B^* satisfies (10.4.10). $\square$

Remark 10.4.2. In (10.4.9), A_0 is the extension of the operator from (10.4.5) to an operator in $\mathcal{L}(H, H_{-1})$. Comments similar to those in Remark 10.3.4 apply also here: In (10.4.9) A_0^2 is not the fourth derivative operator in the sense of distributions. The operator $A_0^2 E$ can only be defined by duality:

$$\langle A_0^2 E \mathrm{v}, \psi \rangle_{H_{-1}, H_1} = \langle A_0 E \mathrm{v}, A_0 \psi \rangle \qquad \forall\, \psi \in H_1.$$

Proposition 10.4.3. *The above boundary control system is well posed.*

Proof. We have seen in the proof of Proposition 10.4.1 that $A = L|_{\mathrm{Ker}\ G}$ generates a unitary group $\mathbb{T}$ on X. Thus we only have to show that the control operator B expressed in the same proposition is admissible for $\mathbb{T}$.

We return to the hinged Euler–Bernoulli equation discussed in Example 6.8.4. With our current notation the state space in Example 6.8.4 is $X_1 = \mathcal{D}(A)$, the semigroup generator is $A|_{\mathcal{D}(A^2)}$, which generates the restriction of $\mathbb{T}$ to X_1, and the observation operator $C : \mathcal{D}(A^2) \to \mathbb{C}$ is given by

$$C \begin{bmatrix} f \\ g \end{bmatrix} = \frac{\mathrm{d}g}{\mathrm{d}x}(0) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A^2) = H_{\frac{5}{2}} \times H_{\frac{3}{2}}.$$

We have shown in Example 6.8.4 that C is an admissible observation operator for $\mathbb{T}$ restricted to X_1. Using the isomorphism $Q = \left[\begin{smallmatrix} A_0 & 0 \\ 0 & A_0 \end{smallmatrix}\right]$ from X_1 to X (which commutes with A and hence with $\mathbb{T}$), we obtain that CQ^{-1} is an admissible observation operator for $\mathbb{T}$ (acting on X). From (10.4.10) we see that $CQ^{-1} = -B^*$. Thus B^* is an admissible observation operator for $\mathbb{T}$. Since $\mathbb{T}$ is invertible, B^* is admissible also for the inverse semigroup, which in our case is $\mathbb{T}^*$. By the duality result in Theorem 4.4.3, B is an admissible control operator for $\mathbb{T}$. $\square$

10.5 An Euler–Bernoulli beam with angular velocity control

In this section we consider a system modeling the vibrations of an Euler–Bernoulli beam which is clamped at the end $x = 1$ whereas it is fixed at the end $x = 0$ and an angular velocity $\frac{\partial \dot{w}}{\partial x}(0,t) = u(t)$ is imposed at this end. More precisely, we study the initial and boundary value problem

$$\frac{\partial^2 w}{\partial t^2}(x,t) = -\frac{\partial^4 w}{\partial x^4}(x,t), \quad 0 < x < 1, t > 0, \tag{10.5.1}$$

$$w(0,t) = 0, \qquad w(1,t) = 0, \tag{10.5.2}$$

$$\frac{\partial \dot{w}}{\partial x}(0,t) = u(t), \qquad \frac{\partial w}{\partial x}(1,t) = 0, \tag{10.5.3}$$

$$w(\cdot,0) = f, \qquad \frac{\partial w}{\partial t}(\cdot,0) = g. \tag{10.5.4}$$

We denote

$$X = V \times L^2[0,1], \quad \text{where} \quad V = \left\{ h \in \mathcal{H}^2(0,1) \mid h(0) = h(1) = \frac{\mathrm{d}h}{\mathrm{d}x}(1) = 0 \right\}.$$

The norm on X is defined by

$$\|z\|^2 = \|z_1\|_V^2 + \|z_2\|_{L^2}^2,$$

where $\|z_1\|_V^2 = \int_0^1 \left| \frac{\mathrm{d}^2 z_1}{\mathrm{d}x^2} \right|^2 \mathrm{d}x$. We introduce the space $Z \subset X$ by

$$Z = \left(V \cap \mathcal{H}^4(0,1) \right) \times V,$$

and we define the operators $L : Z \to X$, $G : Z \to \mathbb{C}$ by

$$L = \begin{bmatrix} 0 & I \\ -\frac{\mathrm{d}^4}{\mathrm{d}x^4} & 0 \end{bmatrix}, \qquad G \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \frac{\mathrm{d}z_2}{\mathrm{d}x}(0).$$

With the above notation, (10.5.1)–(10.5.3) can be written as follows:

$$\dot{z} = Lz, \quad Gz = u.$$

Such equations determine a boundary control system if L and G satisfy certain conditions; see Section 10.1. We prove below that this is indeed the case. Before doing this, we introduce the operator $A = L|_{\mathrm{Ker}\, G}$. It is easy to verify that

$$\mathcal{D}(A) = \mathrm{Ker}\, G = \left(V \cap \mathcal{H}^4(0,1) \right) \times \mathcal{H}_0^2(0,1),$$

which is a closed subspace of V.

Proposition 10.5.1. *The pair (L, G) is a well-posed boundary control system on $\mathbb{C}$, Z and X. Its control operator B is determined by*

$$B^* \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = -\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2}(0) \qquad \forall \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \in \mathcal{D}(A^*) = \mathcal{D}(A). \tag{10.5.5}$$

Proof. It is clear that G is onto. We decompose the state space X into two parts: the null-space of A, denoted X_n, and its orthogonal complement X_r. From a simple computation,

$$X_n = \operatorname{Ker} A = \left\{ \begin{bmatrix} aq(x) \\ 0 \end{bmatrix} \;\middle|\; a \in \mathbb{C} \right\}, \qquad \text{where} \quad q(x) = x(x-1)^2.$$

Now we determine $X_r = X_n^\perp$. If $z = \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] \in X_r$, then $z_1 \in V$ and $z_2 \in L^2[0,1]$. The condition $z \in X_n^\perp$ is equivalent to

$$\langle q, z_1 \rangle_V = 0.$$

We have, using integration by parts, that for every $h \in V$,

$$\langle q, h \rangle_V = \left[\frac{\mathrm{d}^2 q}{\mathrm{d}x^2} \cdot \frac{\mathrm{d}\overline{h}}{\mathrm{d}x} \right]_0^1 - \int_0^1 \frac{\mathrm{d}^3 q}{\mathrm{d}x^3}(x) \cdot \frac{\mathrm{d}\overline{h}}{\mathrm{d}x}(x)\mathrm{d}x.$$

Since $\frac{\mathrm{d}h}{\mathrm{d}x}(1) = 0$, we get, by another integration by parts,

$$\langle q, h \rangle_V = -\frac{\mathrm{d}^2 q}{\mathrm{d}x^2}(0) \cdot \frac{\mathrm{d}\overline{h}}{\mathrm{d}x}(0) - \left[\frac{\mathrm{d}^3 q}{\mathrm{d}x^3} \cdot \overline{h} \right]_0^1 + \int_0^1 \frac{\mathrm{d}^4 q}{\mathrm{d}x^4} \cdot \overline{h}\mathrm{d}x.$$

Using that $\frac{\mathrm{d}^4 q}{\mathrm{d}x^4} = 0$ and $h(0) = h(1) = 0$, we get

$$\langle q, h \rangle_V = -\frac{\mathrm{d}^2 q}{\mathrm{d}x^2}(0) \cdot \frac{\mathrm{d}\overline{h}}{\mathrm{d}x}(0) \qquad \forall\, h \in V.$$

Therefore we have, for z_1 in place of h,

$$\frac{\mathrm{d}^2 q}{\mathrm{d}x^2}(0) \cdot \frac{\mathrm{d}\overline{z_1}}{\mathrm{d}x}(0) = 0.$$

Since $\frac{\mathrm{d}^2 q}{\mathrm{d}x^2}(0) = -4$, it follows that $\frac{\mathrm{d}z_1}{\mathrm{d}x}(0) = 0$, so that $z_1 \in \mathcal{H}_0^2(0,1)$. Thus we get $X_r \subset \mathcal{H}_0^2(0,1) \times L^2[0,1]$. The converse inclusion is proved by the same computation, so that

$$X_r = \mathcal{H}_0^2(0,1) \times L^2[0,1].$$

We denote by A_r the part of A in X_r. Then

$$\mathcal{D}(A_r) = \left(\mathcal{H}_0^2(0,1) \cap \mathcal{H}^4(0,1) \right) \times \mathcal{H}_0^2(0,1), \qquad A_r = \begin{bmatrix} 0 & I \\ -\frac{\mathrm{d}^4}{\mathrm{d}x^4} & 0 \end{bmatrix}.$$

It is easy to see that X_r is invariant under A; i.e., $A_r z \in X_r$ for every $z \in \mathcal{D}(A_r)$. Moreover, by comparing the last two formulas with those in Section 6.10, we see that the operator A_r corresponds to the equations of a beam clamped at both ends. Therefore, according to the remarks at the beginning of Section 6.10, the operator A_r is skew-adjoint, so that it generates a unitary group $\mathbb{S}$ on X_r. Since

$$X = X_r \oplus X_n, \qquad A = \begin{bmatrix} A_r & 0 \\ 0 & 0 \end{bmatrix}, \tag{10.5.6}$$

it follows that A is skew-adjoint and hence it generates a unitary group $\mathbb{T}$ on X given by $\mathbb{T}_t = \left[\begin{smallmatrix} \mathbb{S}_t & 0 \\ 0 & I \end{smallmatrix}\right]$. In particular, it follows that conditions (ii)–(iv) in Definition 10.1.1 are satisfied, so that (L, G) is a boundary control system.

To compute the control operator B we use Remark 10.1.6; i.e., we use the formula (10.1.7) to find B^*. Using the fact that $A^* = -A$, (10.1.7) becomes

$$\langle Gz, B^*\psi\rangle_{\mathbb{C}} = \langle Lz, \psi\rangle_X + \langle z, A\psi\rangle_X \qquad \forall\, z \in Z,\ \psi \in \mathcal{D}(A).$$

Hence, denoting $z = \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right],\ \psi = \left[\begin{smallmatrix} \psi_1 \\ \psi_2 \end{smallmatrix}\right]$,

$$\begin{aligned} Gz \cdot \overline{B^*\psi} &= \left\langle \begin{bmatrix} z_2 \\ -\frac{\mathrm{d}^4 z_1}{\mathrm{d}x^4} \end{bmatrix}, \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \right\rangle_X + \left\langle \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} \psi_2 \\ -\frac{\mathrm{d}^4 \psi_1}{\mathrm{d}x^4} \end{bmatrix} \right\rangle_X \\ &= \langle z_2,\ \psi_1\rangle_V - \left\langle \frac{\mathrm{d}^4 z_1}{\mathrm{d}x^4},\ \psi_2 \right\rangle_{L^2} + \langle z_1, \psi_2\rangle_V - \left\langle z_2, \frac{\mathrm{d}^4 \psi_1}{\mathrm{d}x^4} \right\rangle_{L^2}. \end{aligned}$$

Using integration by parts twice for the above inner products in L^2, many terms cancel and we are left with

$$\frac{\mathrm{d}z_2}{\mathrm{d}x}(0) \cdot \overline{B^*\psi} = -\frac{\mathrm{d}z_2}{\mathrm{d}x}(0) \frac{\mathrm{d}^2 \overline{\psi_1}}{\mathrm{d}x^2}(0) \qquad \forall\, z \in Z,\ \psi \in \mathcal{D}(A).$$

This implies (10.5.5).

In order to show that the boundary control system (L, G) is well posed, it remains to prove that B is an admissible control operator for $\mathbb{T}$, or equivalently, that B^* is an admissible observation operator for $\mathbb{T}^*$. We use again the decomposition (10.5.6) of X. As already mentioned, the operator A_r coincides with the group generator of the clamped Euler–Bernoulli beam in Section 6.10. We define the operators C_r and C_n as the restrictions of B^* to $\mathcal{D}(A_r)$ and to X_n, so that $B^* = \begin{bmatrix} C_r & C_n \end{bmatrix}$. It is easy to see that $C_r = -C$, where C is the operator defined in (6.10.5). Since C is admissible for $\mathbb{S}$ (see Proposition 6.10.1) and since C_n is bounded, it follows that B^* is an admissible observation operator for $\mathbb{T}$. Since $\mathbb{T}$ is invertible, B^* is admissible also for the inverse semigroup, which in our case is $\mathbb{T}^*$. We have thus checked that the boundary control system (L, G) is well posed. $\square$

10.6 The Dirichlet map on an n-dimensional domain

In this section we introduce the Dirichlet map and the boundary trace operators γ_0 and γ_1, which are important tools for the formulation of certain PDEs as boundary control systems (see, for example, Section 10.8).

We consider Ω to be an open bounded subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^2 (as defined in Section 13.5). We denote by $-A_0$ the Dirichlet Laplacian on Ω, as introduced in Section 3.6, so that $A_0 : \mathcal{D}(A_0) \to L^2(\Omega)$, where

$$\mathcal{D}(A_0) = \mathcal{H}^2(\Omega) \cap \mathcal{H}^1_0(\Omega),$$

see Theorem 3.6.2, and $A_0 > 0$. For any $f \in \mathcal{H}^2(\Omega)$ we denote by $\frac{\partial f}{\partial \nu}$ the outward normal derivative of f on $\partial\Omega$ (see Section 13.6 for more details on this concept).

We denote $H = L^2(\Omega)$ and, as in Section 3.4, we define $H_1 = \mathcal{D}(A_0)$, $H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}})$, with the norms $\|z\|_1 = \|A_0 z\|_H$ and $\|z\|_{\frac{1}{2}} = \|A_0^{\frac{1}{2}} z\|_H$. The spaces H_{-1} and $H_{-\frac{1}{2}}$ are defined as the duals of H_1 and of $H_{\frac{1}{2}}$ with respect to the pivot space H, respectively. As explained a little earlier, we have $H_1 = \mathcal{H}^2(\Omega) \cap \mathcal{H}^1_0(\Omega)$ and, according to Proposition 3.6.1, we have $H_{\frac{1}{2}} = \mathcal{H}^1_0(\Omega)$ and $H_{-\frac{1}{2}} = \mathcal{H}^{-1}(\Omega)$.

Proposition 10.6.1. *For every* $\mathrm{v} \in L^2(\partial\Omega)$*, there exists a unique function* $D\mathrm{v} \in L^2(\Omega)$ *such that*

$$\int_\Omega (D\mathrm{v})(x) g(x)\,\mathrm{d}x = -\int_{\partial\Omega} \mathrm{v}\,\frac{\partial(A_0^{-1}g)}{\partial\nu}\,\mathrm{d}\sigma \qquad \forall\, g \in L^2(\Omega). \tag{10.6.1}$$

Moreover, the operator D *defined above (called the Dirichlet map) is linear and bounded from* $L^2(\partial\Omega)$ *into* $L^2(\Omega)$ *and its adjoint* $D^* \in \mathcal{L}(L^2(\Omega), L^2(\partial\Omega))$ *is given by*

$$D^* g = -\frac{\partial(A_0^{-1}g)}{\partial\nu} \qquad \forall\, g \in L^2(\Omega). \tag{10.6.2}$$

Proof. We denote $U = L^2(\partial\Omega)$. Since $A_0^{-1} \in \mathcal{L}(H, H_1)$, and since by Theorem 13.6.6 in Appendix II, the map $\psi \mapsto \frac{\partial\psi}{\partial\nu}$ is in $\mathcal{L}(H_1, U)$, it follows that the expression $\frac{\partial(A_0^{-1}g)}{\partial\nu} \in U$ depends boundedly on $g \in H$. According to the Riesz representation theorem, for every $\mathrm{v} \in U$, there exists a unique $D\mathrm{v} \in H$ such that

$$\langle D\mathrm{v}, g\rangle_H = \left\langle \mathrm{v},\ -\frac{\partial(A_0^{-1}g)}{\partial\nu} \right\rangle_U \qquad \forall\, g \in L^2(\Omega),$$

which is almost (10.6.1). To really get (10.6.1), we have to replace g with its complex conjugate $\overline{g}$. It is clear that the above formula implies (10.6.2). $\square$

Proposition 10.6.2. *For every* $\mathrm{v} \in L^2(\partial\Omega)$ *we have* $D\mathrm{v} \in C^\infty(\Omega)$ *and* $\Delta D\mathrm{v} = 0$.

Proof. First we prove that for every $\mathrm{v} \in L^2(\partial\Omega)$ we have $\Delta D\mathrm{v} = 0$, in the sense of distributions on Ω. Indeed, if we take in (10.6.1) $g = \Delta\varphi$, where $\varphi \in \mathcal{D}(\Omega)$, we obtain

$$\int_\Omega (D\mathrm{v})(x)(\Delta\varphi)(x)\,\mathrm{d}x \;=\; \int_\Gamma \mathrm{v}\,\frac{\partial\varphi}{\partial\nu}\,\mathrm{d}\sigma \;=\; 0 \qquad\qquad \forall\,\varphi\in\mathcal{D}(\Omega)\,.$$

From the definition of the Laplacian of a distribution (see Section 3.6 or Section 13.3), we now see that $\Delta D\mathrm{v}=0$ (in $\mathcal{D}'(\Omega)$).

It follows from Remark 13.5.6 in Appendix II that for any $\mathrm{v}\in L^2(\partial\Omega)$ we have $D\mathrm{v}\in\mathcal{H}^m_{\mathrm{loc}}(\Omega)$, for every $m\in\mathbb{N}$. According to Remark 13.4.5 (also in Appendix II), it follows that $D\mathrm{v}\in C^\infty(\Omega)$. Thus, the formula $\Delta D\mathrm{v}=0$ (which holds in the sense of distributions) must actually hold in the classical sense. $\square$

Remark 10.6.3. The last proposition implies, in particular, that

$$D\,\in\,\mathcal{L}(L^2(\partial\Omega),\mathcal{W}(\Delta))\,,$$

where

$$\mathcal{W}(\Delta)\;=\;\{g\in L^2(\Omega)\mid \Delta g\in\mathcal{H}^{-1}(\Omega)\}\,, \tag{10.6.3}$$

which is a Hilbert space with the norm

$$\|g\|_{\mathcal{W}(\Delta)}\;=\;\sqrt{\|g\|^2_{L^2(\Omega)}+\|\Delta g\|^2_{\mathcal{H}^{-1}(\Omega)}} \qquad\qquad \forall\,g\in\mathcal{W}(\Delta)\,.$$

For Ω as above and for every $f\in C^2(\operatorname{clos}\Omega)$, we introduce the boundary traces

$$\gamma_0 f\;=\;f|_{\partial\Omega}\,,\qquad \gamma_1 f\;=\;\frac{\partial f}{\partial\nu}\,. \tag{10.6.4}$$

It can be shown that the operators γ_0 and γ_1 can be extended such that

$$\gamma_0\in\mathcal{L}(\mathcal{H}^1(\Omega),\mathcal{H}^{\frac12}(\partial\Omega))\,,\qquad \gamma_1\in\mathcal{L}(\mathcal{H}^2(\Omega),\mathcal{H}^{\frac12}(\partial\Omega))\,.$$

For the definition of the space $\mathcal{H}^{\frac12}(\partial\Omega)$ and some of its properties we refer the reader to Section 13.5 in Appendix II. The dual of $\mathcal{H}^{\frac12}(\partial\Omega))$ with respect to the pivot space $L^2(\partial\Omega)$ is denoted by $\mathcal{H}^{-\frac12}(\partial\Omega))$ – for more on this space we refer the reader to Section 13.7.

Formulas (10.6.4) determine γ_0 and γ_1, because $C^2(\operatorname{clos}\Omega)$ is dense in both $\mathcal{H}^1(\Omega)$ and $\mathcal{H}^2(\Omega)$. $\gamma_0 f$ is called the *Dirichlet trace* of f, while $\gamma_1 f$ is called the *Neumann trace* of f. For more details on these trace operators and for references see Section 13.6. It is shown in Section 13.7 that γ_0 has a unique extension such that

$$\gamma_0\,\in\,\mathcal{L}(\mathcal{W}(\Delta),\mathcal{H}^{-\frac12}(\partial\Omega)) \tag{10.6.5}$$

and for every $\varphi\in\mathcal{H}^{\frac12}(\partial\Omega)$ and every $g\in\mathcal{W}(\Delta)$,

$$\langle\gamma_0 g,\overline{\varphi}\rangle_{\mathcal{H}^{-\frac12}(\partial\Omega),\mathcal{H}^{\frac12}(\partial\Omega)}\;=\;\int_\Omega g\,\Delta\widetilde{\varphi}\,\mathrm{d}x\;-\;\langle\Delta g,\overline{\widetilde{\varphi}}\rangle_{\mathcal{H}^{-1}(\Omega),\mathcal{H}^1_0(\Omega)}\,. \tag{10.6.6}$$

Here $\widetilde{\varphi}\in\mathcal{H}^2(\Omega)$ is obtained from φ as in the proof of Proposition 13.7.8, so that

$$\gamma_0\widetilde{\varphi}\;=\;0,\quad \gamma_1\widetilde{\varphi}\;=\;\varphi\,. \tag{10.6.7}$$

Remark 10.6.3 with (10.6.5) imply that $\gamma_0 D$ is well defined.

Proposition 10.6.4. *We have $\gamma_0 D = I$ (the identity on $L^2(\partial\Omega)$).*

Proof. Take $\mathrm{v} \in L^2(\partial\Omega)$, $\varphi \in \mathcal{H}^{\frac{1}{2}}(\partial\Omega)$ and $g = D\mathrm{v}$. According to (10.6.6), we obtain (using Proposition 10.6.2)

$$\langle \gamma_0 D\mathrm{v}, \overline{\varphi} \rangle_{\mathcal{H}^{-\frac{1}{2}}(\partial\Omega), \mathcal{H}^{\frac{1}{2}}(\partial\Omega)} = \int_\Omega (D\mathrm{v})(x)\,(\Delta\widetilde{\varphi})(x)\,\mathrm{d}x\,.$$

Using the definition of D in Proposition 10.6.1, we obtain

$$\langle \gamma_0 D\mathrm{v}, \overline{\varphi} \rangle_{\mathcal{H}^{-\frac{1}{2}}(\partial\Omega), \mathcal{H}^{\frac{1}{2}}(\partial\Omega)} = -\int_{\partial\Omega} \mathrm{v}\,\frac{\partial(A_0^{-1}\Delta\widetilde{\varphi})}{\partial\nu}\,\mathrm{d}\sigma\,.$$

Since $\widetilde{\varphi} \in \mathcal{D}(A_0)$, we have $-A_0^{-1}\Delta\widetilde{\varphi} = \widetilde{\varphi}$. Thus, we get

$$\langle \gamma_0 D\mathrm{v}, \overline{\varphi} \rangle_{\mathcal{H}^{-\frac{1}{2}}(\partial\Omega), \mathcal{H}^{\frac{1}{2}}(\partial\Omega)} = \int_{\partial\Omega} \mathrm{v}\,\varphi\,\mathrm{d}\sigma$$

for all $\varphi \in \mathcal{H}^{\frac{1}{2}}(\partial\Omega)$. Since this space is dense in $L^2(\partial\Omega)$, we get $\gamma_0 D\mathrm{v} = \mathrm{v}$. □

Remark 10.6.5. The name "Dirichlet map" for the operator D is due to the fact that, as we have shown, $z = D\mathrm{v}$ is a solution of the *Dirichlet problem*

$$\Delta z = 0, \qquad \gamma_0 z = \mathrm{v}\,. \tag{10.6.8}$$

Proposition 10.6.6. *For every $\mathrm{v} \in L^2(\partial\Omega)$, $z = D\mathrm{v}$ is the unique solution of the Dirichlet problem* (10.6.8) *in $L^2(\Omega)$.*

Proof. Let $z \in L^2(\Omega)$ be a solution of (10.6.8). Then $g = D\mathrm{v} - z \in L^2(\Omega)$ satisfies

$$\Delta g = 0, \qquad \gamma_0 g = 0\,.$$

From (10.6.6) it follows that

$$\int_\Omega g\,\Delta\widetilde{\varphi}\,\mathrm{d}x = 0 \qquad \forall\,\varphi \in \mathcal{H}^{\frac{1}{2}}(\partial\Omega),$$

where $\widetilde{\varphi} \in \mathcal{H}^2(\Omega)$ is obtained from φ as in the proof of Proposition 13.7.8. Using the definition of the Laplacian in the distributional sense, it follows that

$$\int_\Omega g\,\Delta(\widetilde{\varphi} + \psi)\,\mathrm{d}x = 0 \qquad \forall\,\varphi \in \mathcal{H}^{\frac{1}{2}}(\partial\Omega), \qquad \psi \in \mathcal{D}(\Omega)\,.$$

Recall that $\widetilde{\varphi} \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$, so that $\phi + \psi \in \mathcal{D}(A_0)$ and the last formula implies that

$$\langle g, A_0(\widetilde{\varphi} + \psi) \rangle = 0 \tag{10.6.9}$$

for every $\varphi \in \mathcal{H}^{\frac{1}{2}}(\partial\Omega)$ and $\psi \in \mathcal{D}(\Omega)$.

Let us show that the set of all the functions of the form $\widetilde{\varphi} + \psi$ as above are dense in $\mathcal{H}^2(\Omega) \cap \mathcal{H}^1_0(\Omega)$. Take $f \in \mathcal{H}^2(\Omega) \cap \mathcal{H}^1_0(\Omega)$ and denote $\varphi = \gamma_1 f$. Then, according to (10.6.7), the corresponding $\widetilde{\varphi} \in \mathcal{H}^2(\Omega)$ satisfies $\gamma_1 \widetilde{\varphi} = \gamma_1 f$. It follows that $\psi_0 = f - \widetilde{\varphi}$, which is an element of $\mathcal{H}^2(\Omega) \cap \mathcal{H}^1_0(\Omega)$, satisfies $\gamma_1 \psi_0 = 0$. According to Proposition 13.6.7 in Appendix II, it follows that $\psi_0 \in \mathcal{H}^2_0(\Omega)$. By the definition of $\mathcal{H}^2_0(\Omega)$ given in the same appendix, for every $\varepsilon > 0$ there exists $\psi \in \mathcal{D}(\Omega)$ such that $\|\psi - \psi_0\|_{\mathcal{H}^2(\Omega)} < \varepsilon$. It follows that $\|f - (\widetilde{\varphi} + \psi)\|_{\mathcal{H}^2(\Omega)} < \varepsilon$. This shows that indeed the space of all the functions of the form $\eta = \widetilde{\varphi} + \psi$, where $\varphi \in \mathcal{H}^{\frac{1}{2}}(\partial\Omega)$ and $\psi \in \mathcal{D}(\Omega)$, is dense in $H_1 = \mathcal{H}^2(\Omega) \cap \mathcal{H}^1_0(\Omega)$.

The density result that we have just proved, together with (10.6.9) and the fact that $A_0 \in \mathcal{L}(H_1, H)$, implies that

$$\langle g, A_0 \eta \rangle = 0 \qquad \forall\, \eta \in H_1 .$$

Since A_0 is onto H, it follows that $g = 0$. □

We know from Remark 3.6.3 that A_0 can be uniquely extended to a unitary operator in $\mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$ or in $\mathcal{L}(H, H_{-1})$. These extensions (still denoted by A_0) may also be regarded as strictly positive operators on $H_{-\frac{1}{2}}$ or on H_{-1}, respectively.

Denote $X = H_{-\frac{1}{2}} = \mathcal{H}^{-1}(\Omega)$ and regard A_0 as a positive operator on X, with domain $X_1 = H_{\frac{1}{2}} = \mathcal{H}^1_0(\Omega)$. According to Remark 3.4.7, $X_{\frac{1}{2}} = H = L^2(\Omega)$ and $X_{-\frac{1}{2}} = H_{-1}$ is the dual of $X_{\frac{1}{2}}$ with respect to the pivot space X.

Proposition 10.6.7. *Let $B_0 \in \mathcal{L}(L^2(\partial\Omega), X_{-\frac{1}{2}})$ be defined by $B_0 = A_0 D$. Identifying $L^2(\partial\Omega)$ with its dual, we have that $B_0^* \in \mathcal{L}(X_{\frac{1}{2}}, L^2(\partial\Omega))$ is given by*

$$B_0^* g = -\frac{\partial (A_0^{-1} g)}{\partial \nu} \qquad \forall\, g \in X_{\frac{1}{2}} .$$

Proof. For $\mathrm{v} \in L^2(\partial\Omega)$ and $g \in X_{\frac{1}{2}}$ we have

$$\langle B_0 \mathrm{v}, g \rangle_{X_{-\frac{1}{2}}, X_{\frac{1}{2}}} = \langle A_0 D\mathrm{v}, g \rangle_{X_{-\frac{1}{2}}, X_{\frac{1}{2}}} = \langle A_0^{\frac{1}{2}} D\mathrm{v}, A_0^{\frac{1}{2}} g \rangle_X .$$

Since $A_0^{\frac{1}{2}}$ is a unitary operator from $X_{\frac{1}{2}}$ to X and $X_{\frac{1}{2}} = H$, it follows that for every $\mathrm{v} \in L^2(\partial\Omega)$ and $g \in X_{\frac{1}{2}}$ we have

$$\langle B_0 \mathrm{v}, g \rangle_{X_{-\frac{1}{2}}, X_{\frac{1}{2}}} = \langle D\mathrm{v}, g \rangle_H .$$

The above formula and (10.6.1) imply that

$$\langle B_0 \mathrm{v}, g \rangle_{X_{-\frac{1}{2}}, X_{\frac{1}{2}}} = -\langle \mathrm{v}, \frac{\partial (A_0^{-1} g)}{\partial \nu} \rangle_H \qquad \forall\, g \in X_{\frac{1}{2}}, \quad \mathrm{v} \in L^2(\partial\Omega),$$

which yields the conclusion. □

10.7 The heat and Schrödinger equations with boundary control

In this section $\Omega \subset \mathbb{R}^n$ is open, bounded and with C^2 boundary $\partial\Omega$. Let Γ be a non-empty open subset of $\partial\Omega$. We first consider an initial and boundary value problem corresponding to the heat equation

$$\frac{\partial z}{\partial t} = \Delta z \text{ in } \Omega \times (0, \infty). \tag{10.7.1}$$

We impose the initial and boundary conditions

$$z = u \qquad \text{on } \Gamma \times (0, \infty), \tag{10.7.2}$$
$$z = 0 \qquad \text{on } (\partial\Omega \setminus \Gamma) \times (0, \infty). \tag{10.7.3}$$
$$z(x, 0) = f(x) \quad \text{for } x \in \Omega. \tag{10.7.4}$$

To formulate these equations as a boundary control system, we introduce the following input space U, solution space Z and state space X:

$$U = L^2(\Gamma), \qquad Z = \mathcal{H}_0^1(\Omega) + DU, \qquad X = \mathcal{H}^{-1}(\Omega), \tag{10.7.5}$$

where D is the Dirichlet map introduced in Proposition 10.6.1. The space U is regarded as a (closed) subspace of $L^2(\partial\Omega)$ by extending any element $\mathrm{v} \in U$ to be zero on $\partial\Omega \setminus \Gamma$. Thus, D can be applied to elements of U.

The operators $L \in \mathcal{L}(Z, X)$ and $G \in \mathcal{L}(Z, U)$ are defined by

$$Lz = \Delta z, \qquad Gz = \gamma_0 z, \tag{10.7.6}$$

where γ_0 is the extension of the Dirichlet trace operator to the domain $\mathcal{W}(\Delta)$ introduced in (10.6.3). We have $Z \subset \mathcal{W}(\Delta)$, as follows from the fact that $\Delta : \mathcal{H}_0^1(\Omega) \to \mathcal{H}^{-1}(\Omega)$ and from Remark 10.6.3. Thus, $\gamma_0 z$ in (10.7.6) is well defined.

As in the previous section (and in Section 3.6), we denote by $-A_0$ the Dirichlet Laplacian on Ω and its various extensions. Again, A_0 can be regarded as a strictly positive operator (densely defined) on X. Considering this extension of A_0, we denote (as at the end of the previous section)

$$X_1 = \mathcal{D}(A_0) = \mathcal{H}_0^1(\Omega), \qquad X_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}}) = L^2(\Omega).$$

The space $X_{-\frac{1}{2}}$ is the dual of $X_{\frac{1}{2}}$ with respect to the pivot space X, hence $X_{-\frac{1}{2}}$ is the dual of $\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$ with respect to the pivot space $L^2(\Omega)$. Using Proposition 3.4.5 and Corollary 3.4.6 (with X in place of H), we see that A_0 can be extended also to an operator in $\mathcal{L}(X_{\frac{1}{2}}, X_{-\frac{1}{2}})$ and A_0 is a unitary operator from X_1 to X, from $X_{\frac{1}{2}}$ to $X_{-\frac{1}{2}}$ and from X to X_{-1}. Similarly, $A_0^{\frac{1}{2}}$ is a unitary operator from X_1 to $X_{\frac{1}{2}}$, from $X_{\frac{1}{2}}$ to X and from X to $X_{-\frac{1}{2}}$. Hence,

$$\langle z, w \rangle_X = \langle A_0^{-\frac{1}{2}} z, A_0^{-\frac{1}{2}} w \rangle_{L^2(\Omega)} \qquad \forall\, z, w \in X.$$

Proposition 10.7.1. *The pair* (L, G) *defined by* (10.7.6) *is a well-posed boundary control system on the spaces* U, Z *and* X *defined by* (10.7.5)*. Its generator is* $A = -A_0$ *and its control operator is* $B = A_0 D$ *(this is* B_0 *from Proposition* 10.6.7*).*

Proof. It follows from Proposition 10.6.4 that if we take an arbitrary element of Z, i.e., an element of the form $z = h + D\mathrm{v}$, where $h \in \mathcal{H}_0^1(\Omega)$ and $\mathrm{v} \in U$, then $Gz = \mathrm{v}$. This shows that G is onto U, as required in Definition 10.1.1. To check the other conditions in this definition, introduce $A = L|_{\mathrm{Ker}\ G}$. Clearly $\mathrm{Ker}\ G = \mathcal{H}_0^1(\Omega)$. Recall from (the second part of) Remark 3.6.3 that on $\mathcal{H}_0^1(\Omega)$, $\Delta = -A_0$. Hence, $A = L|_{\mathrm{Ker}\ G}$ is the generator of the heat semigroup; see Remark 3.6.11. Moreover, it follows that conditions (ii)–(iv) in Definition 10.1.1 are satisfied with $\beta = 0$, so that L and G define a boundary control system.

Let us determine the control operator B of this system. We do this directly from (10.1.3) (the definition of B). Indeed, if $\mathrm{v} \in U$, then (according to (10.1.3))

$$LD\mathrm{v} - AD\mathrm{v} = BGD\mathrm{v}.$$

Taking into account the definitions of L and D and using Proposition 10.6.2, we obtain

$$A_0 D\mathrm{v} = BGD\mathrm{v}.$$

Since $GD = I$ (see Proposition 10.6.4), we obtain the desired formula for B.

It remains to show the well-posedness. We have already seen that $A = -A_0$ generates the heat semigroup $\mathbb{T}$. Using the fact that $B^* \in \mathcal{L}(X_{\frac{1}{2}}, U)$ combined with Proposition 5.1.3, we get that B^* is an admissible observation operator for $\mathbb{T} = \mathbb{T}^*$. By applying Theorem 4.4.3 it follows that B is an admissible control operator for $\mathbb{T}$, so that we have indeed a well-posed boundary control system. □

Using the terminology of the PDE literature, the above result can be stated in terms of the existence and uniqueness of weak solutions of (10.7.1)–(10.7.4), without using any operators.

Definition 10.7.2. For $f \in \mathcal{H}^{-1}(\Omega)$ and $u \in L^2([0, \tau]; L^2(\Gamma))$, we say that

$$z \in C([0, \infty), \mathcal{H}^{-1}(\Omega))$$

is a *weak solution* of (10.7.1)–(10.7.4) if

$$\begin{aligned} &\langle z(t), \psi \rangle_{\mathcal{H}^{-1}(\Omega), \mathcal{H}_0^1(\Omega)} - \langle f, \psi \rangle_{\mathcal{H}^{-1}(\Omega), \mathcal{H}_0^1(\Omega)} \\ &\quad = \int_0^t \langle z(s), \Delta\psi \rangle_{\mathcal{H}^{-1}(\Omega), \mathcal{H}_0^1(\Omega)} \, \mathrm{d}s - \int_0^t \int_\Gamma u(s) \frac{\partial \overline{\psi}}{\partial \nu} \, \mathrm{d}\sigma \, \mathrm{d}t \end{aligned} \tag{10.7.7}$$

for every $t \geqslant 0$ and every $\psi \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$ such that $\Delta\psi \in \mathcal{H}_0^1(\Omega)$.

The above definition is motivated by the fact that if z is a smooth solution of (10.7.1)–(10.7.4), then, by taking the product of (10.7.1) with $\overline{\psi}$ and by integrating by parts on Ω and then on $[0, t]$, we easily obtain (10.7.7). Conversely, if z is smooth enough and it satisfies (10.7.7), then z satisfies (10.7.1)–(10.7.4).

Proposition 10.7.3. *For every $f \in \mathcal{H}^{-1}(\Omega)$ and $u \in L^2([0,\infty); L^2(\Gamma))$, the problem* (10.7.1)–(10.7.4) *admits a unique weak solution, in the sense of Definition* 10.7.2.

Proof. Since $B = A_0 D$ is an admissible control operator for the semigroup $\mathbb{T}$ generated by the self-adjoint $A = -A_0$ on X (as proved in the previous proposition), according to Remark 4.2.6, for every $z_0 \in X$ and every $u \in L^2_{\rm loc}([0,\infty); U)$, there exists a unique $z \in C([0,\infty); X)$ such that, for every $t \geqslant 0$,

$$\langle z(t) - z_0, \varphi\rangle_X = \int_0^t [-\langle z(\zeta), A_0\varphi\rangle_X + \langle u(\zeta), B^*\varphi\rangle_U]\, \mathrm{d}\zeta \qquad \forall\, \varphi \in \mathcal{D}(A). \tag{10.7.8}$$

Using the fact that $A_0^{\frac{1}{2}}$ is an isomorphism from H onto $H_{-\frac{1}{2}}$ and Proposition 10.6.7, it follows that for every $t \geqslant 0$,

$$\begin{aligned}&\langle z(t) - z_0, A_0^{-1}\varphi\rangle_{H_{-\frac{1}{2}}, H_{\frac{1}{2}}} \\ &\quad = -\int_0^t \left[\langle z(\zeta), \varphi\rangle_{H_{-\frac{1}{2}}, H_{\frac{1}{2}}} + \int_\Gamma u(x,\zeta) \frac{\partial\overline{(A_0^{-1}\varphi)}}{\partial\nu}(x,\zeta)\, \mathrm{d}\sigma\right] \mathrm{d}\zeta\,.\end{aligned}$$

Setting $A_0^{-1}\varphi = \psi$, the above formula implies that z is a solution of (10.7.1)–(10.7.4), with $f = z_0$, in the sense of Definition 10.7.2.

We show that this weak solution is unique. Let z be a weak solution of (10.7.1)–(10.7.4), in the sense of Definition 10.7.2. Setting $A_0\varphi = \psi$ it follows that z satisfies (10.7.8). Since, according to Remark 4.2.6, such a z is unique in $C([0,\infty); X)$, we obtain the uniqueness of the weak solution of (10.7.1)–(10.7.4). □

Now we consider an initial and boundary value problem corresponding to the Schrödinger equation

$$\frac{\partial z}{\partial t} = i\Delta z \text{ in } \ \Omega \times (0,\infty).$$

We impose the initial and boundary conditions

$$\begin{aligned} z &= u && \text{on } \ \Gamma \times (0,\infty), \\ z &= 0 && \text{on } \ (\partial\Omega \setminus \Gamma) \times (0,\infty), \\ z(x,0) &= f(x) && \text{for } \ x \in \Omega. \end{aligned}$$

Proposition 10.7.4. *The pair (iL, G) defined by* (10.7.6) *is a well-posed boundary control system on the spaces U, Z and X defined by* (10.7.5). *Its control operator is $B = iB_0$, where $B_0 = A_0 D$ is as in Proposition* 10.6.7.

Proof. We have seen in the proof of Proposition 10.7.1 that G maps Z onto U. Moreover, we have $\operatorname{Ker} G = \mathcal{H}_0^1(\Omega)$ and $L|_{\operatorname{Ker} G} = -iA_0$, so that $L|_{\operatorname{Ker} G}$ generates a unitary group $\mathbb{T}$ on X. It follows that conditions (ii)–(iv) in Definition 10.1.1 are satisfied with $\beta = 0$, so that L and G define a boundary control system.

In order to determine the control operator B of this system we use (10.1.3). More precisely, if $\mathrm{v} \in U$ then, according to (10.1.3), we have

$$iLD\mathrm{v} + iA_0 D\mathrm{v} = BGD\mathrm{v}.$$

Since $LD = 0$ and $GD = I$, we obtain the claimed formula for B.

The well-posedness of this boundary control system is equivalent to the fact that B^* is an admissible observation operator for the semigroup $\mathbb{T}^*$, which in our case is the inverse semigroup of $\mathbb{T}$. According to Proposition 10.6.7, we have

$$B^* g = i\,\frac{\partial(A_0^{-1} g)}{\partial \nu} = iC_1(A_0^{-1} g) \qquad \forall\, g \in X_{\frac{1}{2}}\,,$$

where $C_1 f = \frac{\partial f}{\partial \nu}|_\Gamma$. According to Proposition 7.5.1, C_1 is an admissible observation operator for $\mathbb{T}$ acting on $X_1 = H_{\frac{1}{2}}$. This implies that B^* is admissible for $\mathbb{T}$ acting on X, and hence also for its inverse $\mathbb{T}^*$ acting on X. $\square$

Remark 10.7.5. The concept of weak solution of the Schrödinger equation, with the initial and boundary conditions imposed as before Proposition 10.7.4, is a very slight modification of the one in Definition 10.7.2. It follows from the last proposition that for every $f \in \mathcal{H}^{-1}(\Omega)$ and every $u \in L^2([0,\infty); L^2(\Gamma))$, the Schrödinger equation with the initial and boundary conditions mentioned above has a unique weak solution. The proof is a very slight modification of the proof of Proposition 10.7.3.

10.8 The convection-diffusion equation with boundary control

In this section, $\Omega \subset \mathbb{R}^n$ is open, bounded and with C^2 boundary $\partial\Omega$. In Example 5.4.4 we have introduced (with less assumptions on Ω) the operator semigroup $\mathbb{T}^{cl}$ corresponding to the convection-diffusion equation

$$\frac{\partial z}{\partial t} = \Delta z + b \cdot \nabla z + cz \quad \text{in} \;\; \Omega \times (0,\infty), \tag{10.8.1}$$

with the homogeneous boundary condition

$$z = 0 \quad \text{on} \;\; \partial\Omega.$$

In this section we assume that

$$b \in L^\infty(\Omega; \mathbb{C}^n), \qquad c \in L^\infty(\Omega), \qquad \operatorname{div} b \in L^\infty(\Omega) \tag{10.8.2}$$

(this is more restrictive than in Example 5.4.4). We continue to regard $\mathbb{T}^{cl}$ as a perturbation of the heat semigroup, of the type discussed in Section 5.4. However, in this section we need to extend the semigroup $\mathbb{T}^{cl}$ to the larger state space $X = \mathcal{H}^{-1}(\Omega)$. This is needed in order to introduce a boundary control system corresponding to the same PDE with Dirichlet boundary control.

We denote $H = L^2(\Omega)$, A is the Dirichlet Laplacian on Ω, so that (as shown in Section 3.6) $A < 0$, $\mathcal{D}(A) = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$, $H_{\frac{1}{2}} = \mathcal{D}((-A)^{\frac{1}{2}}) = \mathcal{H}_0^1(\Omega)$ and $H_{-\frac{1}{2}} = \mathcal{H}^{-1}(\Omega)$. We define $C \in \mathcal{L}(H_{\frac{1}{2}}, H)$ by

$$Cz = b \cdot \nabla z + cz \qquad \forall\, z \in \mathcal{H}_0^1(\Omega). \tag{10.8.3}$$

Remark 10.8.1. In this remark we discuss the extension of the semigroup $\mathbb{T}^{cl}$ to the space H_{-1}. It is not difficult to verify (using (13.3.1) to express div $(\overline{b}\psi)$, first for $\psi \in \mathcal{D}(\Omega)$ and then for $\psi \in \mathcal{H}_0^1(\Omega)$ by continuous extension) that

$$\begin{aligned}(A + C)^*\psi &= \Delta\psi - \operatorname{div}(\overline{b}\psi) + \overline{c}\psi \\ &= \Delta\psi - \overline{b} \cdot \nabla\psi + \overline{(c - \operatorname{div} b)}\psi\end{aligned}$$

for all $\psi \in \mathcal{D}((A + C)^*) = \mathcal{D}(A)$. Thus, $(A + C)^*$ is a perturbation of A of a similar nature as $A + C$. The graph norms of A, $A + C$ and $(A + C)^*$ on $\mathcal{D}(A)$ are clearly equivalent. It follows that the space H_{-1} for $A + C$ (which is the dual of $H_1^d = \mathcal{D}((A + C)^*)$ with respect to the pivot space H, see Proposition 2.10.2) is the same as the space H_{-1} for A. According to Proposition 2.10.4, $\mathbb{T}^{cl}$ can be extended to a strongly continuous semigroup $\widetilde{\mathbb{T}^{cl}}$ on H_{-1}, and the generator of this extended semigroup is an extension of $A + C$, denoted $\widetilde{A + C}$, with domain H. For us, it is more interesting to understand the extension of $\mathbb{T}^{cl}$ to the smaller space $H_{-\frac{1}{2}}$. This extension of the original $\mathbb{T}^{cl}$ can be understood using the properties of the semigroup $\mathbb{T}^{cl}$ acting on H and H_{-1} and then using the interpolation results from Remark 3.4.10. An alternative, direct approach will be used in what follows.

As in the previous two sections, introduce the state space $X = H_{-\frac{1}{2}} = \mathcal{H}^{-1}(\Omega)$. It is clear that the heat semigroup $\mathbb{T}$ generated by A, originally defined on H, has a continuous extension to an operator semigroup acting on X, still denoted by $\mathbb{T}$. The generator of this semigroup is an extension of the Dirichlet Laplacian, still denoted by A, with $\mathcal{D}(A) = X_1 = H_{\frac{1}{2}}$. Unless otherwise stated, when we write A, we mean this operator. As mentioned in Section 10.6, we have $X_{\frac{1}{2}} = \mathcal{D}((-A)^{\frac{1}{2}}) = L^2(\Omega)$.

Lemma 10.8.2. *We define C by* (10.8.3)*, where b and c satisfy* (10.8.2)*. Then C has an extension C^e such that*

$$C^e \in \mathcal{L}(L^2(\Omega), \mathcal{H}^{-1}(\Omega)). \tag{10.8.4}$$

Proof. We rewrite C, using (13.3.1), as

$$Cz = \operatorname{div}(bz) - (\operatorname{div} b)z + cz\,.$$

This is true for $z \in \mathcal{D}(\Omega)$ and, by the density of $\mathcal{D}(\Omega)$ in $\mathcal{H}_0^1(\Omega)$, it holds for all $z \in \mathcal{H}_0^1(\Omega)$. Since div is a bounded operator from $L^2(\Omega)$ to $\mathcal{H}^{-1}(\Omega)$ (this follows from Proposition 13.4.9), using our assumptions on b and c, (10.8.4) follows. $\square$

In what follows we consider the boundary controlled convection-diffusion equation. Let Γ be a non-empty open subset of $\partial\Omega$. We consider the initial and boundary value problem corresponding to the convection-diffusion equation (10.8.1), where b and c are as in (10.8.2). We impose the initial and boundary conditions (10.7.2)–(10.7.4). To formulate these equations as a boundary control system, we introduce the same input and solution spaces as in the previous section:

$$U = L^2(\Gamma), \qquad Z = \mathcal{H}_0^1(\Omega) + DU,$$

where D is the Dirichlet map. As usual, U is regarded as a subspace of $L^2(\partial\Omega)$.

The operators $L^{cl} \in \mathcal{L}(Z, X)$ and $G \in \mathcal{L}(Z, U)$ are defined by

$$L^{cl} z = \Delta z + C^e z, \qquad Gz = \gamma_0 z, \tag{10.8.5}$$

where Δ is the Laplacian in the sense of distributions, C^e is the operator introduced in Lemma 10.8.2 and γ_0 is a suitable extension of the Dirichlet trace operator (as in the previous section). As explained after (10.7.6), $\gamma_0 z$ in (10.8.5) is well defined. It is clear that indeed L^{cl} corresponds to the convection-diffusion equation (10.8.1).

Theorem 10.8.3. *The pair (L^{cl}, G) defined above is a well-posed boundary control system on the spaces U, Z and X.*

Proof. We denote $L = \Delta$ (as in the previous section), so that (according to Proposition 10.7.1) (L, G) is a well-posed boundary control system on U, Z and X. The generator of (L, G) is A and its control operator is $B_1 = -AD$. According to Proposition 5.1.3 and Lemma 10.8.2, C from (10.8.3) is an admissible observation operator for the heat semigroup $\mathbb{T}$ on X (with output space X). We have $L^{cl} = L + C^e$, where C^e is the extension of C from Lemma 10.8.2. Clearly $Z \subset L^2(\Omega)$ with continuous embedding, so that $C^e \in \mathcal{L}(Z, X)$. To be able to apply Proposition 10.1.10 (with $Y = X$ and $B = I$) we only have to verify that for some $\alpha \in \mathbb{R}$,

$$\|C^e(sI - A)^{-1} B_1\|_{\mathcal{L}(U,X)} \leqslant M \qquad \forall\, s \in \mathbb{C}_\alpha. \tag{10.8.6}$$

Let us factor $C^e = C^b(-A)^{\frac{1}{2}}$, where $C^b \in \mathcal{L}(X)$ (we have used that $(-A)^{\frac{1}{2}}$ is unitary from $X_{\frac{1}{2}}$ to X). Similarly, we factor $B_1 = (-A)^{\frac{1}{2}}(-A)^{\frac{1}{2}}D$, where $(-A)^{\frac{1}{2}}D \in \mathcal{L}(U, X)$. Then (10.8.6) follows if we can show that for some $\alpha \in \mathbb{R}$ we have

$$\|(-A)^{\frac{1}{2}}(sI - A)^{-1}(-A)^{\frac{1}{2}}\|_{\mathcal{L}(X)} \leqslant M \qquad \forall\, s \in \mathbb{C}_\alpha.$$

For every $\alpha \geqslant 0$, the above estimate is an easy consequence of $A < 0$. Thus, the statement follows from Proposition 10.1.10. $\square$

Remark 10.8.4. With the notation of the last theorem, the generator of the well-posed boundary control system (L^{cl}, G) is $A+C$ and its control operator is $-JAD$, where J is the extension of the identity operator introduced in Proposition 10.1.10. These additional statements follow from Proposition 10.1.10, once we have reached the end of the proof of the theorem.

Remark 10.8.5. Let us denote by $\mathbb{T}^{cl}$ the operator semigroup on X corresponding to the well-posed boundary control system in Theorem 10.8.3. As explained in the previous remark, its generator is $A + C$. This semigroup is an extension of the one from Example 5.4.4. This follows from the last part of Proposition 2.4.4 (with $V = L^2(\Omega)$). In particular, it follows that $L^2(\Omega)$ is an invariant subspace for $\mathbb{T}^{cl}$.

10.9 The wave equation with Dirichlet boundary control

The physical system that we have in mind in this section consists of a vibrating membrane which is fixed on a part of the boundary, while the displacement field is controlled on the remaining part of the boundary. A membrane could be modeled in a domain in $\mathbb{R}^2$, but we consider a more general wave equation on a bounded n-dimensional domain Ω. We denote by Γ the part of $\partial\Omega$ where the control acts. Our model is the following initial and boundary value problem:

$$\frac{\partial^2 w}{\partial t^2} = \Delta w \qquad \text{in } \ \Omega \times (0, \infty), \tag{10.9.1}$$

$$w = 0 \qquad \text{on } \ \partial\Omega \setminus \Gamma \times (0, \infty), \tag{10.9.2}$$

$$w = u \qquad \text{on } \ \Gamma \times (0, \infty), \tag{10.9.3}$$

$$w(x, 0) = f(x), \quad \frac{\partial w}{\partial t}(x, 0) = g(x) \quad \text{for } x \in \Omega. \tag{10.9.4}$$

The input of this system is the function u in (10.9.3). At the end of this section we shall define weak solutions for the above initial and boundary value problem and we shall prove the existence and uniqueness of these solutions. We shall also see that lower-order terms can be added in the equation at no extra cost in the difficulty of proving its well-posedness. (This is in contrast to the convection-diffusion equation, where the lower-order terms needed much effort to handle.)

Notation. In this section, $\Omega \subset \mathbb{R}^n$ is bounded and open with boundary $\partial\Omega$ of class C^2. Let Γ be an open subset of $\partial\Omega$ and denote $U = L^2(\Gamma)$. For $\varphi \in \mathcal{H}^1(\Omega)$, we denote by $\varphi|_\Gamma$ the restriction of the boundary trace $\gamma_0\varphi$ to Γ. Similarly, for $\varphi \in \mathcal{H}^2(\Omega)$, we denote by $\frac{\partial\varphi}{\partial\nu}|_\Gamma$ the restriction of the normal derivative $\gamma_1\varphi$ to Γ (γ_0 and γ_1 have been introduced in Section 10.6). We denote $H = L^2(\Omega)$ and the operator A_0 is the Dirichlet Laplacian defined in Section 3.6. With the above smoothness assumptions on $\partial\Omega$, we know from Theorem 3.6.2 that $A_0 : H_1 \to H$

is defined by

$$H_1 = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega), \qquad A_0 f = -\Delta f \qquad \forall\, f \in H_1 .$$

We know from Proposition 3.6.1 that A_0 is strictly positive and that the Hilbert spaces $H_{\frac{1}{2}}$ and $H_{-\frac{1}{2}}$ obtained from H and A_0, according to the definitions in Section 3.4, are given by

$$H_{\frac{1}{2}} = \mathcal{H}_0^1(\Omega), \qquad H_{-\frac{1}{2}} = \mathcal{H}^{-1}(\Omega).$$

We know from Corollary 3.4.6 and Remark 3.4.7 that A_0 can be extended to a unitary operator from $H_{\frac{1}{2}}$ onto $H_{-\frac{1}{2}}$ and from H onto H_{-1}. As usual, these extensions will be denoted also by A_0. The inner products in $H_{\frac{1}{2}}$, H and $H_{-\frac{1}{2}}$ will be denoted by $\langle\cdot,\cdot\rangle_{\frac{1}{2}}$, $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{-\frac{1}{2}}$. We also introduce the spaces

$$X = H \times H_{-\frac{1}{2}} = L^2(\Omega) \times \mathcal{H}^{-1}(\Omega), \qquad \mathcal{D}(A) = H_{\frac{1}{2}} \times H = \mathcal{H}_0^1(\Omega) \times L^2(\Omega)$$

and the operator $A : \mathcal{D}(A) \to X$ defined by

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}. \tag{10.9.5}$$

Since A_0 is strictly positive, we know from Proposition 3.7.6 that A is skew-adjoint, so that it generates a unitary group $\mathbb{T}$ on X. We also know that $0 \in \rho(A)$. Moreover, we have $X_{-1} = H_{-\frac{1}{2}} \times H_{-1}$. Finally, we introduce

$$W = \mathcal{H}_0^1(\Omega) + DU, \tag{10.9.6}$$

where D is the Dirichlet map introduced in Proposition 10.6.1. Note that this space has been denoted by Z in Section 10.7, where it was used as the solution space for the boundary controlled heat and Schrödinger equations. However, in this section we shall need the notation Z for the solution space for the wave equation.

To formulate (10.9.1)–(10.9.4) as a boundary control system, we introduce the solution space

$$Z = W \times H .$$

The operators $L \in \mathcal{L}(Z, X)$ and $G \in \mathcal{L}(Z, U)$ are defined by

$$L \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \Delta f \end{bmatrix}, \qquad G \begin{bmatrix} f \\ g \end{bmatrix} = f|_\Gamma \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in Z . \tag{10.9.7}$$

The fact that L takes values in X follows from the decomposition (10.9.6). Indeed, any $f \in W$ can be written as $f = f_0 + D\mathrm{v}$, with $f_0 \in \mathcal{H}_0^1(\Omega)$ and $\mathrm{v} \in L^2(\Omega)$, which implies (using Proposition 10.6.2) that $\Delta f = \Delta f_0 \in \mathcal{H}^{-1}(\Omega)$.

Proposition 10.9.1. *The pair (L, G) is a well-posed boundary control system on U, Z and X. Its control operator and its adjoint are given by*

$$B\mathrm{v} = \begin{bmatrix} 0 \\ A_0 D\mathrm{v} \end{bmatrix} \qquad \forall\, \mathrm{v} \in U, \tag{10.9.8}$$

$$B^* \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = - \frac{\partial}{\partial \nu} \left(A_0^{-1} \psi \right) \Big|_\Gamma \qquad \forall \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A^*) = \mathcal{D}(A). \tag{10.9.9}$$

Proof. It follows from Proposition 10.6.4 that if we take an arbitrary element of W, i.e., an element of the form $f = f_0 + D\mathrm{v}$, where $f_0 \in \mathcal{H}_0^1(\Omega)$ and $\mathrm{v} \in U$, then $G \left[\begin{smallmatrix} f \\ 0 \end{smallmatrix} \right] = \mathrm{v}$. This shows that G is onto U, as required in Definition 10.1.1. Notice that $\operatorname{Ker} G = \mathcal{D}(A)$ and $L|_{\operatorname{Ker} G} = A$. Indeed, we know from (the second part of) Remark 3.6.3 that $\Delta = -A_0$ on $\mathcal{H}_0^1(\Omega)$. We know that A is skew-adjoint and $0 \in \rho(A)$, so that conditions (ii)–(iv) in Definition 10.1.1 are satisfied with $\beta = 0$. Thus, L and G define a boundary control system.

In order to write a formula for B, we use Remark 10.1.5. For every $\mathrm{v} \in \mathbb{C}$, we solve the following abstract elliptic problem in the unknown $\left[\begin{smallmatrix} f \\ g \end{smallmatrix} \right] \in Z$:

$$L \begin{bmatrix} f \\ g \end{bmatrix} = 0, \qquad G \begin{bmatrix} f \\ g \end{bmatrix} = \mathrm{v}.$$

This problem is equivalent to $g = 0$, $f \in W$, $\Delta f = 0$ and $\gamma_0 f = \mathrm{v}$. It is easy to see that the unique solution of this problem is given by $f = D\mathrm{v}$. (Proposition 10.6.6 is not needed for this.) Using Remark 10.1.5 with $\beta = 0$, we obtain that $(-A)^{-1} B\mathrm{v} = \left[\begin{smallmatrix} D\mathrm{v} \\ 0 \end{smallmatrix} \right]$. Applying A to both sides, we obtain (10.9.8).

In order to express B^*, we use Remark 10.1.6. We take $\left[\begin{smallmatrix} f \\ g \end{smallmatrix} \right] \in Z$ and $\left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix} \right] \in \mathcal{D}(A^*) = \mathcal{D}(A)$. Then, using that $A^* = -A$, we have

$$\begin{aligned} &\left\langle L \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \right\rangle_X - \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, A^* \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \right\rangle_X \\ &\qquad = \langle g, \varphi \rangle + \langle \Delta f, \psi \rangle_{-\frac{1}{2}} + \langle f, \psi \rangle - \langle g, A_0 \varphi \rangle_{-\frac{1}{2}}. \end{aligned} \tag{10.9.10}$$

Using that

$$f = f_0 + D\mathrm{v} \quad \text{with} \quad f_0 \in \mathcal{H}_0^1(\Omega), \quad \mathrm{v} \in U,$$

it follows that the second term on the right-hand side of the above relation can be written as

$$\langle \Delta f, \psi \rangle_{-\frac{1}{2}} = \langle \Delta f_0, \psi \rangle_{-\frac{1}{2}} = \langle -f_0, \psi \rangle,$$

since $A_0^{-1}(\Delta f_0) = -f_0$. Writing $f_0 = f - D\mathrm{v}$ and using (10.6.1), it follows that

$$\langle \Delta f, \psi \rangle_{-\frac{1}{2}} = - \langle f, \psi \rangle - \left\langle \mathrm{v}, \frac{\partial (A_0^{-1} \psi)}{\partial \nu} \right\rangle_U. \tag{10.9.11}$$

The inner product $\langle g, A_0\varphi\rangle_{-\frac{1}{2}}$ from (10.9.10) can be expressed, using that $A_0^{\frac{1}{2}}$ is unitary from $H_{\frac{1}{2}}$ to H, as follows:

$$\langle g, A_0\varphi\rangle_{-\frac{1}{2}} = \langle g, \varphi\rangle . \tag{10.9.12}$$

By combining (10.9.10), (10.9.11) and (10.9.12), it follows that

$$\left\langle L\begin{bmatrix} f \\ g\end{bmatrix}, \begin{bmatrix} \varphi \\ \psi\end{bmatrix}\right\rangle_X - \left\langle \begin{bmatrix} f \\ g\end{bmatrix}, A^*\begin{bmatrix} \varphi \\ \psi\end{bmatrix}\right\rangle_X = -\left\langle \mathrm{v}, \frac{\partial(A_0^{-1}\psi)}{\partial\nu}\right\rangle_U .$$

Comparing this with (10.1.7), we obtain (10.9.9).

To show that our boundary control system is well posed we note that

$$B^*A\begin{bmatrix} f \\ g\end{bmatrix} = \frac{\partial f}{\partial\nu}|_\Gamma \qquad \forall \begin{bmatrix} f \\ g\end{bmatrix} \in \mathcal{D}(A^2) = H_1 \times H_{\frac{1}{2}} .$$

Denote $C = B^*A \in \mathcal{L}(X_2, U)$, where $X_2 = \mathcal{D}(A^2) = H_1 \times H_{\frac{1}{2}}$ with the graph norm (see Remark 2.10.5). We know from Theorem 7.1.3 that C is an admissible observation operator for $\mathbb{T}$ acting on X_1, and hence also for its inverse semigroup (whose generator is $-A$). Since $\mathbb{T}$ is unitary (on any of the spaces X, X_1), it follows that C is admissible for $\mathbb{T}^*$ acting on the space X_1. This implies that $B^* = CA^{-1}$ is an admissible observation operator for the semigroup $\mathbb{T}^*$ acting on X. From the duality result in Theorem 4.4.3, it follows that B is an admissible control operator for $\mathbb{T}$ acting on X. $\square$

Let us express the above result using the terminology commonly used by researchers working on PDEs. First we define a concept of weak solution of (10.9.1)–(10.9.4) in terms of these equations only, without using any operators.

Definition 10.9.2. For $u \in L^2([0,\infty); L^2(\Gamma))$, $f \in L^2(\Omega)$ and $g \in \mathcal{H}^{-1}(\Omega)$, a function

$$w \in C([0,\infty), L^2(\Omega)) \cap C^1([0,\infty), \mathcal{H}^{-1}(\Omega))$$

is called a *weak solution of* (10.9.1)–(10.9.4) if the relation

$$\begin{aligned} &\int_\Omega w(x,t)\varphi(x)\,\mathrm{d}x - \int_\Omega f(x)\varphi(x)\,\mathrm{d}x - t\langle g, \varphi\rangle_{\mathcal{H}^{-1}(\Omega), \mathcal{H}_0^1(\Omega)} \\ &\quad = \int_0^t\int_0^s\int_\Omega w(x,\zeta)\Delta\varphi(x)\,\mathrm{d}x\,\mathrm{d}\zeta\,\mathrm{d}s - \int_0^t\int_0^s\int_\Gamma u(x,\zeta)\frac{\partial\varphi}{\partial\nu}(x)\,\mathrm{d}\sigma\,\mathrm{d}\zeta\,\mathrm{d}s \end{aligned} \tag{10.9.13}$$

holds for every $t \geqslant 0$ and every $\varphi \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$.

This definition is motivated by the fact that if we assume that w is a smooth solution of (10.9.1)–(10.9.4), then, multiplying (10.9.1) by φ and integrating by parts on Ω and then twice in time, we easily obtain (10.9.13). Conversely, if w is smooth enough and it satisfies (10.9.13), then it is easy to see that w satisfies (10.9.1)–(10.9.4).

The main result of this section is the following.

Theorem 10.9.3. *For every $f \in L^2(\Omega)$, $g \in \mathcal{H}^{-1}(\Omega)$ and $u \in L^2\big([0,\infty); L^2(\Gamma)\big)$, the system* (10.9.1)–(10.9.4) *admits a unique weak solution, in the sense of Definition* 10.9.2*. Moreover, for every $\tau > 0$, the map $u \mapsto w$ is bounded from $L^2([0,\tau]; L^2(\Gamma))$ to $C([0,\tau]; L^2(\Omega)) \cap C^1([0,\tau]; \mathcal{H}^{-1}(\Omega))$. This solution coincides with the solution of $\dot z = Az + Bu$, $z(0) = \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$, as given in Proposition* 4.2.5*, if we put $z = \left[\begin{smallmatrix} w \\ \dot w \end{smallmatrix}\right]$.*

Proof. Since B is an admissible control operator for $\mathbb{T}$ acting on X (as proved in the previous proposition), according to Remark 4.2.6, for every $z_0 \in X$ and every $u \in L^2_{\rm loc}([0,\infty); U)$, there exists a unique $z \in C([0,\infty); X)$ such that, for every $t \geqslant 0$,

$$\langle z(t) - z_0, \phi\rangle_X = \int_0^t \left[\langle z(\zeta), A^*\phi\rangle_X + \langle u(\zeta), B^*\phi\rangle_U\right] \mathrm{d}\zeta \qquad \forall\, \phi \in \mathcal{D}(A^*). \tag{10.9.14}$$

Taking here $z(t) = \left[\begin{smallmatrix} w(t) \\ v(t) \end{smallmatrix}\right]$, $\phi = \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix}\right]$ and $z_0 = \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$, we obtain that, for every $t \geqslant 0$,

$$\begin{aligned}&\langle w(t) - f, \varphi\rangle_H + \langle v(t) - g, \psi\rangle_{H_{-\frac12}} \\ &\quad = \int_0^t \left[-\langle w(\zeta), \psi\rangle_H + \langle v(\zeta), A_0\varphi\rangle_{H_{-\frac12}} - \int_\Gamma u(x,\zeta) \frac{\partial(\overline{A_0^{-1}\psi})}{\partial\nu}(x,\zeta)\,\mathrm{d}\sigma\right] \mathrm{d}\zeta\end{aligned}$$

for every $\varphi \in H_{\frac12}$, $\psi \in H$ (we have used (10.9.9) to express B^*). Using the fact that $A_0^{\frac12}$ is an isomorphism from H onto $H_{-\frac12}$, it follows that

$$\begin{aligned}&\langle w(t) - f, \varphi\rangle_H + \langle v(t) - g, A_0^{-1}\psi\rangle_{H_{-\frac12}, H_{\frac12}} \\ &\quad = \int_0^t \left[-\langle w(\zeta), \psi\rangle_H + \langle v(\zeta), \varphi\rangle_{H_{-\frac12}, H_{\frac12}} - \int_\Gamma u(x,\zeta) \frac{\partial(\overline{A_0^{-1}\psi})}{\partial\nu}(x,\zeta)\,\mathrm{d}\sigma\right] \mathrm{d}\zeta\,.\end{aligned} \tag{10.9.15}$$

Choosing $\psi = 0$ in the above relation, it follows that $v(t) = \dot w(t)$. Therefore

$$w \in C([0,\infty); L^2(\Omega)) \cap C^1([0,\infty), \mathcal{H}^{-1}(\Omega))\,.$$

Using $v(t) = \dot w(t)$ in (10.9.15), it follows that for every $\psi \in H$, we have

$$\langle \dot w(t) - g, A_0^{-1}\psi\rangle_{H_{-\frac12}, H_{\frac12}} = -\int_0^t \left[\langle w(\zeta), \psi\rangle_H + \int_\Gamma u(x,\zeta) \frac{\partial(\overline{A_0^{-1}\psi})}{\partial\nu}(x,\zeta)\,\mathrm{d}\sigma\right] \mathrm{d}\zeta\,.$$

Using in the above the fact that A_0 is an isomorphism from H_1 onto H, it follows that for every $\eta \in H_1$, we have

$$\langle \dot w(t) - g, \eta\rangle_{H_{-\frac12}, H_{\frac12}} = -\int_0^t \left[\langle w(\zeta), A_0\eta\rangle_H + \int_\Gamma u(x,\zeta) \frac{\partial\overline{\eta}}{\partial\nu}(x,\zeta)\,\mathrm{d}\sigma\right] \mathrm{d}\zeta\,. \tag{10.9.16}$$

Integrating the last formula with respect to t, it follows that w is a weak solution of (10.9.1)–(10.9.4), in the sense of Definition 10.9.2 (using $\eta = \overline{\varphi}$).

Now we show that this weak solution is unique. Indeed, let w be a weak solution of (10.9.1)–(10.9.4), in the sense of Definition 10.9.2. By differentiating (10.9.13) with respect to t, it follows that w satisfies (10.9.16) (with $\eta = \overline{\varphi}$). From here it is easy to check that $z(t) = \left[\begin{smallmatrix} w(t) \\ \dot{w}(t) \end{smallmatrix}\right]$ satisfies (10.9.14). Since, according to Remark 4.2.6, such a z is unique in $C([0,\infty);X)$, we obtain the uniqueness of the weak solution of (10.9.1)–(10.9.4), in the sense of Definition 10.9.2. $\square$

10.10 Remarks and bibliographical notes on Chapter 10

Section 10.1. The abstract theory of boundary control systems started with Fattorini [61] and it was significantly developed by Salamon [203]. Our exposition follows the ideas of [203], but in a more concise form. Relevant earlier references on the translation of boundary control systems into the semigroup language can be found in [203] and also in the survey of Emirsajlow and Townley [56]. Interesting recent papers on passive and conservative boundary control systems are Malinen and Staffans [165, 166]. As already mentioned, most references consider also an output given by $y = Kz$, where $K \in \mathcal{L}(Z, Y)$, and a boundary control system is defined as the triple (L, G, K). Without such an output, the discussion of passivity in [165, 166] would not be possible. In this chapter we are only concerned with the pair (L, G) and the translation of (10.1.1) into the standard form $\dot{z}(t) = Az(t) + Bu(t)$.

The definition of a boundary control system in [203] (see assumption (B0) there) is not exactly the same as ours. Apart from the fact that we do not consider outputs, the difference is that instead of our assumption (iii) the following weaker requirement appears: "$\beta I - L$ is onto". From the subsequent text in [203] it is clear that Salamon believed his assumptions to imply that $\beta I - A$ is invertible. Unfortunately, this is not the case. For example, consider $U = \mathbb{C}^2$, $Z = \mathcal{H}^1(0,1)$, $X = L^2[0,1]$, $Lz = z'$, $Gz = [z(0)\ z(1)]$, then A is dissipative but not m-dissipative.

Sections 10.2–10.5. These examples of systems in one space dimension are classical and we are not able to trace their origin. Our treatment of the beam from Section 10.5 is a particular case of the arguments in Section 4 of Zhao and Weiss [242].

Section 10.6. The existence, uniqueness and regularity properties for the Laplacian with homogeneous Dirichlet boundary conditions are classical topics, presented in most of the standard books on PDEs (see, for instance, Brezis [22], Evans [59] or Taylor [217]). The Laplace equation with non-homogeneous Dirichlet boundary conditions can be reduced to the homogeneous case if the boundary trace is in $H^{\frac{1}{2}}(\partial\Omega)$, but things get more complicated for less regular boundary data. The study of the latter case is more difficult to find in classical books. Our presentation of the Dirichlet map, defined on $L^2(\partial\Omega)$, is close to the "transposition method" as described in Lions and Magenes [157]. However, some of the properties we derive (such as Proposition 10.6.4) have not been published before, as far as we know.

Sections 10.7 and 10.8. The semigroup approach to parabolic equations with non-homogeneous Dirichlet boundary conditions (in view of control) in $L^2(\partial\Omega)$ has been introduced (as far as we know) in Balakrishnan [12] and Washburn [225]. An alternative definition of weak solutions, which was also extended for non-linear equations, relies on taking test functions that depend both on the time and on the space variables. We refer the reader to Amann [4] for a concise presentation of this approach.

Sections 10.9. The use of the Dirichlet map and of semigroup theory for hyperbolic equations with non-homogeneous Dirichlet boundary conditions in $L^2(\partial\Omega)$ goes back to Lasiecka and Triggiani [144, 145]. The fact that, for every $\tau > 0$, the corresponding boundary control system defines a bounded map from $L^2(\partial\Omega)$ to

$$C([0,\tau]; L^2(\Omega)) \cap C^1([0,\tau]; \mathcal{H}^{-1}(\Omega)),$$

has been shown in [145]. A different notion of weak solution for the wave equation with non-homogeneous Dirichlet boundary conditions in $L^2(\partial\Omega)$ has been introduced in Lions [156]. This notion of weak solution can be defined briefly as follows: For $u \in L^2([0,\infty); L^2(\Gamma))$, $f \in L^2(\Omega)$ and $g \in \mathcal{H}^{-1}(\Omega)$, a function

$$w \in C([0,\tau], L^2(\Omega)) \cap C^1([0,\tau], \mathcal{H}^{-1}(\Omega))$$

is called a *weak solution of* (10.9.1)–(10.9.4) if the relation

$$\begin{aligned} &\int_0^\tau \int_\Omega w(x,t)\left(\ddot{\theta}(x,t) - \Delta\theta(x,t)\right)\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_\Omega f(x)\theta(x,0)\,\mathrm{d}x - \langle g, \theta(\cdot,0)\rangle_{\mathcal{H}^{-1}(\Omega),\mathcal{H}_0^1(\Omega)} \\ &= -\int_0^\tau \int_\Gamma u(x,t)\frac{\partial\theta}{\partial\nu}(x,t)\,\mathrm{d}\sigma\,\mathrm{d}t \end{aligned}$$

holds for every function θ satisfying

$$\theta \in C([0,\tau]; \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)) \cap C^1([0,\tau]; \mathcal{H}_0^1(\Omega)),$$
$$\theta(\cdot,\tau) = \dot{\theta}(\cdot,\tau) = 0.$$

It is not difficult to check that the above notion of weak solution coincides with the concept from Definition 10.9.2.

Chapter 11

Controllability

Notation. Throughout this chapter, U, X and Y are complex Hilbert spaces which are identified with their duals. $\mathbb{T}$ is a strongly continuous semigroup on X, with generator $A : \mathcal{D}(A) \to X$ and growth bound $\omega_0(\mathbb{T})$. Remember that we use the notation A and $\mathbb{T}_t$ also for the extension of the original generator to X and for the extension of the original semigroup to X_{-1}. Recall also that X_1^d is $\mathcal{D}(A^*)$ with the norm $\|z\|_1^d = \|(\overline{\beta}I - A^*)z\|$ and X_{-1}^d is the completion of X with respect to the norm $\|z\|_{-1}^d = \|(\overline{\beta}I - A^*)^{-1}z\|$. Recall that X_{-1} is the dual of X_1^d with respect to the pivot space X.

For $u \in L^2_{\mathrm{loc}}([0,\infty);U)$ and $\tau \geqslant 0$, the truncation of u to $[0,\tau]$ is denoted by $\mathbf{P}_\tau u$. This is regarded as an element of $L^2([0,\infty);U)$ which is zero for $t > \tau$.

For any open interval J, the spaces $\mathcal{H}^1(J;U)$ and $\mathcal{H}^2(J;U)$ are defined as at the beginning of Chapter 2. $\mathcal{H}^1_{\mathrm{loc}}(0,\infty;U)$ is the space of those functions on $(0,\infty)$ whose restriction to $(0,n)$ is in $\mathcal{H}^1(0,n;U)$, for every $n \in \mathbb{N}$. The space $\mathcal{H}^2_{\mathrm{loc}}(0,\infty;U)$ is defined similarly. Recall that $\mathbb{C}_\alpha$ is the half-plane where $\mathrm{Re}\, s > \alpha$.

11.1 Some controllability concepts

For infinite-dimensional systems we have at least three important controllability concepts, each depending on the time τ. In this section we introduce these concepts and explore how they are related to each other.

We assume that U is a complex Hilbert space and $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$. According to the definition in Section 4.2, this means that for every $\tau \geqslant 0$, the formula

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma} B u(t)\, \mathrm{d}\sigma \tag{11.1.1}$$

defines a bounded operator $\Phi_\tau : L^2([0,\infty);U) \to X$.

Definition 11.1.1. Let $\tau > 0$.

- The pair (A, B) is *exactly controllable in time* τ if $\operatorname{Ran} \Phi_\tau = X$.
- The pair (A, B) is *approximately controllable in time* τ if $\operatorname{Ran} \Phi_\tau$ is dense in X.
- The pair (A, B) is *null-controllable in time* τ if $\operatorname{Ran} \Phi_\tau \supset \operatorname{Ran} \mathbb{T}_\tau$.

It is easy to see that exact controllability in time τ is equivalent to the following property: For any $z_0, z_1 \in X$ there exists $u \in L^2([0, \tau]; U)$ such that the solution z of

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z_0, \tag{11.1.2}$$

satisfies $z(\tau) = z_1$. Approximate controllability in time τ is equivalent to the following: For any $z_0, z_1 \in X$ and any $\varepsilon > 0$, there exists $u \in L^2([0, \tau]; U)$ such that the solution z of (11.1.2) satisfies $\|z(\tau) - z_1\| < \varepsilon$. Null-controllability in time τ is equivalent to the following: For any $z_0 \in X$, there exists a $u \in L^2([0, \tau]; U)$ such that the solution z of (11.1.2) satisfies $z(\tau) = 0$. Indeed, all this follows from (4.2.7). Often we need the above controllability concepts without having to specify the time τ. For this reason we introduce the following.

Definition 11.1.2. (A, B) is *exactly controllable* if it is exactly controllable in some finite time $\tau > 0$. (A, B) is *approximately controllable* if it is approximately controllable in some finite time $\tau > 0$. (A, B) is *null-controllable* if it is null-controllable in some finite time $\tau > 0$.

Remark 11.1.3. It is easy to see that if $\mathbb{T}$ is right-invertible, then (A, B) is exactly controllable in time τ iff (A, B) is null-controllable in time τ. Another simple observation is that if $\operatorname{Ran} \mathbb{T}_\tau$ is dense in X and (A, B) is null-controllable in time τ, then (A, B) is approximately controllable in time τ.

Remark 11.1.4. The following simple observations are often useful. If the pair (A, B) has one of the controllability properties introduced in Definition 11.1.1 and $\lambda \in \mathbb{C}$, then also $(A - \lambda I, B)$ has the same controllability property. If $\mathbb{T}$ is invertible, and if (A, B) has one of the controllability properties introduced in Definition 11.1.1, then also $(-A, B)$ has the same property.

The following proposition shows that if the system described by (11.1.2) is null-controllable in time τ, then there exists a bounded operator $\mathbf{F}_\tau$ which, when applied to z_0, provides the input function u that drives $z(\tau)$ to zero.

Proposition 11.1.5. *Suppose that (A, B) is null-controllable in time τ. Then there exist operators $\mathbf{F}_\tau \in \mathcal{L}(X, L^2([0, \infty); U))$ such that*

$$\mathbb{T}_\tau + \Phi_\tau \mathbf{F}_\tau = 0.$$

Indeed, this follows from Proposition 12.1.2 by taking $F = -\mathbb{T}_\tau$ and $G = \Phi_\tau$.

11.2 The duality between controllability and observability

In this section we show that the observability concepts introduced in Definition 6.1.1 are dual to the controllability concepts introduced in Definition 11.1.1 and we give several applications of this duality to systems governed by PDEs.

Theorem 11.2.1. *We assume that $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$, the semigroup generated by A, and let $\tau > 0$.*

(1) *The pair (A, B) is exactly controllable in time τ if and only if (A^*, B^*) is exactly observable in time τ.*
(2) *The pair (A, B) is approximately controllable in time τ if and only if (A^*, B^*) is approximately observable in time τ.*
(3) *The pair (A, B) is null-controllable in time τ if and only if (A^*, B^*) is final state observable in time τ.*

Proof. We know from Theorem 4.4.3 that B^* is an admissible observation operator for the semigroup $\mathbb{T}^*$ generated by A^*. Using the reflection operators $\boldsymbol{Я}_\tau$ introduced at the beginning of this chapter, (4.4.1) can be written as follows:

$$\Phi_\tau^* = \boldsymbol{Я}_\tau \Psi_\tau^d, \tag{11.2.1}$$

where Ψ_τ^d is the output map corresponding to (A^*, B^*):

$$(\Psi_\tau^d z_0)(t) = \begin{cases} B^* \mathbb{T}_t^* z_0 & \text{for } \ t \in [0, \tau], \\ 0 & \text{for } \ t > \tau. \end{cases}$$

In order to prove statement (1) note that, according to Proposition 12.1.3, Φ_τ is onto iff Φ_τ^* is bounded from below. By (11.2.1), this is equivalent to Ψ_τ^d being bounded from below, i.e., to the fact that (A^*, B^*) is exactly observable in time τ.

In order to prove statement (2) note that, according to Remark 2.8.2, Ran Φ_τ is dense in X iff Ker $\Phi_\tau^* = \{0\}$. By (11.2.1), this is equivalent to Ker $\Psi_\tau^d = \{0\}$, i.e., to the fact that (A^*, B^*) is approximately observable in time τ.

In order to prove statement (3) we note that, according to Proposition 12.1.2, Ran $\Phi_\tau \supset$ Ran $\mathbb{T}_\tau$ iff there exists a $c > 0$ such that $c\|\Phi_\tau^* z\| \geqslant \|\mathbb{T}_\tau^* z\|$ for every $z \in X$. By (11.2.1), this is equivalent to $c\|\Psi_\tau^d z\| \geqslant \|\mathbb{T}_\tau^* z\|$ for all $z \in X$, i.e., to the fact that (A^*, B^*) is final state observable in time τ. □

Example 11.2.2. We consider the problem of controlling the vibrations of an elastic membrane by a force field acting on a part of this membrane. More precisely, let $n \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\partial\Omega$ of class C^2 or let Ω be a rectangular domain. The physical problem described above can be modeled by the equations

$$\frac{\partial^2 w}{\partial t^2} - \Delta w = u \qquad \text{in } \ \Omega \times (0, \infty), \tag{11.2.2}$$

$$w = 0 \qquad \text{on} \quad \partial\Omega \times (0, \infty), \tag{11.2.3}$$

$$w(x, 0) = f(x), \quad \frac{\partial w}{\partial t}(x, 0) = g(x) \quad \text{for} \quad x \in \Omega, \tag{11.2.4}$$

where f is the initial displacement and g is the initial velocity. Let $\mathcal{O}$ be a nonempty open subset of Ω and let $u \in L^2([0, \infty); L^2(\mathcal{O}))$ be the input function. For any such u we consider that $u(x, t) = 0$ for $x \in \Omega \setminus \mathcal{O}$.

Equations (11.2.2)–(11.2.4) can be put in the form (11.1.2) using the following spaces and operators:

$$X = \mathcal{H}_0^1(\Omega) \times L^2(\Omega), \quad \mathcal{D}(A) = \left[\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)\right] \times \mathcal{H}_0^1(\Omega), \tag{11.2.5}$$

$$A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \Delta f \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A), \tag{11.2.6}$$

$$U = L^2(\mathcal{O}) \subset L^2(\Omega) \quad \text{and} \quad Bu = \begin{bmatrix} 0 \\ u \end{bmatrix} \qquad \forall\, u \in U. \tag{11.2.7}$$

The space X and the operator A coincide with those introduced in the preamble of Chapter 7 so that, as mentioned there, A is skew-adjoint and, consequently, it generates a unitary group $\mathbb{T}$. Moreover we clearly have $B \in \mathcal{L}(U, X)$, so that B is an admissible control operator for $\mathbb{T}$ and

$$\left\langle Bu, \begin{bmatrix} f \\ g \end{bmatrix} \right\rangle_X = \langle u, g \rangle_U \qquad \forall\, u \in U, \ \begin{bmatrix} f \\ g \end{bmatrix} \in X.$$

From the above formula it follows that

$$B^* \begin{bmatrix} f \\ g \end{bmatrix} = g|_{\mathcal{O}} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X,$$

so that $B^* = C$, where C is the operator introduced at the beginning of Section 7.4.

Let Γ be a relatively open subset of $\partial\Omega$. From the above facts it follows that if Γ and $\mathcal{O}$ satisfy the assumptions in Theorem 7.4.1, then the pair (A, B) is exactly controllable in the same time τ as in Theorem 7.4.1. In particular, by combining Theorems 7.4.1 and 7.2.4, we get that the above controllability property holds if there exist $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ such that

$$\mathcal{N}_\varepsilon(\{x \in \partial\Omega \mid (x - x_0) \cdot \nu(x) > 0\}) \subset \operatorname{clos} \mathcal{O}. \tag{11.2.8}$$

Here we have used the notation $\mathcal{N}_\varepsilon$ from (7.4.1). If (11.2.8) holds, then the pair (A, B) is exactly controllable in any time $\tau > 2r(x_0)$, where $r(x_0) = \sup_{x \in \Omega} |x - x_0|$.

Example 11.2.3. Let the open sets Ω, $\mathcal{O}$ and the space U be as in the previous example. We denote $H = L^2(\Omega)$ (so that $U \subset H$) and $\mathcal{D}(A_0) = H_1$ is the Sobolev space $\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$. The strictly positive operator $A_0 : \mathcal{D}(A_0) \to H$ is defined by $A_0\varphi = -\Delta\varphi$ for all $\varphi \in \mathcal{D}(A_0)$. Let $B_0 \in \mathcal{L}(U, H)$ be defined by

$$B_0 u = u \qquad \forall\, u \in U.$$

Then the pair $(-iA_0, B_0)$ is exactly controllable in any time $\tau > 0$ provided that one of the following assumptions hold:

(A1) The boundary $\partial\Omega$ of Ω is of class C^2 and $\mathcal{O}$ satisfies the assumption in Proposition 7.5.3.

(A2) The set Ω is a rectangle in $\mathbb{R}^2$ (with no restrictions on $\mathcal{O}$).

In terms of PDEs this means that if (A1) or (A2) holds, then for every $f \in L^2(\Omega)$ there exists $u \in L^2([0,\infty); L^2(\mathcal{O}))$ such that the solution of the Schrödinger equation

$$\frac{\partial z}{\partial t} = i\Delta z + u \quad \text{in} \ \ \Omega \times (0,\infty), \tag{11.2.9}$$

$$z = 0 \qquad \text{on} \ \ \partial\Omega \times (0,\infty), \tag{11.2.10}$$

$$z(x,0) = 0 \qquad \text{for} \ \ x \in \Omega \tag{11.2.11}$$

satisfies $z(\cdot,\tau) = f$.

To prove the above assertions we notice that $(-iA_0)^* = iA_0$ and $B_0^* = C_0$, where

$$C_0 f = f|_{\mathcal{O}} \qquad \forall\, f \in H\,.$$

With the assumption (A1) we know from Proposition 7.5.3 that the pair (iA_0, C_0) is exactly observable for any time $\tau > 0$, whereas under the assumption (A2) the same property holds thanks to Theorem 8.5.1. Consequently, the claimed assertions follow by applying Theorem 11.2.1.

Example 11.2.4. Let $n \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\partial\Omega$ of class C^2 or let Ω be a rectangular domain and let $\mathcal{O}$ be an open subset of Ω. We consider the problem of controlling the vibrations of an elastic plate occupying the domain Ω by a force field acting on $\mathcal{O}$. More precisely, we consider the initial and boundary value problem

$$\frac{\partial^2 w}{\partial t^2} + \Delta^2 w = u \quad \text{in} \ \ \Omega \times (0,\infty), \tag{11.2.12}$$

$$w = \Delta w = 0 \quad \text{on} \ \ \partial\Omega \times (0,\infty), \tag{11.2.13}$$

$$w(x,0) = 0, \qquad \frac{\partial w}{\partial t}(x,0) = 0 \quad \text{for} \ \ x \in \Omega, \tag{11.2.14}$$

where $u \in L^2([0,\infty); L^2(\mathcal{O}))$ is the input function. As usual, we consider $u(x,t) = 0$ for $x \in \Omega \setminus \mathcal{O}$. Equations (11.2.12)–(11.2.14) determine a system with state space $X = \left[\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)\right] \times L^2(\Omega)$ and input space $U = L^2(\Omega)$, which is exactly controllable in any time $\tau > 0$ if the pair $(\Omega, \mathcal{O})$ satisfies one of the assumptions (A1) or (A2) in Example 11.2.3. Indeed, let us use the same notation for H, A_0 and H_1 as in Example 11.2.3 and let $H_2 = \mathcal{D}(A_0^2)$, endowed with the graph norm. Let $\mathcal{X}$ be the Hilbert space $H_1 \times H$, consider the dense subspace of $\mathcal{X}$ defined by $\mathcal{D}(\mathcal{A}) = H_2 \times H_1$ and let the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ be defined by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0^2 & 0 \end{bmatrix}.$$

It is not difficult to see that (11.2.12)–(11.2.14) can be written in the form

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t), \quad z(0) = 0,$$

where $\mathcal{B} \in \mathcal{L}(U, \mathcal{X})$ is defined by $\mathcal{B}u = \left[\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right]$ for all $u \in U$. We have seen at the beginning of Section 7.5 that $\mathcal{A}$ is skew-adjoint. Moreover, it is not difficult to see that $\mathcal{B}^* = \mathcal{C}_0$, where $\mathcal{C}_0$ is the operator introduced in Proposition 7.5.7. From Proposition 7.5.7 and Theorem 8.5.1 it follows that the pair $(\mathcal{A}, \mathcal{C}_0)$ is exactly observable in any time $\tau > 0$ if one of the assumptions (A1) or (A2) in Example 11.2.3 holds, so that the conclusion follows by applying Theorem 11.2.1.

Example 11.2.5. We consider the problem of controlling the temperature of a rod by means of the heat flux at its left end. The equations describing this problem have been formulated as a well-posed boundary control system in Subsection 10.2.1. Here we continue to use the notation of Subsection 10.2.1. Thus,

$$H = L^2[0, \pi], \qquad \mathcal{H}^1_R(0, \pi) = \left\{\phi \in \mathcal{H}^1(0, \pi) \mid \phi(\pi) = 0\right\},$$
$$H_1 = \left\{ f \in \mathcal{H}^2(0, \pi) \cap \mathcal{H}^1_R(0, \pi) \;\middle|\; \frac{\mathrm{d}f}{\mathrm{d}x}(0) = 0 \right\}$$

and the operator $A : H_1 \to H$ is defined by

$$Af = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \qquad \forall\, f \in H_1 .$$

Recall that $A < 0$ and the control operator of this system satisfies

$$B^* \psi = -\psi(0) \qquad \forall\, \varphi \in H_1 ,$$

so that $B^* = C_0$, where C_0 is the observation operator in Example 9.2.4. We have seen in Example 9.2.4 that the pair (A, C_0) is final state observable in any time $\tau > 0$. According to Theorem 11.2.1, it follows that the pair (A, B) is null-controllable in any time $\tau > 0$. In terms of PDEs, this means that for any $z_0 \in L^2[0, \pi]$ and for any $\tau > 0$ there exists $u \in L^2[0, \tau]$ such that the weak solution of (10.2.2) (in the sense of Remark 10.2.2) satisfies $z(\cdot, \tau) = 0$.

Example 11.2.6. We consider the problem of controlling the vibrations of a string occupying the interval $[0, \pi]$ by means of a force $u(t)$ acting at its left end. The equations describing this problem have been formulated as a well-posed boundary control system in Subsection 10.2.2. Here we continue to use the notation of Subsection 10.2.2. Thus, $X = \mathcal{H}^1_R(0, \pi) \times L^2[0, \pi]$ and $A : \mathcal{D}(A) \to X$ is the skew-adjoint operator defined by

$$\mathcal{D}(A) = \left\{ f \in \mathcal{H}^2(0, \pi) \cap \mathcal{H}^1_R(0, \pi) \;\middle|\; \frac{\mathrm{d}f}{\mathrm{d}x}(0) = 0 \right\} \times \mathcal{H}^1_R(0, \pi),$$
$$A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A).$$

We know from Proposition 10.2.3 that the control operator of this boundary control system satisfies

$$B^* \begin{bmatrix} f \\ g \end{bmatrix} = -g(0) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A),$$

which means that $B^* = -C$, where C is the observation operator considered in Proposition 6.2.5. According to this proposition, (A, C) is exactly observable in any time $\tau \geqslant 2\pi$ and (A, C) is not approximately observable in any time $\tau < 2\pi$. According to Theorem 11.2.1, it follows that (A, B) is exactly controllable in time τ if $\tau \geqslant 2\pi$ and that for $\tau < 2\pi$ the pair (A, B) is not approximately controllable.

In terms of PDEs, the above results imply that for every $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} w_0 \\ w_1 \end{smallmatrix}\right] \in X$ and $\tau \geqslant 2\pi$, there exists $u \in L^2[0, \infty)$ such that the weak solution w of (10.2.5) (in the sense of Remark 10.2.4) satisfies $w(\cdot, \tau) = w_0$ and $\frac{\partial w}{\partial t}(\cdot, t) = w_1$.

Example 11.2.7. We return to the boundary control of the non-homogeneous elastic string that has been considered in Section 10.3. The model consists of (10.3.1)–(10.3.3). Here the coefficients functions are such that $a \in C^2[0, \pi]$, $b \in L^\infty[0, \pi]$, $a(x) \geqslant m > 0$ and $b(x) \geqslant 0$ for all $x \in [0, \pi]$.

We know from Proposition 10.3.3 that these equations correspond to a well-posed boundary control system with state space

$$X = L^2[0, \pi] \times \mathcal{H}^{-1}(0, \pi)$$

and input space $\mathbb{C}$. The generator of this boundary control system is

$$A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -A_0 f \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A) = \mathcal{H}_0^1(0, \pi) \times L^2[0, \pi],$$

where, as in in Section 10.3, $A_0 \in \mathcal{L}(\mathcal{H}_0^1(0, \pi), \mathcal{H}^{-1}(0, \pi))$ is defined by

$$A_0 f = -\frac{\mathrm{d}}{\mathrm{d}x}\left(a \frac{\mathrm{d}f}{\mathrm{d}x}\right) + bf \qquad \forall\, f \in \mathcal{H}_0^1(0, \pi).$$

The operator A is skew-adjoint, so that it generates a unitary group $\mathbb{T}$ on X.

The control operator B of this boundary control system is determined by

$$B^* \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = a(0) \frac{\mathrm{d}}{\mathrm{d}x} \left(A_0^{-1} \psi\right) \Big|_{x=0} \qquad \forall \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A^*) = \mathcal{D}(A).$$

We claim that the pair (A, B) is exactly controllable in any time

$$\tau > 2 \int_0^\pi \frac{\mathrm{d}x}{\sqrt{a(x)}}. \tag{11.2.15}$$

To prove this claim, notice that

$$B^* A z = -a(0) C z \qquad \forall\, z \in \mathcal{D}(A^2),$$

where C is the operator from Proposition 8.2.2. We know from this proposition that C is an admissible observation operator for the semigroup $\mathbb{T}$ restricted to $X_1 = \mathcal{D}(A)$ (with the graph norm). According to the same proposition, the pair (A, C) is exactly observable in any time τ satisfying (11.2.15). Since A is a unitary operator from X_1 to X, it follows that B^* is an admissible observation operator for the semigroup $\mathbb{T}$ on X and the pair (A, B^*) is exactly observable in any time τ satisfying (11.2.15). Since $\mathbb{T}$ is invertible, the same conclusions remain valid if we replace A with $-A$. Since $-A = A^*$, we obtain that the pair (A^*, B^*) is exactly observable in any time τ satisfying (11.2.15). The claim follows by applying Theorem 11.2.1.

We refer the reader to Corollary 11.3.9 for further controllability properties of this system.

Example 11.2.8. We consider the problem of controlling the vibrations of a beam occupying the interval $[0, \pi]$ by means of a torque $u(t)$ acting at its left end. The equations describing this problem have been formulated as a well-posed boundary control system in Section 10.4. We briefly recall what we need from Section 10.4. We denote $H = L^2[0, \pi]$ and $A_0 : H_1 \to H$ is the operator defined by

$$H_1 = \mathcal{H}^2(0, \pi) \cap \mathcal{H}_0^1(0, \pi), \qquad A_0 f = -\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \qquad \forall\, f \in H_1 .$$

We have $A_0 > 0$. The Hilbert spaces $H_{\frac{1}{2}}$ and $H_{-\frac{1}{2}}$ are given by

$$H_{\frac{1}{2}} = \mathcal{H}_0^1(0, \pi), \qquad H_{-\frac{1}{2}} = \mathcal{H}^{-1}(0, \pi).$$

The unique extensions of A_0 to unitary operators from $H_{\frac{1}{2}}$ onto $H_{-\frac{1}{2}}$ and from H onto H_{-1} are still denoted by A_0. The space $H_{\frac{3}{2}} = A_0^{-1} H_{\frac{1}{2}}$ is

$$H_{\frac{3}{2}} = \left\{ g \in \mathcal{H}^3(0, \pi) \cap \mathcal{H}_0^1(0, \pi) \;\middle|\; \frac{\mathrm{d}^2 \psi}{\mathrm{d}x^2}(0) = \frac{\mathrm{d}^2 \psi}{\mathrm{d}x^2}(\pi) = 0 \right\}.$$

We set

$$X = H_{\frac{1}{2}} \times H_{-\frac{1}{2}}, \qquad \mathcal{D}(A) = H_{\frac{3}{2}} \times H_{\frac{1}{2}},$$
$$A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ -A_0^2 f \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A),$$

and A is skew-adjoint. We know from Proposition 10.4.1 that the control operator B of this boundary control system is determined by

$$B^* \begin{bmatrix} f \\ g \end{bmatrix} = -\frac{\mathrm{d}}{\mathrm{d}x}(A_0^{-1} g)\bigg|_{x=0} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A^*) = \mathcal{D}(A).$$

As in the proof of Proposition 10.4.3, we return to the hinged Euler–Bernoulli equation discussed in Example 6.8.4. With our current notation the state space in

Example 6.8.4 is $X_1 = \mathcal{D}(A)$, the semigroup generator is $A|_{\mathcal{D}(A^2)}$, which generates the restriction of $\mathbb{T}$ to X_1, and the observation operator $C : \mathcal{D}(A^2) \to \mathbb{C}$ is given by

$$C \begin{bmatrix} f \\ g \end{bmatrix} = \frac{\mathrm{d}g}{\mathrm{d}x}(0) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A^2) = H_{\frac{5}{2}} \times H_{\frac{3}{2}} .$$

We have shown in Example 6.8.4 that C is an admissible observation operator for $\mathbb{T}$ restricted to X_1 and (A, C) is exactly observable in any time $\tau > 0$. Using again the isomorphism $Q = \left[\begin{smallmatrix} A_0 & 0 \\ 0 & A_0 \end{smallmatrix}\right]$ from X_1 to X, we obtain that (A, CQ^{-1}) is exactly observable in any time $\tau > 0$. From (10.4.10) we see that $CQ^{-1} = -B^*$. Thus, (A, B^*) is exactly observable in any time $\tau > 0$. Since $\mathbb{T}$ is invertible, also $(-A, B^*)$ is exactly observable in any time $\tau > 0$. In our case $-A = A^*$, so that by the duality result in Theorem 11.2.1, (A, B) is exactly controllable in any time $\tau > 0$.

Example 11.2.9. We consider the problem of controlling the vibrations of a beam occupying the interval $[0, 1]$ by means of an angular velocity $u(t)$ applied at its left end. The equations describing this problem have been formulated as a well-posed boundary control system in Section 10.5. Here we continue to use the notation of Section 10.5, so that we know from Proposition 10.5.1 that the control operator B of this boundary control system is determined by

$$B^* \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = -\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2}(0) \qquad \forall \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \in \mathcal{D}(A^*) = \mathcal{D}(A).$$

Recall from Section 10.5 that $X = X_r \oplus X_n$ where $X_n = \operatorname{Ker} A$ and $X_r = X_n^\perp$. We have seen in the proof of Proposition 10.5.1 that

$$A = \begin{bmatrix} A_r & 0 \\ 0 & 0 \end{bmatrix}, \qquad B^* = \begin{bmatrix} C_r & C_n \end{bmatrix},$$

where A_r is the part of A in X_r while C_r and C_n are the restrictions of B^* to $\mathcal{D}(A_r)$ and to X_n, respectively. As shown in the proof of Proposition 10.5.1, the pair (A_r, C_r) coincides with $(A, -C)$ from Section 6.10. We have seen in Proposition 6.10.1 that this pair is exactly observable in any time $\tau > 0$. Moreover, since $X_n = \operatorname{span}\left\{\left[\begin{smallmatrix} q \\ 0 \end{smallmatrix}\right]\right\}$ with $q(x) = x(x-1)^2$ and $C_n \left[\begin{smallmatrix} q \\ 0 \end{smallmatrix}\right] \neq 0$, the finite-dimensional system (A_n, C_n) is observable. Since 0 is not an eigenvalue of A_r, from Theorem 6.4.2 we get that the pairs (A_r, C_r) and (A_n, C_n) are simultaneously exactly observable in any time $\tau > 0$. This means that (A, B^*) is exactly observable in any time $\tau > 0$. Since A generates a strongly continuous group, it follows that also $(-A, B^*)$ is exactly observable in any time $\tau > 0$. Since $-A = A^*$, according to Theorem 11.2.1, it follows that the pair (A, B) is exactly controllable in any time $\tau > 0$.

11.3 Simultaneous controllability and the reachable space with $\mathcal{H}^1$ inputs

Definition 11.3.1. For $j \in \{1, 2\}$, let A_j be the generator of a strongly continuous semigroups $\mathbb{T}^j$ acting on the Hilbert space X^j. Let U be a Hilbert space and let $B_j \in \mathcal{L}(U, X_{-1}^j)$ be an admissible control operator for $\mathbb{T}^j$.

The pairs (A_j, B_j) are called *simultaneously exactly controllable* in time $\tau > 0$, if for every $(z_1, z_2) \in X^1 \times X^2$ there exists a function $u \in L^2([0, \tau]; U)$ such that

$$\int_0^\tau \mathbb{T}_{T-\sigma}^j B_j u(\sigma) \mathrm{d}\sigma = z_j, \quad j \in \{1, 2\}.$$

The same pairs are called *simultaneously approximately controllable* in time $\tau > 0$, if the property described above holds for (z_1, z_2) in a dense subspace of $X^1 \times X^2$.

It is clear that the concepts introduced in the last definition are equivalent to the exact (approximate) controllability in time τ of the pair

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

Using Theorem 11.2.1, it is easy to see that the concept of simultaneous exact (respectively, approximate) observability is the dual of the concept of simultaneous exact (respectively, approximate) controllability. More precisely, we have the following proposition.

Proposition 11.3.2. *With the notation of Definition* 6.4.1 *we have*

1. *The pairs (A_1, C_1) and (A_2, C_2) are simultaneously exactly observable in time τ if and only if the pairs (A_1^*, C_1^*) and (A_2^*, C_2^*) are simultaneously exactly controllable in time τ.*
2. *The pairs (A_1, C_1) and (A_2, C_2) are simultaneously approximately observable in time τ if and only if the pairs (A_1^*, C_1^*) and (A_2^*, C_2^*) are simultaneously approximately controllable in time τ.*

By combining the above result with Theorem 6.4.2 we obtain the following:

Corollary 11.3.3. *Let A be the generator of the strongly continuous semigroup $\mathbb{T}$ acting on the Hilbert space X. Let $B \in \mathcal{L}(\mathbb{C}^m, X)$ be an admissible control operator for $\mathbb{T}$ and assume that (A, B) is exactly controllable in time τ_0. Let $a \in \mathbb{C}^{n \times n}$ and $b \in \mathbb{C}^{n \times m}$ be matrices such that (a, b) is controllable. Assume that A^* and a^* have no common eigenvalues. Then the pairs (A, B) and (a, b) are simultaneously exactly controllable in any time $\tau > \tau_0$.*

A useful application of the simultaneous exact controllability concept is the characterization of the reachable subspaces of an exactly controllable system, when the input function is restricted to Sobolev-type spaces strictly included in L^2. The remaining part of this section is devoted to this issue.

Suppose that the pair (A, B) is *exactly controllable* in time τ. This means that the range of the operator Φ_τ defined by (4.2.1) is equal to X. A natural question is the characterization of the states which can be reached by more regular inputs. Define

$$\mathcal{H}^1_L((0,\tau);U) = \{\psi \in \mathcal{H}^1((0,\tau);U) \mid \psi(0) = 0\}. \tag{11.3.1}$$

The existence and uniqueness result in Lemma 4.2.8 shows that the space reachable by means of controls in $\mathcal{H}^1_L((0,\tau);U)$ cannot be larger than Z defined in (4.2.9).

In the case of a finite-dimensional input space, we can now characterize the states which are reachable by means of input functions in $\mathcal{H}^1_L((0,\tau);U)$, as follows.

Proposition 11.3.4. *Suppose that the pair (A, B) is exactly controllable in time τ_0 and that U is finite dimensional. Then for every $\tau > \tau_0$, the reachable space by means of input functions $u \in \mathcal{H}^1_L((0,\tau);U)$ is*

$$Z = X_1 + (\beta I - A)^{-1}BU = (\beta I - A)^{-1}(X + BU). \tag{11.3.2}$$

Proof. We know from Lemma 4.2.8 that the reachable space is included in Z. To show that for $\tau > \tau_0$, Z is contained in the reachable space, take $\beta \in \rho(A)$ and consider two systems with states $w(t) \in X$ and $v(t) \in U$ and with the input u_1, described by

$$\dot{w} = (A - \beta I)w + Bu_1, \qquad \dot{v} = u_1. \tag{11.3.3}$$

For an arbitrary $z^0 \in Z$ choose $w^0 \in X,\ v^0 \in U$ such that

$$z^0 = (\beta I - A)^{-1}[w^0 - Bv^0]. \tag{11.3.4}$$

Since 0 is not an eigenvalue of $A - \beta I$, by Corollary 11.3.3, the systems in (11.3.3) are simultaneously exactly controllable in any time $\tau > \tau_0$. Hence we can find $u_1 \in L^2([0,\tau];U)$ such that the solutions $w,\ v$ of (11.3.3) satisfy

$$w(0) = 0, \quad w(\tau) = e^{-\beta\tau}w^0, \quad v(0) = 0, \quad v(\tau) = e^{-\beta\tau}v^0. \tag{11.3.5}$$

We define the function z_1 by

$$z_1(t) = (\beta I - A)^{-1}(w(t) - Bv(t)) \qquad \forall\, t \in [0,\tau].$$

Then it is easy to see that

$$z_1(0) = 0, \quad z_1(\tau) = e^{-\beta\tau}z^0. \tag{11.3.6}$$

Moreover, after a simple calculation, (11.3.3) implies that

$$\dot{z}_1(t) = -w(t) = (A - \beta I)z_1(t) - Bv(t) \qquad \forall\, t \in (0,\tau). \tag{11.3.7}$$

If we define now

$$z(t) = e^{\beta t}z_1(t), \quad u(t) = e^{\beta t}v(t),$$

relations (11.3.6) and (11.3.7) imply that z and u satisfy (4.2.10) together with $z(0) = 0$ and $z(\tau) = z^0$. This means that Z is included in the space reachable by means of input functions $u \in \mathcal{H}^1_L((0,\tau);U)$, as claimed. $\square$

The above result remains true also if U is an arbitrary Hilbert space, but then the proof becomes much longer. For this, we need the following lemma on simultaneous exact observability, which is related to Theorem 6.4.2.

Lemma 11.3.5. *Let A be the generator of the strongly continuous exponentially stable semigroup $\mathbb{T}$ on X. Let Y be another Hilbert space, let $C \in \mathcal{L}(X_1, Y)$ be an admissible observation operator for $\mathbb{T}$ and assume that (A, C) is exactly observable in time $\tau_0 > 0$. Assume that $\lambda > 0$, let $c \in \mathcal{L}(Y)$ be the identity, $c = I$, and let $a \in \mathcal{L}(Y)$ be defined by $a = \lambda I$.*

Then the pairs (A, C) and (a, c) are simultaneously exactly observable in any time

$$\tau > \tau_0 + \frac{1}{\lambda} \log \frac{K}{k_{\tau_0}}, \tag{11.3.8}$$

where K is the infinite-time admissibility constant of (A, C), as in (4.6.6), and k_{τ_0} is the exact observability constant of (A, C) in time τ_0, as in (6.1.1).

Notice that $K \geqslant k_{\tau_0}$, so that in (11.3.8) $\tau > \tau_0$.

Proof. Let e_λ denote the exponential function $e_\lambda(t) = e^{\lambda t}$ (for all $t \geqslant 0$). Assume that the claimed simultaneous observability property is not true (to show that this leads to a contradiction). Then there exists τ satisfying (11.3.8) such that the two systems are not simultaneously exactly observable in time τ. This means that the expression $\Psi_\tau z_0 + e_\lambda x_0 \in L^2([0,\tau]; Y)$ can be made as small as we wish (in norm), for some $(z_0, x_0) \in X \times Y$ with $\|z_0\|^2 + \|x_0\|^2 = 1$. Thus, for each $\varepsilon > 0$ there exist $(z_0, x_0) \in X \times Y$ with $\|z_0\|^2 + \|x_0\|^2 = 1$ such that

$$\Psi_\tau z_0 = \; - e_\lambda x_0 + \delta, \qquad \|\delta\|_{L^2[0,\tau]} \leqslant \varepsilon. \tag{11.3.9}$$

We shall now derive two estimates that link $\|x_0\|$, $\|z_0\|$ and ε, if x_0, z_0 and ε satisfy (11.3.9). Estimating the norm in $L^2([0,\tau_0]; Y)$, we get from (11.3.9)

$$k_{\tau_0}\|z_0\| \leqslant \|\Psi_{\tau_0} z_0\| \leqslant \|e_\lambda x_0\|_{L^2[0,\tau_0]} + \|\delta\|_{L^2[0,\tau_0]} \leqslant \sqrt{\frac{e^{2\lambda\tau_0} - 1}{2\lambda}} \cdot \|x_0\| + \varepsilon.$$

This implies

$$\sqrt{\frac{e^{2\lambda\tau_0} - 1}{2\lambda}} \cdot \|x_0\| \geqslant k_{\tau_0}\|z_0\| - \varepsilon. \tag{11.3.10}$$

On the other hand, estimating norms in $L^2([0,\tau]; Y)$, we get from (11.3.9)

$$\sqrt{\frac{e^{2\lambda\tau} - 1}{2\lambda}} \cdot \|x_0\| = \|e_\lambda x_0\|_{L^2[0,\tau]} = \| - \Psi_\tau z_0 + \delta\|_{L^2[0,\tau]} \leqslant \|\Psi_\tau z_0\|_{L^2[0,\tau]} + \varepsilon.$$

It follows from the above estimate and the definition of K that

$$\sqrt{\frac{e^{2\lambda\tau}-1}{2\lambda}} \cdot \|x_0\| \leqslant K\|z_0\| + \varepsilon . \tag{11.3.11}$$

This resembles (11.3.10), but the inequality is reversed.

The next step is to show that $\|z_0\|$ cannot be very small. Notice from the Taylor expansion of $e^{2\lambda\tau}$ that

$$\sqrt{\frac{e^{2\lambda\tau}-1}{2\lambda}} > \sqrt{\tau} .$$

Let us agree that we shall only use $\varepsilon < \frac{\sqrt{\tau}}{2}$. Define $\varphi \in (0, \frac{\pi}{2})$ such that

$$\|x_0\| = \cos\varphi, \qquad \|z_0\| = \sin\varphi .$$

Then (11.3.11) implies that $\sqrt{\tau}\cos\varphi < K\sin\varphi + \frac{\sqrt{\tau}}{2}$. By elementary considerations, this inequality can only hold for $\varphi > \varphi_{\min} > 0$, where $\varphi_{\min}$ depends on τ and K. It follows that $\|z_0\| \geqslant \sin\varphi_{\min} > 0$.

If we divide the sides of (11.3.11) by the sides of (11.3.10), we obtain

$$\sqrt{\frac{e^{2\lambda\tau}-1}{e^{2\lambda\tau_0}-1}} \leqslant \frac{K\|z_0\| + \varepsilon}{k_{\tau_0}\|z_0\| - \varepsilon} . \tag{11.3.12}$$

We take a sequence of possible choices for ε that converges to zero. For each ε there exist corresponding z_0 and x_0 with all the properties explained earlier, including (11.3.12). We know from the previous step of the proof that the sequence of $\|z_0\|$ is bounded from below. Therefore, in the limit, (11.3.12) implies that

$$\sqrt{\frac{e^{2\lambda\tau}-1}{e^{2\lambda\tau_0}-1}} \leqslant \frac{K}{k_{\tau_0}} .$$

By elementary manipulations, this implies that

$$\frac{e^{2\lambda\tau}}{e^{2\lambda\tau_0}} \leqslant \frac{K^2}{k_{\tau_0}^2}, \qquad \text{hence} \qquad \lambda(\tau-\tau_0) \leqslant \log\frac{K}{k_{\tau_0}} .$$

The last inequality contradicts (11.3.8). It follows that our assumption at the beginning of this proof was false, hence the statement in the lemma is true. □

Now we can state and prove the promised generalization of Proposition 11.3.4.

Theorem 11.3.6. *Suppose that the pair* (A, B) (*with input space* U *and state space* X) *is exactly controllable in time* τ_0. *Then for every* $\tau > \tau_0$, *the reachable space by means of input functions* $u \in \mathcal{H}^1_L((0,\tau);U)$ *is* Z *from* (11.3.2).

Proof. First, notice that we may assume, without loss of generality, that A is exponentially stable. Indeed, otherwise we replace A with $A - \mu I$, with $\mu > 0$ sufficiently large, and this does not change the reachable space.

We know from Lemma 4.2.8 that the reachable space with $\mathcal{H}^1_L$ inputs is included in Z (for every $\tau > 0$). To show that Z is included in the reachable space for all $\tau > \tau_0$, we choose a fixed $\tau > \tau_0$. Then we can find $\lambda > 0$ such that (11.3.8) holds. Consider two systems with states $w(t) \in X$ and $u(t) \in U$ and the common input u_1, described by

$$\dot{w} = Aw + Bu_1 , \qquad \dot{u} = \lambda u + u_1 . \tag{11.3.13}$$

For an arbitrary $z^0 \in Z$ choose $w^0 \in X,\ v^0 \in U$ such that

$$z^0 = A^{-1}[w^0 - Bv^0]. \tag{11.3.14}$$

By Lemma 11.3.5 translated into its dual form, the systems in (11.3.13) are simultaneously exactly controllable in time τ. Hence, there exists an input signal $u_1 \in L^2([0,\tau];U)$ such that the solutions $w,\ u$ of (11.3.13) satisfy

$$w(0) = 0, \quad w(\tau) = w^0 - \lambda z^0, \quad u(0) = 0, \quad u(\tau) = v^0 . \tag{11.3.15}$$

It is clear that $u \in \mathcal{H}^1_L((0,\tau);U)$.

We define the function $z \in C([0,\tau];X)$ by

$$z(t) = (A - \lambda I)^{-1}[w(t) - Bu(t)].$$

It is clear that $z(0) = 0$. It is easy to see that

$$z(\tau) = (A - \lambda I)^{-1}[w^0 - Bv^0 - \lambda z^0] = (A - \lambda I)^{-1}[Az^0 - \lambda z^0] = z^0 .$$

The proof will be complete if we show that z is a solution in X_{-1} of

$$\dot{z}(t) = Az(t) + Bu(t).$$

First we verify that z satisfies the differential equation

$$\dot{z}(t) = \lambda z(t) + w(t) \qquad \forall\, t \in [0,\tau]. \tag{11.3.16}$$

Indeed, we have (using the definition of z)

$$\begin{aligned}\dot{z}(t) &= (A - \lambda I)^{-1}[\dot{w}(t) - B\dot{u}(t)] = (A - \lambda I)^{-1}[Aw(t) - \lambda Bu(t)] \\ &= (A - \lambda I)^{-1}[(A - \lambda I)w(t) + \lambda(w(t) - Bu(t))] = w(t) + \lambda z(t).\end{aligned}$$

Note that (11.3.16) implies that $z \in C^1([0,\tau];X)$. Now from (11.3.16) we get, using again the definition of z,

$$\begin{aligned}\dot{z}(t) &= (\lambda I - A + A)(A - \lambda I)^{-1}[w(t) - Bu(t)] + w(t) \\ &= -[w(t) - Bu(t)] + A(A - \lambda I)^{-1}[w(t) - Bu(t)] + w(t) \\ &= Az(t) + Bu(t).\end{aligned}$$

□

In the case of boundary control systems, which have been studied in Chapter 10, the above theorem yields the following controllability result.

Proposition 11.3.7. *Let (L, G) be a well-posed boundary control system on U, Z and X. Assume that this system is exactly controllable in time $\tau_0 > 0$. Then for every $\tau > \tau_0$ and every $f \in Z$ there exists $u \in \mathcal{H}^1_L((0,\tau);U)$ such that the solution z of*

$$\dot{z}(t) = Lz(t), \qquad Gz(t) = u(t), \qquad z(0) = 0 \tag{11.3.17}$$

satisfies $z(\tau) = f$.

Proof. We know from Proposition 10.1.8 that for every $\tau > 0$ and every $u \in \mathcal{H}^1_L((0,\tau);U)$, equations (11.3.17) admit a unique solution $z \in C([0,\tau];Z)$, so that the reachable space is included in the solution space Z. We denote by A and B the generator and the control operator of this system. According to Remark 10.1.4, we have $\dot{z}(t) = Az(t) + Bu(t)$ for all $t \in [0,\tau]$. To show that Z is included in the reachable space it suffices to note that, according to Remark 10.1.3, the solution space of our boundary control system coincides with Z from (11.3.2) and then to apply Theorem 11.3.6 for this (A, B). $\square$

Note that if U is finite dimensional, then in the above proof we do not need Theorem 11.3.6, it is enough to use the simpler Proposition 11.3.4.

We now give a proposition that is analogous to Proposition 11.3.4, but it considers the following smoother space of input functions:

$$\mathcal{H}^2_L((0,\tau);U) = \{u \in \mathcal{H}^2((0,\tau);U) \mid u(0) = \dot{u}(0) = 0\}.$$

Proposition 11.3.8. *Suppose that the pair (A, B) (with input space U and state space X) is exactly controllable in time τ_0 and that U is finite dimensional. Then for every $\tau > \tau_0$, the reachable space by means of input functions in $\mathcal{H}^2_L((0,\tau);U)$ is*

$$Z_2 = X_2 + (\beta I - A)^{-2}BU + (\beta I - A)^{-1}BU, \tag{11.3.18}$$

where $\beta \in \rho(A)$ is arbitrary.

Proof. We may assume, without loss of generality, that $0 \in \rho(A)$ (otherwise, we replace A with $A - \mu I$). First we prove that the reachable space is contained in Z_2. For $u \in \mathcal{H}^2_L((0,\tau);U)$ we consider the new input $\widetilde{u} = \dot{u}$ (which is in $\mathcal{H}^1_L((0,\tau);U)$) and the corresponding state trajectory $\widetilde{z} = \dot{z}$. Then (from $z(0) = 0$ and $\widetilde{u}(0) = 0$) we have $\widetilde{z}(0) = 0$ so that, according to Lemma 4.2.8, $\widetilde{z}(\tau) \in Z = \mathcal{D}(A) + A^{-1}BU$. From $\widetilde{z}(\tau) = \dot{z}(\tau) = Az(\tau) + Bu(\tau)$ we can easily see that $z(\tau) \in Z_2$.

Conversely, suppose that we want to reach (at time τ) $z_1 \in Z_2$, so that

$$z_1 = z_0 + A^{-2}Bu_0 - A^{-1}Bu_1, \quad \text{where} \quad z_0 \in \mathcal{D}(A^2),\ u_0, u_1 \in U.$$

Consider the following two systems with the common input signal $\widetilde{u}$:

$$\dot{\widetilde{z}} = A\widetilde{z} + B\widetilde{u}, \qquad \dot{u} = \widetilde{u}.$$

According to Corollary 11.3.3, these systems are simultaneously controllable in any time $\tau > \tau_0$. It follows from Proposition 11.3.4 that the reachable space for the pair $\left[\begin{smallmatrix} \widetilde{z}(\tau) \\ u(\tau) \end{smallmatrix}\right]$ using $\widetilde{u} \in \mathcal{H}_L^1((0,\tau);U)$ is

$$\widetilde{Z} = \begin{bmatrix} (\beta I - A)^{-1} & 0 \\ 0 & \frac{1}{\beta} I \end{bmatrix} \left(X \times U + \begin{bmatrix} B \\ I \end{bmatrix} U \right),$$

where $\beta \in \rho(A)$, $\beta \neq 0$. A simple argument shows that $\widetilde{Z} = Z \times U$.

Let $\widetilde{u} \in \mathcal{H}_L^1((0,\tau);U)$ be the input that causes

$$\widetilde{z}(\tau) = Az_0 + A^{-1}Bu_0, \qquad u(\tau) = u_1,$$

and let $u \in \mathcal{H}_L^2((0,\tau);U)$ be the corresponding input (the integral of $\widetilde{u}$). The state trajectory z corresponding to the input u and satisfying $z(0) = 0$ satisfies $\widetilde{z}(\tau) = \dot{z}(\tau) = Az(\tau) + Bu(\tau)$, which becomes

$$Az_0 + A^{-1}Bu_0 = Az(\tau) + Bu_1.$$

Applying A^{-1}, we easily get that $z(\tau) = z_1$. □

We think that the above proposition remains valid for infinite-dimensional U.

We now describe an application of Propositions 11.3.4 and 11.3.8 to the non-homogeneous string equations (10.3.1)–(10.3.3). As in Section 10.3 we denote

$$\mathcal{H}_R^1(0,\pi) = \{\psi \in \mathcal{H}^1(0,\pi) \mid \psi(\pi) = 0\}.$$

The notations $\mathcal{H}_L^1(0,\tau)$ and $\mathcal{H}_L^2(0,\tau)$ are as defined earlier, but now $U = \mathbb{C}$.

Corollary 11.3.9. *For every $\tau > 2\int_0^\pi \frac{\mathrm{d}x}{\sqrt{a(x)}}$ the space of states $\left[\begin{smallmatrix} w(\cdot,\tau) \\ \dot{w}(\cdot,\tau) \end{smallmatrix}\right]$ which can be reached by using inputs $u \in \mathcal{H}_L^1(0,\tau)$, from the initial state $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ by solving (10.3.1)–(10.3.3), is $\mathcal{H}_R^1((0,\pi)) \times L^2[0,\pi]$.*

Moreover, for every τ as above, the space of states which can be reached by using inputs $u \in \mathcal{H}_L^2(0,\tau)$, from the initial state $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, is $\left(\mathcal{H}^2(0,\pi) \cap \mathcal{H}_R^1(0,\pi)\right) \times \mathcal{H}_R^1(0,\pi)$.

Proof. We have seen in Proposition 10.3.3 that (10.3.1)–(10.3.3) correspond to a well-posed boundary control system (L,G) with solution space $Z = \mathcal{H}_R^1(0,\pi) \times L^2[0,\pi]$. Moreover, we know from Example 11.2.7 that this system is exactly controllable in any time $\tau > 2\int_0^\pi \frac{\mathrm{d}x}{\sqrt{a(x)}}$, so that the first assertion in the corollary follows by applying Proposition 11.3.7.

To prove the second assertion, let A and B be the semigroup generator and the control operator of this system, as expressed in Section 10.3, and let τ be as in the corollary. According to Proposition 11.3.8 the reachable space by inputs

$u \in \mathcal{H}_L^2(0,\tau)$, starting from the initial state 0, is Z_2 from (11.3.18). It follows easily from the material in Section 10.3 that

$$X_2 = \mathcal{D}(A^2) = \left(\mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi)\right) \times \mathcal{H}_0^1(0,\pi).$$

It is easy to see from Proposition 10.3.3 that

$$A^{-1}B = \begin{bmatrix} -D \\ 0 \end{bmatrix}, \qquad A^{-2}B = \begin{bmatrix} 0 \\ -D \end{bmatrix},$$

where $D \in \mathcal{L}(\mathbb{C}, \mathcal{H}^2(0,\pi) \cap \mathcal{H}_R^1(0,\pi))$ is the operator from Proposition 10.3.1. Putting these facts together, we obtain that

$$Z_2 = \left(\mathcal{H}^2(0,\pi) \cap \mathcal{H}_R^1(0,\pi)\right) \times \mathcal{H}_R^1(0,\pi). \qquad \square$$

11.4 An example of a coupled system

Consider a vertical string whose horizontal displacement in a given plane is described by the one-dimensional wave equation on the domain $(0,\pi)$. The upper end (corresponding to $x=\pi$) is kept fixed and an object of mass M is attached at the lower end (corresponding to $x=0$). The external input is a horizontal force v acting on the object, and it is contained in the plane mentioned earlier. We neglect the moment of inertia of the object (i.e., we imagine the object to be very small). From simple physical considerations, and taking a certain constant to be one, we obtain that this system is described by the following equations, valid for all $x \in (0,\pi)$ and for all $t \in (0,\infty)$:

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2}(x,t) = \dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial w}{\partial x}(x,t)\right), & w(\pi,t) = 0, \\ M\dfrac{\partial^2 w}{\partial t^2}(0,t) + a(0)w_x(0,t) = v(t), & t > 0, \\ w(x,0) = \dfrac{\partial w}{\partial t}(x,0) = 0, & x \in (0,\pi). \end{cases} \tag{11.4.1}$$

Here, w is the controlled wave (horizontal displacement) and $\frac{\partial w}{\partial t}$ is the horizontal velocity. Due to the weight of the string the function a is strictly increasing, even for a homogeneous string. For technical reasons, we assume that $a \in C^2[0,\pi]$ and that there exists $m > 0$ such that

$$a(x) \geqslant m \qquad \forall\, x \in [0,\pi].$$

The appropriate spaces for all these functions will be specified later. The point $x=\pi$ is just reflecting waves, while the active end $x=0$ is where both the observation and the control take place. We shall often write $w(t)$ to denote a function of x, meaning that $w(t)(x) = w(x,t)$, and similarly for other functions.

A direct analysis of the well-posedness, controllability and observability of this system is not trivial, in spite of the simplicity of the system. We shall show below that we can obtain a sharp result by simply applying the results in the previous section. First we investigate an auxiliary Hilbert space and an operator generating a group in this space. Denote

$$\mathcal{X} = \left\{ \begin{bmatrix} f \\ g \\ h \\ \kappa \end{bmatrix} \in \mathcal{H}^1_R(0,\pi) \times L^2[0,\pi] \times \mathbb{C} \times \mathbb{C} \;\middle|\; f(0) = h \right\}.$$

On $\mathcal{X}$ we consider the inner product

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \\ h_1 \\ \kappa_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_1 \\ h_2 \\ \kappa_2 \end{bmatrix} \right\rangle_X = \int_0^\pi \left(a \frac{\mathrm{d}f_1}{\mathrm{d}x} \frac{\mathrm{d}\overline{f}_2}{\mathrm{d}x} + g_1 \overline{g_2} \right) \mathrm{d}x + \kappa_1 \overline{\kappa_2}.$$

Lemma 11.4.1. *Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ be the operator defined by*

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{bmatrix} f \\ g \\ h \\ \kappa \end{bmatrix} \in X \;\middle|\; f \in \mathcal{H}^2(0,\pi),\; g \in \mathcal{H}^1(0,\pi), g(0) = \kappa \right\},$$

$$\mathcal{A} \begin{bmatrix} f \\ g \\ h \\ \kappa \end{bmatrix} = \begin{bmatrix} g \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(a \frac{\mathrm{d}f}{\mathrm{d}x} \right) \\ \kappa \\ -a(0)\frac{\mathrm{d}f}{\mathrm{d}x}(0) \end{bmatrix}. \tag{11.4.2}$$

Then $\mathcal{A}$ generates a unitary group on $\mathcal{X}$.

Proof. We have

$$\left\langle \mathcal{A} \begin{bmatrix} f \\ g \\ h \\ \kappa \end{bmatrix}, \begin{bmatrix} f \\ g \\ h \\ \kappa \end{bmatrix} \right\rangle = \int_0^\pi \left(a \frac{\mathrm{d}g}{\mathrm{d}x} \frac{\mathrm{d}\bar{f}}{\mathrm{d}x} + g \frac{\mathrm{d}^2 \bar{f}}{\mathrm{d}x^2} \right) \mathrm{d}x - a(0) \frac{\mathrm{d}f}{\mathrm{d}x}(0)\bar{\kappa}.$$

If we integrate by parts, we take real parts and we use the fact that $g(0) = \kappa$, we obtain that

$$\mathrm{Re} \left\langle \mathcal{A} \begin{bmatrix} f \\ g \\ h \\ \kappa \end{bmatrix}, \begin{bmatrix} f \\ g \\ h \\ \kappa \end{bmatrix} \right\rangle = 0 \qquad \forall \begin{bmatrix} f \\ g \\ h \\ \kappa \end{bmatrix} \in \mathcal{D}(\mathcal{A}),$$

so that $\mathcal{A}$ is skew-symmetric. To show that $\mathcal{A}$ is onto, we take $\begin{bmatrix} \varphi \\ \psi \\ \delta \\ \gamma \end{bmatrix} \in \mathcal{X}$ and we note that there exists a unique $f \in \mathcal{H}^2(0,\pi)$ such that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\left(a\dfrac{\mathrm{d}f}{\mathrm{d}x}\right) = \psi, \\ f(\pi) = 0, \\ a(0)\dfrac{\mathrm{d}f}{\mathrm{d}x}(0) = -\gamma. \end{cases}$$

It follows that

$$\begin{bmatrix} f \\ \varphi \\ f(0) \\ \varphi(0) \end{bmatrix} \in \mathcal{D}(\mathcal{A}) \quad \text{and} \quad \mathcal{A}\begin{bmatrix} f \\ \varphi \\ f(0) \\ \varphi(0) \end{bmatrix} = \begin{bmatrix} \varphi \\ \psi \\ \delta \\ \gamma \end{bmatrix}.$$

We have shown that $\mathcal{A}$ is skew-symmetric and onto so that, by Proposition 3.7.3, $\mathcal{A}$ is skew-adjoint. By Stone's theorem, $\mathcal{A}$ generates a unitary group on $\mathcal{X}$. $\square$

Corollary 11.4.2. *For every $v \in C^1[0,\infty)$ the initial and boundary value problem* (11.4.1) *admits a unique solution*

$$w \in C([0,\infty);\mathcal{H}^1_R(0,\pi)\cap\mathcal{H}^2(0,\pi))\cap C^1([0,\infty);\mathcal{H}^1_R(0,\pi)). \tag{11.4.3}$$

The result below gives the natural state space of (11.4.1).

Proposition 11.4.3. *Suppose that $v \in L^2[0,\tau]$. Then the initial and boundary value problem* (11.4.1) *admits a unique solution*

$$w \in C([0,\infty);\mathcal{H}^1_R(0,\pi)\cap\mathcal{H}^2(0,\pi))\cap C^1([0,\infty);\mathcal{H}^1_R(0,\pi)). \tag{11.4.4}$$

Proof. By using Lemma 11.4.1, it is easy to prove that, for all $v \in L^2[0,T]$, problem (11.4.1) admits a unique solution

$$w \in C([0,\infty);\mathcal{H}^1_R(0,\pi))\cap C^1([0,\infty);L^2[0,\pi]), \tag{11.4.5}$$

which satisfies the first equation from (11.4.1) in $\mathcal{D}'((0,\pi)\times(0,\infty))$ and the second in $\mathcal{D}'(0,\infty)$ (notice that $w_x(0,\cdot)$ makes sense in $H^{-2}(0,\infty)$). Consider a sequence (v_n) in $\mathcal{D}(0,\infty)$ such that $v_n \to v$ in $L^2[0,\tau]$. If we denote by (w_n) the corresponding sequence of smooth solutions of (11.4.1) (see Corollary 11.4.2 for the existence and uniqueness of these solutions), it is clear that

$$w_n \to w \quad \text{in} \quad C([0,\tau];\mathcal{H}^1_L(0,\pi))\cap C^1([0,\tau];L^2[0,\pi]), \tag{11.4.6}$$

$$w_n(0,t) = \frac{\partial w_n}{\partial t}(0,t) = 0, \qquad \forall\, n \geqslant 1. \tag{11.4.7}$$

Moreover, by multiplying the equation

$$\frac{\partial^2}{\partial t^2}(w_m - w_n)(x,t) = \frac{\partial^2}{\partial x^2}(w_m - w_n)(x,t)$$

by $(x-1)\frac{\partial}{\partial x}(w_m - w_n)(x,t)$ and by integrating over $[0,\pi]\times[0,\tau]$ we obtain, after some integrations by parts, the existence of a constant $C>0$ such that

$$\int_0^\tau \left|\frac{\partial}{\partial x}(w_m - w_n)(0,t)\right|^2 \mathrm{d}t \\ \leqslant C\left(\|w_n - w_m\|_{C([0,\tau];\mathcal{H}^1(0,\pi))} + \left\|\frac{\partial w_n}{\partial t} - \frac{\partial w_m}{\partial t}\right\|_{C([0,\tau];L^2[0,\pi])}\right). \tag{11.4.8}$$

Since

$$M\ddot{w}_n(0,t) + a(0)\frac{\partial w_n}{\partial x}(0,t) = v_n(t),$$

relation (11.4.8) implies that $\frac{\partial^2 w_n}{\partial t^2}(0,\cdot)$ is a Cauchy sequence in $L^2[0,\tau]$. By using (11.4.6) and (11.4.7) we obtain that $w(0,\cdot)\in\mathcal{H}^2_L(0,\tau)$. The regularity (11.4.4) follows now from Proposition 11.3.9. □

Proposition 11.4.4. *Assume that* $\tau > 2\pi$. *Then system* (11.4.1) *is exactly controllable in time* τ *in the state space* $X = [\mathcal{H}^1_R(0,\pi)\cap\mathcal{H}^2(0,\pi)]\times\mathcal{H}^1_R(0,\pi)$. *In other words,* $(w_0,w_1)\in[\mathcal{H}^1_R(0,\pi)\cap\mathcal{H}^2(0,\pi)]\times\mathcal{H}^1_L(0,\pi)$ *if and only if there exists* $v\in L^2[0,\tau]$ *such that the solution of* (11.4.1) *satisfies*

$$w(\cdot,\tau) = w_0, \quad \frac{\partial w}{\partial t}(\cdot,\tau) = w_1\,. \tag{11.4.9}$$

Proof. By Proposition 11.3.9, for any $(w_0,w_1)\in[\mathcal{H}^1_R(0,\pi)\cap\mathcal{H}^2(0,\pi)]\times\mathcal{H}^2_R(0,\pi)$ there exist

$$w\in C([0,\infty);\mathcal{H}^2(0,\pi)),\;\; u\in\mathcal{H}^2_L(0,\tau) \tag{11.4.10}$$

satisfying (10.3.1)–(10.3.3) and (11.4.9). From (11.4.10) it obviously follows that if we define

$$v(t) = M\ddot{u}(t) + a(0)w_x(0,t),$$

then $v\in L^2[0,\tau]$ and w,v satisfy (11.4.1) and (11.4.9). □

11.5 Null-controllability for heat and convection-diffusion equations

In this section we consider systems governed by the heat equation or by the convection-diffusion equation, with an input function given either by a source/sink term supported on an open set or by a Dirichlet boundary condition on a part of the boundary. Recall that the null-controllability of a one-dimensional heat equation with Neumann boundary control has been considered in Example 11.2.5.

In this section, $\Omega \subset \mathbb{R}^n$ is an open bounded and connected set with boundary of class C^2. We denote $X = L^2(\Omega)$ and for a while we consider the operator $A : \mathcal{D}(A) \to X$ introduced in Example 5.4.4 (and discussed also in Section 10.8):

$$\begin{aligned} \mathcal{D}(A) &= \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega), \\ Af &= \Delta f + b \cdot \nabla f + cf \qquad \forall\, f \in \mathcal{D}(A), \end{aligned}$$

where $b \in L^\infty(\Omega; \mathbb{R}^n)$ and $c, \operatorname{div} b \in L^\infty(\Omega)$.

Let $\mathcal{O}$ be an open subset of Ω and let $U = L^2(\mathcal{O})$. We regard U as a closed subspace of X by considering functions in U to be zero on $\Omega \setminus \mathcal{O}$. Let $B \in \mathcal{L}(U, X)$ be defined by $Bu = u$ (i.e., B is the embedding of U into X).

Proposition 11.5.1. *The pair (A, B) is null-controllable in any time $\tau > 0$.*

Proof. We have seen in Remark 10.8.1 that the adjoint of A is given by

$$\begin{aligned} \mathcal{D}(A^*) &= \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega), \\ A^* f &= \Delta f - \overline{b} \cdot \nabla f + \overline{(c - \operatorname{div} b)} f \qquad \forall\, f \in \mathcal{D}(A^*), \end{aligned}$$

so that A^* is of the same nature as A, only with different coefficient functions.

It is easy to check that the adjoint of B is given by

$$B^* f = f|_{\mathcal{O}} \qquad \forall\, f \in X .$$

We have seen in Theorem 9.5.1 that the pair (A^*, B^*) is final state observable in any time $\tau > 0$, so that the conclusion follows by applying Theorem 11.2.1. $\square$

Remark 11.5.2. In terms of PDEs, the above proposition means that for any $\tau > 0$ and any $f \in L^2(\Omega)$ there exists $u \in L^2([0, \tau]; L^2(\mathcal{O}))$ such that the solution of

$$\frac{\partial z}{\partial t} = \Delta z + b \cdot \nabla z + cz + u \quad \text{in } \Omega \times (0, \infty), \tag{11.5.1}$$

$$z = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{11.5.2}$$

$$z(x, 0) = f(x) \quad \text{for } x \in \Omega \tag{11.5.3}$$

satisfies $z(x, \tau) = 0$ for all $x \in \Omega$. This result can be interpreted in physical terms by asserting that the temperature field of a body occupying the domain Ω can be driven to zero (the choice of the temperature level zero is arbitrary) by using a heat source/sink localized in an arbitrary subset $\mathcal{O}$ of Ω.

Remark 11.5.3. For $b = 0$ it is not difficult to check, by using Theorem 11.2.1 and Proposition 9.1.1, that the pair (A, B) is not exactly controllable.

A natural question is controlling the temperature of a body by acting on the temperature field on a part of its boundary. Such a system is modeled by the equations

$$\frac{\partial z}{\partial t} = \Delta z \quad \text{in } \ \Omega \times (0, \infty), \tag{11.5.4}$$

$$z = u \quad \text{on } \ \Gamma \times (0, \infty), \tag{11.5.5}$$

$$z = 0 \quad \text{on } \ (\partial\Omega \setminus \Gamma) \times (0, \infty), \tag{11.5.6}$$

$$z(x, 0) = f(x) \quad \text{for } \ x \in \Omega, \tag{11.5.7}$$

where Γ is a non-empty open subset of $\partial\Omega$.

Our aim is to control the above system by inputs $u \in L^2([0,\tau]; L^2(\Gamma))$. We have seen in Section 10.7 that the above equations determine a well-posed boundary control system with input space $U = L^2(\Gamma)$, state space $X = \mathcal{H}^{-1}(\Omega)$, generator $A = -A_0$ (the Dirichlet Laplacian) and control operator $B = A_0 D$, where D is the Dirichlet map. We have seen in the same section that the weak solutions of (11.5.4)–(11.5.7) are in fact the solutions of $\dot{z} = Az + Bu$ (with the same initial conditions).

Proposition 11.5.4. *For every initial state $f \in \mathcal{H}^{-1}(\Omega)$ and $\tau > 0$, there exists $u \in L^2([0,\tau]; L^2(\Gamma))$ such that the weak solution z of* (11.5.4)–(11.5.7) *satisfies $z(\tau) = 0$. In other words, the pair (A, B) is null-controllable in any time $\tau > 0$.*

Proof. We shall now construct a larger open set $\tilde{\Omega}$ which is like Ω with a little hump $\mathcal{O}$ glued to Γ; see Figure 11.1. For the precise definition of $\tilde{\Omega}$ we take a point $x_0 \in \Gamma$ and a rectangular open neighborhood V of x_0 in $\mathbb{R}^n$ as in the definition of the boundary of class C^2 (see Section 13.5 in Appendix II). In a suitable system of orthonormal coordinates $(y_1, \ldots, y_n)$, the set V can be written as $V' \times [-a_n, a_n]$, where

$$V' = \{(y_1, \ldots, y_{n-1}) \mid -a_i < y_j < a_j, \ 1 \leqslant j \leqslant n-1\},$$

and there exists a real-valued $\varphi \in C^2(V')$ such that $|\varphi(y')| \leqslant \frac{a_n}{2}$ for every $y' \in V'$,

$$\Omega \cap V = \{y = (y', y_n) \in V \mid y_n < \varphi(y')\},$$
$$\partial\Omega \cap V = \{y = (y', y_n) \in V \mid y_n = \varphi(y')\}.$$

We choose V sufficiently small such that $V \cap \partial\Omega \subset \Gamma$. We choose a non-zero function $\psi \in \mathcal{D}(V')$ with values in $\left[0, \frac{a_n}{2}\right)$. We define the hump by

$$\mathcal{O} = \{y = (y', y_n) \in V \mid \varphi(y') < y_n < \varphi(y') + \psi(y')\}.$$

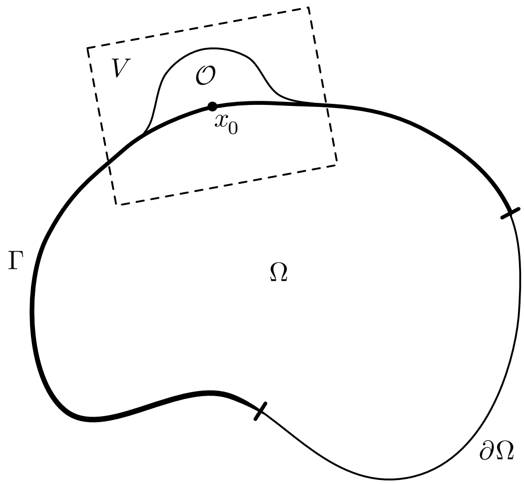

Figure 11.1: The domain Ω with the hump $\mathcal{O}$ which is glued to the part Γ of the boundary in such a way that the enlarged domain $\tilde{\Omega}$ has again a C^2 boundary.

We define the enlarged domain by

$$\tilde{\Omega} = \operatorname{int}(\operatorname{clos}\mathcal{O} \cup \operatorname{clos}\Omega),$$

and this has again a C^2 boundary.

Let $\mathbb{T}$ be the heat semigroup generated by A and let $\tau > 0$. As we have seen in Remark 3.6.11, we have $\mathbb{T}_{\tau/2}f \in \mathcal{H}_0^1(\Omega)$. We extend $\mathbb{T}_{\tau/2}f$ to a function, denoted by g, defined on $\tilde{\Omega}$ by setting $g(x) = 0$ for $x \in \tilde{\Omega} \setminus \operatorname{clos}\Omega$. From Lemma 13.4.11 it follows that $g \in \mathcal{H}_0^1(\tilde{\Omega})$. According to Remark 11.5.2, it follows that there exists $\widetilde{u} \in L^2([0,\tau]; L^2(\mathcal{O}))$ such that the solution $\widetilde{z}$ of

$$\frac{\partial \widetilde{z}}{\partial t} = \Delta z + \widetilde{u} \quad \text{in} \quad \tilde{\Omega} \times (0,\infty), \tag{11.5.8}$$

$$\widetilde{z} = 0 \qquad \text{on} \quad \partial\tilde{\Omega} \times (0,\infty), \tag{11.5.9}$$

$$\widetilde{z}(x,0) = g(x) \qquad \text{for} \quad x \in \tilde{\Omega} \tag{11.5.10}$$

satisfies $\widetilde{z}(x,\tau/2) = 0$ for all $x \in \tilde{\Omega}$. Note that $\widetilde{z} \in C([0,\infty), \mathcal{H}_0^1(\tilde{\Omega}))$ so that, by the trace theorem, we have $\widetilde{z}_{|\partial\Omega} \in C([0,\infty); L^2(\Gamma))$. Define

$$u(t) = \begin{cases} 0 & \text{if} \quad t \in [0,\tau/2], \\ \widetilde{z}(t-\tau/2)_{|\Gamma} & \text{if} \quad t \in [\tau/2,\tau], \end{cases}$$

$$z(t) = \begin{cases} \mathbb{T}_t f & \text{if} \quad t \in [0,\tau/2], \\ \widetilde{z}(t-\tau/2) & \text{if} \quad t \in [\tau/2,\tau]. \end{cases}$$

Then the pair (u, z) satisfies (11.5.4)–(11.5.7) (in the sense of Definition 10.7.2) and $z(\tau) = 0$. $\square$

Remark 11.5.5. By duality (using Theorem 11.2.1) we can obtain the following final state observability result from the last proposition: If z is the solution of

$$\begin{aligned} \frac{\partial z}{\partial t} &= \Delta z && \text{in } \ \Omega \times (0, \infty), \\ z &= 0 && \text{on } \ \partial\Omega \times (0, \infty), \\ z(x, 0) &= f(x) && \text{for } \ x \in \Omega, \end{aligned}$$

with $f \in \mathcal{H}_0^1(\Omega)$, then for every non-empty open set $\Gamma \subset \partial\Omega$ and for every $\tau > 0$, there exists a constant $k_\tau > 0$ (independent of f) such that

$$\int_0^\tau \int_\Gamma \left| \frac{\partial z}{\partial \nu} \right|^2 \mathrm{d}\sigma \mathrm{d}t \geqslant k_\tau^2 \|z(\tau)\|^2_{\mathcal{H}_0^1(\Omega)} .$$

To obtain this, we have used Proposition 10.6.7 to express B^* and then the fact that A_0^{-1} is an isomorphism from $\mathcal{H}^{-1}(\Omega)$ to $\mathcal{H}_0^1(\Omega)$.

11.6 Boundary controllability for Schrödinger and wave equations

Notation. Throughout this section, Ω denotes a bounded open set in $\mathbb{R}^n$, where $n \in \mathbb{N}$, with boundary $\partial\Omega$ of class C^2. Let Γ be a non-empty open subset of $\partial\Omega$ and denote $U = L^2(\Gamma)$. For $\varphi \in \mathcal{H}^1(\Omega)$ we denote by $\varphi|_\Gamma$ the restriction of the boundary trace $\gamma_0\varphi$ to Γ. Similarly, for $\varphi \in \mathcal{H}^2(\Omega)$, we denote by $\frac{\partial \varphi}{\partial \nu}|_\Gamma$ the restriction of the normal derivative of φ to Γ (the precise definitions of these trace operators are given in Section 10.6 and in Appendix II). We denote $H = L^2(\Omega)$ and the operator A_0 is the Dirichlet Laplacian defined in Section 3.6. With the above smoothness assumptions on $\partial\Omega$, we know from Theorem 3.6.2 that $A_0 : H_1 \to H$ is defined by

$$H_1 = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega), \quad A_0 f = -\Delta f \qquad \forall\, f \in H_1 .$$

We know from from Proposition 3.6.1 that A_0 is strictly positive and that the Hilbert spaces $H_{\frac{1}{2}}$ and $H_{-\frac{1}{2}}$ obtained from H and A_0, according to the definitions in Section 3.4, are given by

$$H_{\frac{1}{2}} = \mathcal{H}_0^1(\Omega), \quad H_{-\frac{1}{2}} = \mathcal{H}^{-1}(\Omega).$$

We know from Corollary 3.4.6 and Remark 3.4.7 that A_0 can be extended to a strictly positive (densely defined) operator on $H_{-\frac{1}{2}}$, also denoted by A_0, with domain $H_{\frac{1}{2}}$. The operator A_0 can also be regarded as a unitary operator from $H_{\frac{1}{2}}$ to $H_{-\frac{1}{2}}$ and from H onto H_{-1}, where H_{-1} is the dual of H_1 with respect to the pivot space H.

11.6.1 Boundary controllability for the Schrödinger equation

We consider a system governed by the Schrödinger equation, with the input function being the Dirichlet boundary condition on a part of the boundary:

$$\frac{\partial z}{\partial t} = i\Delta z \quad \text{in} \quad \Omega \times (0,\infty), \tag{11.6.1}$$

$$z = u \quad \text{on} \quad \Gamma \times (0,\infty), \tag{11.6.2}$$

$$z = 0 \quad \text{on} \quad (\partial\Omega \setminus \Gamma) \times (0,\infty), \tag{11.6.3}$$

$$z(x,0) = f(x) \quad \text{for} \quad x \in \Omega. \tag{11.6.4}$$

Define $X = H_{-\frac{1}{2}} = \mathcal{H}^{-1}(\Omega)$. We have seen in Section 10.7 that the above equations determine a well-posed boundary control system with input space U, solution space $Z = \mathcal{H}_0^1(\Omega) + DU$ (where D is the Dirichlet map), state space X, generator $A = -iA_0$ and control operator $B = iA_0 D$. We have seen in the same section that the weak solution of (11.6.1)–(11.6.4), with an initial state in $f \in X$, is in fact the solution of $\dot{z} = Az + Bu$ (with the same initial state).

Proposition 11.6.1. *Assume that Γ satisfies the assumption in Proposition 7.5.1 (i.e., the wave equation with Neumann boundary observation defines an exactly observable system). Then the pair (A,B) is exactly controllable in any time $\tau > 0$.*

Proof. According to Proposition 10.6.7, we have

$$B^* g = i\frac{\partial(A_0^{-1} g)}{\partial \nu} \qquad \forall\, g \in L^2(\Omega).$$

As usual, we denote $X_1 = \mathcal{D}(A)$ with the graph norm. Since A is skew-adjoint, the generator of $\mathbb{T}^*$ is $-A = iA_0$, so that for any $w_0 \in X_1$ we have

$$\|B^*\mathbb{T}_t^* w_0\|_U = \left\| \frac{\partial(A_0^{-1} w(t))}{\partial \nu} \right\|_U \qquad \forall\, t \geqslant 0, \tag{11.6.5}$$

where w is the solution of the initial value problem

$$\dot{w}(t) = iA_0 w(t), \quad w(0) = w_0 .$$

If we set $\eta(t) = A_0^{-1} w(t)$, then (11.6.5) becomes

$$\|B^*\mathbb{T}_t^* w_0\|_U = \left\| \frac{\partial \eta(t)}{\partial \nu} \right\|_U \qquad \forall\, t \geqslant 0, \tag{11.6.6}$$

where

$$\dot{\eta}(t) = iA_0 \eta(t), \quad \eta(0) = A_0^{-1} w_0 .$$

We know from Remark 7.5.2 that for any $\tau > 0$ there exists $k_\tau > 0$ such that

$$\int_0^\tau \left\| \frac{\partial \eta(t)}{\partial \nu} \right\|_U^2 \mathrm{d}t \geqslant k_\tau^2 \|A_0^{-1} w_0\|_{X_1}^2 \qquad \forall\, w_0 \in X_1 .$$

The above estimate, combined with (11.6.6) and with the fact that A_0 is unitary from X_1 to X, implies that

$$\int_0^\tau \|B^*\mathbb{T}_t^* w_0\|_U^2 \,\mathrm{d}t \geqslant k_\tau^2 \|w_0\|_X^2 ,$$

so that the pair (A^*, B^*) is exactly observable in time τ. From Theorem 11.2.1 it follows that the pair (A, B) is exactly controllable in time τ. □

Remark 11.6.2. The above proposition can be formulated in terms of PDEs as follows: For every $f, g \in X$ and $\tau > 0$, there exists $u \in L^2([0,\tau]; U)$ such that the weak solution of the Schrödinger equation (in the sense of Remark 10.7.5) with initial data f and Dirichlet boundary control u satisfies $z(\tau) = g$.

11.6.2 Boundary controllability for the wave equation

As in Section 10.9, we consider the following initial and boundary value problem:

$$\frac{\partial^2 w}{\partial t^2} = \Delta w \quad \text{in} \ \ \Omega \times (0,\infty), \tag{11.6.7}$$

$$w = 0 \quad \text{on} \ \ \partial\Omega \setminus \Gamma \times (0,\infty), \tag{11.6.8}$$

$$w = u \quad \text{on} \ \ \Gamma \times (0,\infty), \tag{11.6.9}$$

$$w(x,0) = f(x), \quad \frac{\partial w}{\partial t}(x,0) = g(x) \quad \text{for} \ \ x \in \Omega. \tag{11.6.10}$$

The input of this system is the function u in (11.6.9).

We also set $X = H \times H_{-\frac{1}{2}}$, $\mathcal{D}(A) = H_{\frac{1}{2}} \times H$ and we define $A : \mathcal{D}(A) \to X$ by

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}. \tag{11.6.11}$$

By Proposition 3.7.6, A is skew-adjoint. By Stone's theorem, A generates a unitary group $\mathbb{T}$. As usual, the semigroup $\mathbb{T}$ can be restricted to an operator semigroup on $X_1 = H_{\frac{1}{2}} \times H$ (which is $\mathcal{D}(A)$ with the graph norm). The generator of this restriction is $A|_{\mathcal{D}(A^2)}$, where $\mathcal{D}(A^2) = H_1 \times H_{\frac{1}{2}}$. For this restricted semigroup we consider the observation operator $C \in \mathcal{L}(H_1 \times H_{\frac{1}{2}}, U)$ defined by

$$C \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \frac{\partial \varphi}{\partial \nu}|_\Gamma \qquad \forall\, \varphi \in H_1 \times H_{\frac{1}{2}}. \tag{11.6.12}$$

We have seen in Theorem 7.1.3 that C is admissible for $\mathbb{T}$ acting on X_1.

We have seen in Section 10.9 that (11.6.7)–(11.6.10) correspond to a well-posed boundary control system with input pace U and state space X. Hence, according to Theorem 10.9.3, these equations have a unique weak solution.

The main result of this subsection is the following.

Theorem 11.6.3. *If τ and Γ are such that the pair (A,C), with state space X_1, is exactly observable in time τ, then for every f, $\widetilde{f} \in L^2(\Omega)$, g, $\widetilde{g} \in \mathcal{H}^{-1}(\Omega)$ there exists $u \in L^2\left([0,\tau];L^2(\Gamma)\right)$ such that the weak solution of* (11.6.7)–(11.6.10) *satisfies*

$$w(\cdot,\tau) = \widetilde{f}, \qquad \frac{\partial w}{\partial t}(\cdot,\tau) = \widetilde{g}. \tag{11.6.13}$$

Proof. We have seen in Proposition 10.9.1 that (11.6.7)–(11.6.10) correspond to a well-posed boundary control system whose generator is A and whose control operator B satisfies

$$B^* \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = -\frac{\partial}{\partial \nu}\left(A_0^{-1}\psi\right)\Big|_{\Gamma} \qquad \forall \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A^*) = \mathcal{D}(A).$$

Notice that $B^*A = C$. Since (A,C) is exactly observable in time τ on the state space X_1 and since A is a unitary operator from X_1 to X, it follows that (B^*,A) is exactly observable in time τ, on the state space X. Since $A^* = -A$, the pair (B^*,A^*) is also exactly observable in time τ, on the state space X. By using Theorem 11.2.1, it follows that (A,B) is exactly controllable in time τ (on the state space X). As mentioned after Definition 11.1.1, this means that for any $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} \widetilde{f} \\ \widetilde{g} \end{smallmatrix}\right] \in X$, there exists $u \in L^2([0,\tau];U)$ such that the solution of $\dot{z} = Az + Bu$, with $z(0) = \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$, satisfies $z(\tau) = \left[\begin{smallmatrix} \widetilde{f} \\ \widetilde{g} \end{smallmatrix}\right]$. We know from Theorem 10.9.3 that this solution coincides with the weak solution of (11.6.7)–(11.6.10) if we put $z = \left[\begin{smallmatrix} w \\ \dot{w} \end{smallmatrix}\right]$. □

By combining the above result with Theorem 7.2.4, we obtain the following.

Corollary 11.6.4. *Assume that there exists $x_0 \in \mathbb{R}^n$ such that*

$$\Gamma \supset \{x \in \partial\Omega \mid (x - x_0)\cdot \nu(x) > 0\},$$

and denote

$$r(x_0) = \sup_{x\in\Omega} |x - x_0|.$$

Then the conclusion in Theorem 11.6.3 *holds for every $\tau > 2r(x_0)$.*

11.7 Remarks and bibliographical notes on Chapter 11

General remarks. As far as we know, the first approaches of controllability for systems governed by PDEs were based on the moment method, already mentioned at the beginning of Section 8.6. We refer the reader to Fattorini and Russell [62, 63] and to Russell [199, 197] for early contributions in this direction. The method of moments has then been developed and systematically applied to systems governed by partial differential equations in the book of Avdonin and Ivanov [9].

We give below a more precise formulation of this method, using the notation introduced in this chapter. Let A be the generator of a semigroup $\mathbb{T}$ on the Hilbert space X, let U be a Hilbert space and let $B \in \mathcal{L}(U, X_{-1})$ be an admissible control operator for $\mathbb{T}$. If we assume that A is diagonalizable, with an orthonormal basis $(\phi_k)_{k\in\mathbb{N}}$ of eigenvectors corresponding to the eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$, then the pair (A, B) is exactly controllable in time τ iff for every sequence $(c_k) \in l^2$ there exists $u \in L^2([0,\tau]; U)$ such that

$$\int_0^\tau \langle u(t), e^{\overline{\lambda_k}\, t} B^* \phi_k \rangle_U \, \mathrm{d}t = c_k \qquad \forall\, k \in \mathbb{N}. \tag{11.7.1}$$

Indeed, by combining (11.1.1) and (2.6.9), it is not difficult to check that the condition

$$\Phi_\tau u = \sum_{k\in\mathbb{N}} c_k \phi_k$$

is equivalent to (11.7.1). By taking $c = e_l$, for every $l \in \mathbb{N}$ (where (e_l) is the standard basis of l^2), we obtain that a necessary condition for exact observability is the existence of a family $(\Psi_k)_{k\in\mathbb{N}}$ which is biorthogonal (in $L^2([0,\tau]; U)$) to the family $\left(e^{\overline{\lambda_k}\, t} B^* \phi_k\right)_{k\in\mathbb{N}}$, i.e., a family satisfying

$$\int_0^\tau \left\langle \Psi_l(t), e^{\overline{\lambda_k}\, t} B^* \phi_k \right\rangle_U = \delta_{lk} \qquad \forall\, k,\ l \in \mathbb{N}.$$

(See also Lemma 9.2.1.) The existence of a family $(\Psi_k)_{k\in\mathbb{N}}$ as above is sufficient for a weaker property of controllability. This property, usually called *spectral controllability*, means that for each $k \in \mathbb{N}$ there exists $u_k \in L^2([0,\tau]; U)$ such that $\Phi_\tau u_k = \phi_k$. For a detailed study of spectral controllability, which is weaker than exact controllability but stronger than approximate controllability, we refer the reader to [9]. For an interesting study of this property in the case of Euler–Bernoulli plate equation we refer the reader to Haraux and Jaffard [95].

Section 11.2. The duality of controllability and of observability has been first formulated in an infinite-dimensional setting in Dolecki and Russell [51], but it has been used for proving the exact controllability of PDEs systems only several years later. We refer the reader to Lions [155, 154] and Triggiani [221] for early contributions using this approach for the exact boundary controllability of the wave equation with Dirichlet boundary control. This duality approach has been mainly developed under the name Hilbert Uniqueness Method (HUM) in the book of J.-L. Lions [156] and then used on various PDEs. For more information on the examples in Section 11.2, we refer the reader to the comments in Sections 7.7, 8.6 and 9.6 on the corresponding observability problems.

Section 11.3. Simultaneous exact controllability was first considered by Russell in [200] and it is the subject of Lions [156, Chapter 5]. The simultaneous controllability of two Riesz spectral systems (one hyperbolic and one parabolic) was studied in Hansen [87, Section 4] (see also Hansen and Zhang [90]). Our presentation follows closely Tucsnak and Weiss [222]. There are now papers which extend the results from [222] to the simultaneous controllability of two infinite-dimensional systems;

see Avdonin and Tucsnak [11] (for two strings) and Avdonin and Moran [10] (for several strings or beams with a common endpoint). Theorem 11.3.6 is new.

Section 11.4. The study of the controllability properties of systems coupling PDEs in one space dimension with ODEs (sometimes called *hybrid systems*) has been probably initiated by Littman and Markus in [159]. This paper was at the origin of a considerable number of articles on this subject (see, for instance, Guo and Ivanov [78], Hansen and Zuazua [91], Morgul, Rao and Conrad [173], Rao [187]). Our approach, following [222], is based on simultaneous exact controllability results.

Section 11.5. As already mentioned, the first results on null-controllability of the heat equation, in one space dimension, have been obtained in [62, 63] by using the moment method. The duality approach combined with various Carleman estimates has been initiated by the works of Fursikov and Imanuvilov in [69] and of Lebeau and Robbiano [151]. We refer the reader to the paragraph on Section 9.5 from Section 9.6 for comments on the dual observability properties.

The result in Proposition 11.5.1 has been generalized recently in Ammar-Khodja et al. [6]. Their result refers to the system described by the equations

$$\dot{z} = D\Delta z + Az + Bu \qquad \text{in } \ \Omega \times (0,T),$$
$$z = 0 \qquad \text{in } \ \partial\Omega \times (0,T),$$

where $z(t) \in L^2(\Omega)^n$ is the state at time $t \geqslant 0$ and $u \in L^2([0,T]; L^2(\mathcal{O})^m)$ is the input function (Ω and $\mathcal{O}$ are as in Proposition 11.5.1). The matrix D is assumed to be real, diagonal and constant (i.e., independent of x and t). The matrices A and B are also constant, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Let us denote by $-A_0$ the Dirichlet Laplacian on Ω. The result is that the above system is null-controllable in any time $\tau > 0$ iff for every $\lambda \in \sigma(A_0)$ the finite-dimensional pair $(A - \lambda D, B)$ is controllable. In particular, if $D = I$, then the condition reduces to the controllability of (A, B).

Section 11.6. The first results on the exact controllability of the wave equation have been first obtained by using the method of moments; see Russell [197]. This approach has been extended to the wave equation in a spherical region by Graham and Russell [76]. We refer the reader to [197, 199] and to Littman [158] for a method based on solving first the initial value problem in the whole space.

For more general spatial domains, the exact controllability for the n-dimensional wave equation with control acting on the whole boundary has been established, via Russell's "stabilizability implies controllability" argument (see [198]), in Lasiecka and Triggiani [146]. The fact that only a part of the boundary might be sufficient for the boundary exact controllability of the wave equation has been first proved by Lions in [154], by using the duality approach which he called HUM. For further information on the dual exact observability problem we refer the reader to the paragraph on Section 7.2 from Section 7.7.

The results of B. Jacob, R. Rebarber and H. Zwart on the spectrum of optimizable systems. An important concept in distributed parameter systems theory that has not been touched in this book is optimizability. Suppose that A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X and $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$. We call (A, B) *optimizable* if for every $z_0 \in X$ there exists $u \in L^2([0,\infty); U)$ such that the corresponding state trajectory z is in $L^2([0,\infty); X)$. Clearly null-controllability implies optimizability. Much material on optimizability can be found, among other sources, in Jacob and Zwart [118], Rebarber and Zwart [189] and Weiss and Rebarber [234]. We mention here two interesting results from [189] and [118].

Theorem 11.7.1. *Suppose that U is finite dimensional and (A, B) is optimizable. Then there exists $\varepsilon > 0$ such that all elements $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda > -\varepsilon$ are isolated and they are eigenvalues of A with finite algebraic multiplicity.*

A point $\lambda \in \sigma(A)$ is called *isolated* if there exists $r > 0$ such that the disk $B(\lambda, r)$ contains no other points from $\sigma(A)$ besides λ.

Theorem 11.7.2. *With the notation of the previous theorem, denote $\Lambda = \sigma(A) \cap \mathbb{C}_0$ (this set is at most countable). For every $\lambda \in \Lambda$ we denote by $m(\lambda)$ its algebraic multiplicity. Then $\sum_{\lambda\in\Lambda} \frac{m(\lambda)}{|\lambda|^2} < \infty$.*

The results of B. Jacob, J. Partington and S. Pott on controllability for systems with diagonal semigroups. Applications of Hardy space interpolation and the theory of Carleson measures to the controllability of systems with a diagonal semigroup has been discussed recently in three papers.

Jacob and Partington [113] considers a one-dimensional input space and a (possibly unbounded) control operator. A priori it is not assumed that the input operator is admissible. Necessary and sufficient conditions for different notions of controllability such as null-controllability, exact controllability and approximate controllability are presented. These conditions, which are given in terms of the eigenvalues of the diagonal generator and in terms of the control operator, are linked with the theory of interpolation in Hardy spaces. Specifically, given a sequence of positive weights (w_n) and a sequence (z_n) in the open unit disk $\mathcal{D}$ of $\mathbb{C}$, the existence, for each sequence (a_n) with $\sum_{n=1}^{\infty} |a_n w_n|^2 < \infty$, of a function $f \in \mathcal{H}^2(\mathcal{D})$ solving the interpolation problem $f(z_n) = a_n$ $(n = 1, 2, 3, \ldots)$ is equivalent to the controllability of a diagonal system with eigenvalues $\lambda_n = (z_n - 1)/(z_n + 1)$.

This work is extended in Jacob, Partington and Pott [117], where norm estimates are obtained for the problem of minimal-norm tangential interpolation by vector-valued analytic functions (solving $G_n F(z_n) = a_n$, where G_n are given linear mappings), expressed in terms of the Carleson constants of related scalar measures. Again, applications are given to the controllability properties of systems with a diagonal semigroup, where now the input space is finite dimensional.

Finally, in Jacob, Partington and Pott [114], norm estimates are obtained for the problem of minimal-norm tangential interpolation by vector-valued analytic functions in weighted $\mathcal{H}^p$ spaces, expressed in terms of the Carleson constants of related scalar measures. Applications are given to the notion of p-controllability of linear systems and controllability by functions in certain Sobolev spaces.

Chapter 12

Appendix I: Some Background on Functional Analysis

12.1 The closed-graph theorem and some consequences

In this section we state the closed-graph theorem without proof, and then we prove a few applications that are needed in the book.

Let X and Y be Banach spaces and let $T : X \to Y$ be a linear operator. T is called *closed* if for every convergent sequence (x_n) with terms in X the following holds: If $\lim Tx_n$ exists, then $\lim Tx_n = T \lim x_n$.

Theorem 12.1.1. *If $T : X \to Y$ is closed, then T is bounded.*

This is a non-trivial result called the *closed-graph theorem*. Its proof can be found in all the standard textbooks on functional analysis.

A typical application of this theorem is the following: Suppose that V and X are Banach spaces such that $V \subset X$, with continuous embedding (i.e., the identity operator on V belongs to $\mathcal{L}(V, X)$). If $T \in \mathcal{L}(X)$ and $TV \subset V$, then $T|_V \in \mathcal{L}(V)$. Here, $T|_V$ denotes the restriction of T to V. Indeed, the assumptions imply that $T|_V$ is closed, so that according to the closed-graph theorem, $T|_V$ is bounded.

Another application concerns inverse operators. If X and Y are Banach spaces and $T \in \mathcal{L}(X, Y)$ is invertible, then it is easy to see that T^{-1} is closed. It follows from the closed-graph theorem that the inverse operator is bounded: $T^{-1} \in \mathcal{L}(Y, X)$. In particular, it follows that T is bounded from below.

We need the following consequence of the closed-graph theorem.

Proposition 12.1.2. *Suppose that Z_1, Z_2 and Z_3 are Hilbert spaces, $F \in \mathcal{L}(Z_1, Z_3)$ and that $G \in \mathcal{L}(Z_2, Z_3)$. Then the following statements are equivalent:*

(a) Ran $F \subset$ Ran G;

(b) *There exists a $c > 0$ such that* $\|F^* z\|_{Z_1} \leqslant c\|G^* z\|_{Z_2} \quad \forall\, z \in Z_3$.

(c) *There exists an operator $L \in \mathcal{L}(Z_1, Z_2)$ such that $F = GL$.*

Proof. To show that (a) implies (c), we suppose that Ran $F \subset$ Ran G. For $x \in Z_1$ we have $Fx \in$ Ran $F \subset$ Ran G, so there exists a unique $y \in (\text{Ker } G)^\perp$ such that $Gy = Fx$. By setting $Lx = y$, we have $F = GL$. It remains to prove that L is bounded from Z_1 to Z_2. Since L is defined on all of Z_1, it suffices to show that L has a closed graph. Let (x_n, y_n) be a sequence in the graph of L such that $\lim(x_n, y_n) = (x, y)$ in $Z_1 \times Z_2$, then $\lim Fx_n = Fx$ and $\lim Gy_n = Gy$. Thus, $Fx = Gy$ and, since $(\text{Ker } G)^\perp$ is closed, $y \in (\text{Ker } G)^\perp$, so that $Lx = y$.

It is clear that assertion (c) implies assertion (a).

To show that (b) implies (c), suppose that (b) holds. Define a mapping K from Ran G^* to Ran F^* so that $K(G^*z) = F^*z$, for all $z \in Z_3$. Then K is well defined, since if $G^*z_1 = G^*z_2$, then $\|F^*(z_1 - z_2)\|_{Z_1} \leqslant c\|G^*(z_1 - z_2)\|_{Z_2} = 0$, so that $F^*z_1 = F^*z_2$. Moreover, the same calculation shows that

$$\|K(G^*z)\|_{Z_1} \leqslant c\|G^*z\|_{Z_2} \qquad \forall\, z \in Z_3\,.$$

Hence, K has a uniquely continuous extension to the closure $\overline{\text{Ran } G^*}$. If we define K on $(\text{Ran } G^*)^\perp$ in an arbitrary bounded way (for example, as zero), then we still have $KG^* = F^*$. If we set $L = K^*$, then $F = GL$.

It is easy to see that (c) implies (b). Indeed, if $F = GL$, then

$$\|F^*z\|_{Z_1} = \|L^*G^*z\|_{Z_1} \leqslant \|L^*\|_{\mathcal{L}(Z_2,Z_1)}\|G^*z\|_{Z_2} \qquad \forall\, z \in Z_3\,.$$

Thus, (a), (b) and (c) are equivalent. □

Proposition 12.1.3. *If Z, X are Hilbert spaces and $G \in \mathcal{L}(Z, X)$, then the following statements are equivalent:*

(a) *G is onto.*

(b) *G^* is bounded from below; i.e., there exists a constant $m > 0$ such that*

$$\|G^*x\|_Z \geqslant m\|x\|_X \qquad \forall\, x \in X\,.$$

(c) *$GG^* > 0$ (as defined in Section* 3.3*).*

Moreover, if these statements are true, then $\|(GG^*)^{-1}\| \leqslant \frac{1}{m^2}$, where m is the constant appearing in statement (b)*.*

Proof. The equivalence of (a) and (b) follows from Proposition 12.1.2 by taking $Z_1 = Z_3 = X$, $Z_2 = Z$, $F = I$ and $c = 1/m$.

We show that (b) implies (c). If (b) holds, then from

$$\langle GG^*x, x\rangle = \|G^*x\|^2 \geqslant m^2\|x\|^2$$

we see that $GG^* \geqslant m^2 I > 0$. By Proposition 3.3.2, GG^* is invertible and satisfies $\|(GG^*)^{-1}\| \leqslant \frac{1}{m^2}$. Conversely, it is obvious that (c) implies (b). □

12.2 Compact operators

In this section we gather, for easy reference, some results on compact operators on a Hilbert space. For a more detailed presentation of this topic we refer the reader, for instance, to Akhiezer and Glazman [2], Dowson [52], Kato [127] or Rudin [195].

Recall that a subset M of a Hilbert space H is said to be *relatively compact* if every sequence in M has a convergent subsequence. It is well known that a set $M \subset H$ is relatively compact iff it has the following property, known as *total boundedness*: For every $n \in \mathbb{N}$ there exists a finite set $F_n \subset H$ such that

$$\min_{f \in F_n} \|x - f\| \leqslant \frac{1}{n} \qquad \forall\, x \in M\,. \tag{12.2.1}$$

M is called *compact* if it is relatively compact and closed. We denote by B_1 the closed unit ball of H. It is well known that B_1 is compact iff $\dim H < \infty$.

Definition 12.2.1. Let H and Y be Hilbert spaces. $K \in \mathcal{L}(H, Y)$ is *compact* if the set KB_1 is relatively compact in Y.

It is clear that the compact operators form a subspace of $\mathcal{L}(H, Y)$. Let U be another Hilbert space. It is easy to see that if $K \in \mathcal{L}(H, Y)$, $T \in \mathcal{L}(Y, U)$ and K is compact, then TK is compact. Similarly, if $T \in \mathcal{L}(U, H)$ and K is as before, then KT is compact. It is easy to see (from what we said earlier in this section) that I (the identity operator on H) is compact iff $\dim H < \infty$.

Proposition 12.2.2. *For any $K \in \mathcal{L}(H, Y)$ the following statements are equivalent:*

(a) *K is compact.*

(b) *There exists a sequence (K_n) in $\mathcal{L}(H, Y)$ such that*

$$\dim \operatorname{Ran} K_n < \infty\,, \qquad \lim K_n = K\,. \tag{12.2.2}$$

Proof. Suppose that K is compact, so that $M = KB_1$ is relatively compact in Y. For every $n \in \mathbb{N}$ let F_n be the finite set with the property (12.2.1). Denote by P_n the orthogonal projector from Y onto the finite-dimensional space $\operatorname{span} F_n$ and define $K_n = P_n K$. Then it is easy to see that $\|K - K_n\| \leqslant 1/n$.

Conversely, suppose that (K_n) is a sequence in $\mathcal{L}(H, Y)$ such that (12.2.2) holds and choose $m \in \mathbb{N}$. We can find $n \in \mathbb{N}$ such that $\|K - K_n\| \leqslant \frac{1}{2m}$. Since $M = K_n B_1$ is relatively compact, according to (12.2.1) we can find a finite set $F_{2m} \subset Y$ such that $\min_{f \in F_{2m}} \|x - f\| \leqslant \frac{1}{2m}$ for all $x \in M$. It follows that

$$\min_{f \in F_{2m}} \|x - f\| \leqslant \frac{1}{m} \qquad \forall\, x \in KB_1\,.$$

This holds for every $m \in \mathbb{N}$, so that KB_1 is relatively compact in Y. □

Corollary 12.2.3. *If $K \in \mathcal{L}(H, Y)$ is compact, then also K^* is compact.*

Proof. If K is compact then, as we have seen in the first part of the proof of Proposition 12.2.2, there exists a sequence (P_n) of orthogonal projectors onto finite-dimensional subspaces of Y such that $\lim P_k K = K$. It follows that $\lim K^* P_k = K^*$. Since $\dim \operatorname{Ran} K^* P_k < \infty$, according to Proposition 12.2.2, K^* is compact. $\square$

We need to recall the following fact from functional analysis, which is a particular case of *Alaoglu's theorem.*

Lemma 12.2.4. *If (x_n) is a bounded sequence in H, then there exists a subsequence (x_{n_k}) and a vector $x_0 \in H$ such that*

$$\lim_{k \to \infty} \langle x_{n_k}, \varphi \rangle = \langle x_0, \varphi \rangle \qquad \forall\, \varphi \in H\,. \tag{12.2.3}$$

A sequence that behaves like (x_{n_k}) in the above lemma is called *weakly convergent* to x_0 (in H). An equivalent way to state the above lemma is the following: The unit ball of any Hilbert space is weakly sequentially compact. We refer the reader to Brezis [22, p. 46] or to Rudin [195, Theorem 3.17] for the proof.

Now we show that the compact operators are precisely those that map weakly convergent sequences into convergent sequences.

Proposition 12.2.5. *For any $K \in \mathcal{L}(H, Y)$ the following statements are equivalent:*

(a) *K is compact.*

(b) *If (x_k) is a sequence in H that converges weakly to an element $x_0 \in H$, i.e.,*

$$\lim_{k \to \infty} \langle x_k, \varphi \rangle = \langle x_0, \varphi \rangle \qquad \forall\, \varphi \in H\,, \tag{12.2.4}$$

then $\lim_{k \to \infty} K x_k = K x_0$.

Proof. Suppose that K is compact and (x_k), x_0 are as in (12.2.4). From the uniform boundedness theorem we know that the sequence (x_k) is bounded: for all $k \in \mathbb{N}$, $\|x_k - x_0\| \leqslant M$. We have to show that for every $\varepsilon > 0$ and for sufficiently large $k \in \mathbb{N}$ we have $\|K(x_k - x_0)\| \leqslant \varepsilon$. According to Proposition 12.2.2, for any given $\varepsilon > 0$, we can choose an operator $K_\varepsilon \in \mathcal{L}(H, Y)$ such that $V_\varepsilon = \operatorname{Ran} K_\varepsilon$ is finite dimensional and $\|K_\varepsilon - K\| \leqslant \varepsilon/(2M)$. It is easy to see (using orthonormal coordinates in V_ε) that $\lim_{k \to \infty} K_\varepsilon(x_k - x_0) = 0$. In the simple estimate

$$\|K(x_k - x_0)\| \leqslant \|K_\varepsilon(x_k - x_0)\| + \|(K - K_\varepsilon)(x_k - x_0)\|$$

we see that for sufficiently large k, both terms on the right-hand side are $\leqslant \varepsilon/2$. Thus, we have shown that $\lim_{k \to \infty} K x_k = K x_0$, so that (b) holds.

Conversely, suppose that statement (b) holds. Let $(K x_n)$ be an arbitrary sequence in $K B_1$. Since $x_n \in B_1$, according to Lemma 12.2.4, there exists a weakly convergent subsequence (x_{n_k}). According to (b), the sequence $(T X_{n_k})$ is convergent. This shows that $K B_1$ is relatively compact in Y, i.e., K is compact. $\square$

In what follows, we look at spectral properties of compact operators. For this, we consider compact operators in $\mathcal{L}(H)$.

Remark 12.2.6. If $\dim H = \infty$ and $K \in \mathcal{L}(H)$ is compact, then H is neither left-invertible nor right-invertible. In particular, $0 \in \sigma(K)$. Indeed, if K were left-invertible, then there would be a $Q \in \mathcal{L}(H)$ such that $QK = I$, whence I would be compact, which is absurd. A similar reasoning applies for right-invertibility.

Proposition 12.2.7. *If $K \in \mathcal{L}(H)$ is compact and λ is a non-zero complex number, then* Ran $(\lambda I - K)$ *is closed.*

Proof. Denote $V = (\text{Ker } (\lambda I - K))^{\perp}$, so that S, defined as the restriction of $\lambda I - K$ to V, is one-to-one. We claim that S is actually bounded from below. Indeed, suppose that this is not the case. Then there exists a sequence (x_n) in V such that $\|x_n\| = 1$ and $\lim Sx_n = 0$. Because of the compactness of K, (Kx_n) has a convergent subsequence (Kx_{n_k}). Thus, $\lim Kx_{n_k} = z$, which implies that $\lambda \lim x_{n_k} = z$, hence $\|z\| = |\lambda|$. By the continuity of S we obtain $Sz = 0$, which is a contradiction. We conclude from this contradiction that S is bounded from below, and hence Ran S is closed. Finally, notice that Ran $(\lambda I - K) =$ Ran S. $\square$

If $K \in \mathcal{L}(H)$ and $\lambda \in \sigma_p(K)$ (recall that $\sigma_p(K)$ denotes the set of all the eigenvalues of K), then the *geometric multiplicity* of λ is $\dim \text{Ker } (\lambda I - K)$.

Proposition 12.2.8. *If $K \in \mathcal{L}(H)$ is compact and $\lambda \in \sigma(K)$, $\lambda \neq 0$, then λ is an eigenvalue of K, with finite geometric multiplicity.*

Proof. If $\dim H < \infty$, then this is a well-known fact from linear algebra. Now consider $\dim H = \infty$. Take $\lambda \in \sigma(K)$ with $\lambda \neq 0$. First we prove that $\lambda \in \sigma_p(K)$. Suppose that this were not true, so that $\lambda I - K$ would be one-to-one. According to Proposition 12.2.7, the space $V =$ Ran $(\lambda I - K)$ is closed, so that $\lambda I - K$ would be invertible as an operator in $\mathcal{L}(H, V)$. The inverse could be extended to an operator $Q \in \mathcal{L}(H)$ by defining it to be zero on $V^{\perp}$, and then $QK = I$, so that K is left-invertible. According to Remark 12.2.6, this is absurd. Thus, in fact $\lambda \in \sigma_p(K)$.

Now we show that λ has finite geometric multiplicity. The restriction of K to the space $E =$ Ker $(\lambda I - K)$ is λI. This must be compact, which implies (as remarked earlier in this section) that $\dim E < \infty$. $\square$

Proposition 12.2.9. *If $K \in \mathcal{L}(H)$ is compact and (λ_k) is a sequence of distinct eigenvalues of K, then* $\lim \lambda_k = 0$.

Proof. Let (x_k) be a sequence of eigenvectors corresponding to the sequence of eigenvalues (λ_k). Suppose that the assertion $\lim \lambda_k = 0$ is not true. Then we can extract from (λ_k) a subsequence with the property that each term in the subsequence satisfies $|\lambda_k| \geqslant \delta > 0$. For the sake of simplicity, we denote this subsequence also by (λ_k). Clearly, the vectors x_k are independent, so that if we denote

$$M_n = \operatorname{span}\{x_1, x_2, \dots, x_n\},$$

then we have the strict inclusions $M_{n-1} \subset M_n$ and $KM_n \subset M_n$. For each $n \in \mathbb{N}$, take z_n to be in the orthogonal complement of M_{n-1} in M_n and such that $\|z_n\| = 1$. For $m, n \in \mathbb{N}$ with $m < n$ we have

$$Kz_n - Kz_m = \lambda_n z_n - q, \quad \text{where} \quad q = (\lambda_n I - K)z_n + Kz_m .$$

Notice that $q \in M_{n-1}$, so that $\langle z_n, q\rangle = 0$. Hence

$$\|Kz_n - Kz_m\| \geqslant |\lambda_n| \cdot \|z_n\| \geqslant \delta .$$

This shows that (Kz_k) does not have any convergent subsequence, which contradicts the definition of a compact operator. Thus, $\lim \lambda_k = 0$. □

Corollary 12.2.10. *Let $K \in \mathcal{L}(H)$ be a diagonalizable operator with the sequence of eigenvalues (λ_k), as in* (2.6.5)*. Then K is compact if and only if $\lim \lambda_k = 0$.*

Proof. The "if" part follows from Proposition 12.2.2 (by truncating the sequence in (2.6.5)). The "only if" part follows from Proposition 12.2.9. □

Theorem 12.2.11. *Assume that $K \in \mathcal{L}(H)$ is compact and self-adjoint. Then there exists in H an at most countable orthonormal set $\mathcal{B}$ consisting of eigenvectors of K,*

$$\mathcal{B} = \{\varphi_k \mid k \in \mathcal{I}\}, \qquad \text{where} \qquad \mathcal{I} \subset \mathbb{Z},$$

with the following properties: If μ_k is the eigenvalue corresponding to φ_k, then

$$Kz = \sum_{k\in\mathcal{I}} \mu_k \langle z, \varphi_k\rangle \varphi_k \qquad \forall\, z \in H, \tag{12.2.5}$$

$\mu_k \in \mathbb{R}$, $\mu_k \neq 0$ and if $\mathcal{I}$ is infinite, then $\lim_{|k|\to\infty} \mu_k = 0$. Moreover,

$$\mathcal{B}^\perp = \operatorname{Ker} K . \tag{12.2.6}$$

Proof. It will be convenient to denote $\sigma_0(K) = \sigma(K) \setminus \{0\}$. According to Propositions 12.2.8 and 12.2.9, $\sigma_0(K)$ consists of an at most countable set of eigenvalues of K, with zero as the only possible accumulation point. Denote

$$E_\lambda = \operatorname{Ker}(\lambda I - K) \qquad \forall\, \lambda \in \sigma_0(K).$$

We know from Proposition 12.2.8 that $\dim E_\lambda < \infty$. We know from Proposition 3.2.6 that $\sigma(K) \subset \mathbb{R}$. For each $\lambda \in \sigma_0(K)$ let $\mathcal{B}_\lambda$ be an orthonormal basis in E_λ and put

$$\mathcal{B} = \bigcup_{\lambda\in\sigma_0(K)} \mathcal{B}_\lambda .$$

We can now construct sequences (μ_k) and (φ_k) indexed by a set $\mathcal{I} \subset \mathbb{Z}$ such that

(1) for each $k \in \mathcal{I}$, $\mu_k \in \sigma_0(K)$ and $\varphi_k \in \mathcal{B}_{\mu_k}$,
(2) there are no repetitions in the sequence (φ_k),
(3) $\mathcal{B} = \{\varphi_k \mid k \in \mathcal{I}\}$.

Thus, each $\lambda \in \sigma_0(K)$ appears in the sequence (μ_k) repeated $\dim E_\lambda$ times. If $\mathcal{I}$ is infinite, then (by a simple rearranging of the indices from Proposition 12.2.9) we see that $\lim_{|k|\to\infty} \mu_k = 0$. An easy consequence of $K^* = K$ is the following: If ϕ, ψ are eigenvectors of K corresponding to different eigenvalues, then $\langle \phi, \psi \rangle = 0$. This implies that the set $\{\varphi_k \mid k \in \mathcal{I}\}$ is orthonormal.

Let K_0 be the operator defined by the sum in (12.2.5), so that K_0 is self-adjoint, as it is easy to verify. It is also easy to check that

$$\operatorname{span} \mathcal{B} \subset \operatorname{Ker}\,(K - K_0). \tag{12.2.7}$$

Denote $E_0 = (\operatorname{span} \mathcal{B})^\perp$. Since $K(\operatorname{span} \mathcal{B}) \subset \operatorname{span} \mathcal{B}$, it is easy to see that $KE_0 \subset E_0$. Let us denote by N the restriction of K to E_0, regarded as an element of $\mathcal{L}(E_0)$. It is easy to see that N is self-adjoint and compact. N cannot have non-zero eigenvalues, because the corresponding eigenvectors would have to belong to F, which is absurd. It follows that $\sigma(N) \subset \{0\}$, so that its spectral radius is $r(N) = 0$. According to Proposition 3.2.7, we obtain that $N = 0$. On the other hand, it is clear that the restriction of K_0 to E_0 is also zero. Thus,

$$E_0 \subset \operatorname{Ker}\,(K - K_0).$$

Combining this with (12.2.7), we obtain that $K = K_0$.

Finally, we have to prove (12.2.6). The inclusion $\mathcal{B}^\perp \subset \operatorname{Ker} K$ is clear from (12.2.5). Conversely, suppose that $x \in \operatorname{Ker} K$ and for each $k \in \mathcal{I}$ denote $x_k = \langle x, \varphi_k \rangle$. If

$$\tilde{x} = \sum_{k \in \mathcal{I}} \operatorname{sign}(\mu_k) x_k \varphi_k,$$

then

$$0 = \langle Kx, \tilde{x} \rangle = \sum_{k \in \mathcal{I}} \mu_k x_k \operatorname{sign}(\mu_k) \overline{x_k} = \sum_{k \in \mathcal{I}} |\mu_k| \cdot |x_k|^2.$$

This shows that $x_k = 0$, so that $x \in \mathcal{B}^\perp$. □

12.3 The square root of a positive operator

In this section we introduce the square root of a bounded positive operator. This is needed in Section 3.4 in order to define the square root of an unbounded positive operator. In this section, H is a Hilbert space.

Lemma 12.3.1. *If $P \in \mathcal{L}(H)$, $P \geqslant 0$, $x \in H$ and $\langle Px, x \rangle = 0$, then $Px = 0$.*

Proof. Denote $z = (\lambda I - P)x$, where $\lambda > 0$. We have

$$\langle Pz, z\rangle = \langle P(\lambda^2 I - 2\lambda P + P^2)x, x\rangle = -2\lambda\|Px\|^2 + \langle P^3 x, x\rangle .$$

If we had $\|Px\| > 0$, then the above expression would become negative for large λ, which is absurd. Hence, $\|Px\| = 0$, so that $Px = 0$. $\square$

One of the uses of the above lemma is in the proof of the following slight generalization of the Cauchy–Schwarz inequality: If $P \in \mathcal{L}(H)$ and $P \geqslant 0$, then

$$|\langle Px, y\rangle|^2 \leqslant \langle Px, x\rangle \cdot \langle Py, y\rangle \qquad \forall\, x, y \in H . \tag{12.3.1}$$

The proof of this follows the same argument as for the classical Cauchy–Schwarz inequality, but eliminating first the case when $\langle Px, x\rangle = 0$.

By using the inequality (12.3.1), it is easy to show that

$$P, Q \in \mathcal{L}(H) \quad \text{and} \quad 0 \leqslant P \leqslant Q \quad \Rightarrow \quad \|P\| \leqslant \|Q\| . \tag{12.3.2}$$

Lemma 12.3.2. *Let (Q_n) be a sequence of bounded and positive operators on H such that $Q_n \geqslant Q_{n+1}$ for all $n \in \mathbb{N}$. Then there exists a positive $Q \in \mathcal{L}(H)$ such that*

$$\lim Q_n x = Qx \qquad \forall\, z \in H .$$

Moreover, we have $Q \leqslant Q_n$ for all $n \in \mathbb{N}$.

Proof. For $m, n \in \mathbb{N}$, $n > m$, using (12.3.1) and the fact that $Q_1 \geqslant Q_m - Q_n \geqslant 0$, we have that for every $x \in H$,

$$\begin{aligned} \|(Q_m - Q_n)x\|^4 &= \langle (Q_m - Q_n)x, (Q_m - Q_n)x\rangle^2 \\ &\leqslant \langle (Q_m - Q_n)x, x\rangle \cdot \langle (Q_m - Q_n)^2 x, (Q_m - Q_n)x\rangle \\ &\leqslant (\langle Q_m x, x\rangle - \langle Q_n x, x\rangle) \cdot \|Q_1\|^3 \cdot \|x\|^2 . \end{aligned} \tag{12.3.3}$$

The sequence $(\langle Q_n x, x\rangle)$ is positive and decreasing and hence convergent. Thus (12.3.3) shows that the sequence $(Q_n x)$ is convergent in H. Define

$$Qx = \lim Q_n x \qquad \forall\, x \in H .$$

Clearly Q is linear. Moreover, since $Q_n \leqslant Q_1$, by using (12.3.2) we have $\|Q_n\| \leqslant \|Q_1\|$ for all $n \in \mathbb{N}$, so that Q is bounded and $\|Q\| \leqslant \|Q_1\|$. Since $\langle Qx, x\rangle = \lim\langle Q_n x, x\rangle \geqslant 0$, it follows that $0 \leqslant Q \leqslant Q_n$ for all $n \in \mathbb{N}$. $\square$

If $S, T \in \mathcal{L}(X)$, we say that S *commutes with* T if $ST = TS$.

Lemma 12.3.3. *If M, $N \in \mathcal{L}(H)$ are positive operators which commute and $M^2 = N^2$, then $M = N$.*

Proof. It is easy to verify the identity

$$(M-N)M(M-N)+(M-N)N(M-N) = 0.$$

Both terms are positive, which implies that both terms are in fact zero. Hence, we have for every $x \in H$ that $\langle M(M-N)x, (M-N)x\rangle = 0$. According to Lemma 12.3.1, it follows that $M(M-N)x = 0$, so that $M(M-N) = 0$. By a similar argument we also have that $N(M-N) = 0$. Combining these facts we can see that $(M-N)^2 = 0$. Since $M-N$ is self-adjoint, it follows that $M-N = 0$. □

Theorem 12.3.4. *If $P \in \mathcal{L}(H)$ is positive, then there exists a unique positive operator $P^{\frac{1}{2}}$, called the* square root of P*, such that $(P^{\frac{1}{2}})^2 = P$. Moreover, $P^{\frac{1}{2}}$ commutes with every operator that commutes with P.*

Proof. If $P = 0$, then the statements are trivially true. Assuming that $P \neq 0$, introduce $S = P/\|P\|$, so that $S \leqslant I$. Define the sequence (Q_n) in $\mathcal{L}(H)$ recursively:

$$Q_1 = I \quad \text{and} \quad Q_{n+1} = Q_n + \frac{1}{2}(S - Q_n^2) \qquad \forall\, n \in \mathbb{N}.$$

Each term Q_n is a real polynomial in S, hence it is self-adjoint and it commutes with every operator that commutes with P. Notice that we have

$$I - Q_{n+1} = \frac{1}{2}(I - Q_n)^2 + \frac{1}{2}(I - S) \qquad \forall\, n \in \mathbb{N}. \tag{12.3.4}$$

From here we can show by induction that for every $n \in \mathbb{N}$, $I - Q_n = p_n(I - S)$, where p_n is a polynomial with positive coefficients. It follows that for every $n \in \mathbb{N}$,

$$I - \frac{1}{2}(Q_{n+1} + Q_n) = \frac{1}{2}(p_{n+1} + p_n)(I - S), \tag{12.3.5}$$

where $p_{n+1} + p_n$ is another polynomial with positive coefficients.

Subtracting the defining (recursive) formula of Q_{n+2} from that of Q_{n+1}, we obtain

$$Q_{n+1} - Q_{n+2} = (Q_n - Q_{n+1}) \cdot \left[I - \frac{1}{2}(Q_{n+1} + Q_n)\right] \qquad \forall\, n \in \mathbb{N}.$$

We have $Q_1 - Q_2 = \frac{1}{2}(I - S)$, and the above formula, together with (12.3.5), implies (by induction) that for every $n \in \mathbb{N}$, $Q_n - Q_{n+1}$ can be expressed as a polynomial with positive coefficients in the variable $I - S \geqslant 0$. This implies that $Q_n - Q_{n+1} \geqslant 0$, which is one of the conditions in Lemma 12.3.2.

Now we show that $Q_n \geqslant 0$. From (12.3.4) we see that

$$\|I - Q_{n+1}\| \leqslant \frac{1}{2}\|I - Q_n\|^2 + \frac{1}{2}\|I - S\| \qquad \forall\, n \in \mathbb{N}. \tag{12.3.6}$$

From $0 \leqslant I - S \leqslant I$ we know that $\|I - S\| \leqslant 1$. This, together with (12.3.6), implies (again by induction) that $\|I - Q_n\| \leqslant 1$ for all $n \in \mathbb{N}$. We have seen earlier that $I - Q_n = p_n(I - S)$, so that $I - Q_n \geqslant 0$. This implies that $I - Q_n \leqslant I$, i.e., $Q_n \geqslant 0$.

We have shown that the sequence (Q_n) satisfies all the assumptions of Lemma 12.3.2. Thus, by the lemma, there exists a positive $Q \in \mathcal{L}(H)$ such that

$$\lim Q_n x = Qx \qquad \forall\, x \in H\,.$$

Since Q_n commutes with every operator that commutes with P, it follows that also Q has this property. Similarly, since $Q_n \leqslant I$, we have $Q \leqslant I$.

Applying the recursive definition of Q_n to $x \in H$ and taking limits, we obtain that $\lim Q_n^2 x = Sx$ for all $x \in H$. On the other hand, it is easy to see that $\lim Q_n^2 x = Q^2 x$ for all $x \in H$ (because $Q_n^2 - Q^2 = (Q_n + Q)(Q_n - Q)$). It follows that $Q^2 = S$, so that the operator

$$P^{\frac{1}{2}} = \|P\|^{\frac{1}{2}}\, Q$$

is positive and $(P^{\frac{1}{2}})^2 = P$. The only thing left to prove is the unicity of $P^{\frac{1}{2}}$. If $M \geqslant 0$ is such that $M^2 = P$, then clearly M commutes with P and hence $P^{\frac{1}{2}}$ commutes with M. Now $M = P^{\frac{1}{2}}$ follows from Lemma 12.3.3. □

Remark 12.3.5. With the notation of the last theorem, it follows from Proposition 2.2.12 that

$$\sigma(P^{\frac{1}{2}}) = \sigma(P)^{\frac{1}{2}}\,.$$

This implies further properties of $P^{\frac{1}{2}}$. For example, since $\|P\| = r(P)$ (see Proposition 3.2.7), it follows that $\|P^{\frac{1}{2}}\| = \|P\|^{\frac{1}{2}}$. Another consequence is the following: If $\lambda \geqslant 0$ is such that $P \geqslant \lambda I$, then $P^{\frac{1}{2}} \geqslant \lambda^{\frac{1}{2}} I$. Indeed, by Remark 3.3.4 we have $\sigma(P) \subset [\lambda, \infty)$, hence $\sigma(P^{\frac{1}{2}}) \subset [\lambda^{\frac{1}{2}}, \infty)$, hence (using again Remark 3.3.4) $P^{\frac{1}{2}} \geqslant \lambda^{\frac{1}{2}} I$. Of course, there are also alternative proofs for these statements.

12.4 The Fourier and Laplace transformations

In this section we recall some facts about the Fourier and Laplace transformations, we introduce the Hardy space $\mathcal{H}^2(\mathbb{C}_0)$, we state the Plancherel theorem, the Paley–Wiener theorem for $\mathcal{H}^2(\mathbb{C}_0)$ and the Carleson measure theorem. We do not give proofs, but the reader can find the material on the Fourier and Laplace transformations in a large number of books, such as Akhiezer and Glazman [2], Arendt et al. [8], Bochner and Chandrasekaran [20], Dautray and Lions [42], Doetsch [50], Dym and McKean [55], Nikol'skii [178], Rudin [194] and Young [240]. We shall give separate references for the Carleson measure theorem.

The Fourier transformation on $L^1(\mathbb{R})$. Denote by $\mathcal{D}(\mathbb{R})$ the space of C^∞ functions on $\mathbb{R}$ that have compact support. We define the *Fourier transformation* initially as an operator $\mathcal{F} : \mathcal{D}(\mathbb{R}) \to C(\mathbb{R})$ as follows:

$$(\mathcal{F}u)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} u(t) \mathrm{d}t \qquad \forall\, \omega \in \mathbb{R}. \tag{12.4.1}$$

The function $\mathcal{F}u$ is much better than just continuous: It is easy to see that $\mathcal{F}u$ can be extended to an analytic function on all of $\mathbb{C}$, and its derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\mathcal{F}u = \mathcal{F}v, \quad \text{where} \quad v(t) = -itu(t) \qquad \forall\, t \in \mathbb{R}.$$

It is an easy consequence of the Hölder inequality that

$$|(\mathcal{F}u)(\omega)| \leqslant \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |u(t)|\,\mathrm{d}t \qquad \forall\, \omega \in \mathbb{R}. \tag{12.4.2}$$

It is easy to check that the function $\omega \mapsto i\omega(\mathcal{F}u)(\omega)$ is the Fourier transform of the derivative $\dot{u} \in \mathcal{D}(\mathbb{R})$. This, together with the last estimate applied to $\dot{u}$, shows that $\lim_{|\omega| \to \infty}(\mathcal{F}u)(\omega) = 0$. Introduce the space

$$C_0(\mathbb{R}) = \left\{ f \in C(\mathbb{R}) \mid \lim_{|\omega| \to \infty} f(\omega) = 0 \right\},$$

which is a Banach space with the norm

$$\|f\|_\infty = \sup_{\omega \in \mathbb{R}} |f(\omega)|.$$

(We know that $\mathcal{D}(\mathbb{R})$ is dense in $C_0(\mathbb{R})$, but this is not needed here.) Then the preceding discussion shows that, in fact, $\mathcal{F} : \mathcal{D}(\mathbb{R}) \to C_0(\mathbb{R})$. Moreover, (12.4.2) shows that $\mathcal{F}$ is bounded if we consider on $\mathcal{D}(\mathbb{R})$ the norm $\|\cdot\|_1$ and on $C_0(\mathbb{R})$ the norm $\|\cdot\|_\infty$ introduced a little earlier. Since $\mathcal{D}(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, it follows that $\mathcal{F}$ has a unique extension to $L^1(\mathbb{R})$ (denoted by the same symbol) such that

$$\mathcal{F} \in \mathcal{L}(L^1(\mathbb{R}), C_0(\mathbb{R})), \qquad \|\mathcal{F}\| = \frac{1}{\sqrt{2\pi}}.$$

(The estimate (12.4.2) only tells us that $\|\mathcal{F}\| \leqslant \frac{1}{\sqrt{2\pi}}$, but if $u(t) \geqslant 0$ for all $t \geqslant 0$, then $(\mathcal{F}u)(0) = \frac{1}{\sqrt{2\pi}}\|u\|_1$, which shows that in fact we have equality.)

The Fourier transformation on $L^2(\mathbb{R})$. The following subtle formula holds:

$$\int_{-\infty}^{\infty} |u(t)|^2\,\mathrm{d}t = \int_{-\infty}^{\infty} |(\mathcal{F}u)(\omega)|^2\,\mathrm{d}\omega \qquad \forall\, u \in \mathcal{D}(\mathbb{R}). \tag{12.4.3}$$

Since $\mathcal{D}(\mathbb{R})$ is dense also in $L^2(\mathbb{R})$, (12.4.3) implies that $\mathcal{F}$ has a unique extension to an isometric operator from $L^2(\mathbb{R})$ to itself:

$$\mathcal{F} \in \mathcal{L}(L^2(\mathbb{R})), \qquad \mathcal{F}^*\mathcal{F} = I.$$

This extended operator $\mathcal{F}$ is no longer given by (12.4.1), since the integral does not converge in general. This can be overcome by writing

$$(\mathcal{F}u)(\omega) = \frac{1}{\sqrt{2\pi}} \lim_{T \to \infty} \int_{-T}^{T} e^{-i\omega t} u(t)\,\mathrm{d}t,$$

where the limit is taken in the norm of the space $L^2(\mathbb{R})$.

It can be checked that for $\varphi, f \in \mathcal{D}(\mathbb{R})$ we have $\langle \mathcal{F}\varphi, f\rangle = \langle \varphi, \mathcal{F}^* f\rangle$, where

$$(\mathcal{F}^* f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} f(\omega)\,\mathrm{d}\omega \qquad \forall\, t \in \mathbb{R}.$$

Thus, $\mathcal{F}^*$ is the same operator as $\mathcal{F}$, except for the change of $-i$ into i. Using a similar reasoning as for $\mathcal{F}$, we can show that $\mathcal{F}\mathcal{F}^* = I$. Thus we get the following result, known as *the Plancherel theorem.*

Theorem 12.4.1. *The Fourier transformation $\mathcal{F} \in \mathcal{L}(L^2(\mathbb{R}))$ is unitary.*

The space $\mathcal{H}^2(\mathbb{C}_0)$. For every $\alpha \in \mathbb{R}$, we denote by $\mathbb{C}_\alpha$ the open right half-plane where $\mathrm{Re}\, s > 0$. The space $\mathcal{H}^2(\mathbb{C}_0)$ consists of all the analytic functions $f : \mathbb{C}_0 \to \mathbb{C}$ for which

$$\sup_{\alpha>0} \int_{-\infty}^{\infty} |f(\alpha + i\omega)|^2\,\mathrm{d}\omega < \infty. \tag{12.4.4}$$

The norm of f in this space is, by definition,

$$\|f\|_{\mathcal{H}^2} = \left(\frac{1}{2\pi} \sup_{\alpha>0} \int_{-\infty}^{\infty} |f(\alpha + i\omega)|^2\,\mathrm{d}\omega \right)^{\frac{1}{2}}.$$

Such a space is also called a *Hardy space* (Hardy spaces are defined also for disks and sometimes for other domains, and the powers 2 and $\frac{1}{2}$ in the above formula are sometimes replaced by $p \geqslant 1$ and $\frac{1}{p}$, respectively).

If $f \in \mathcal{H}^2(\mathbb{C}_0)$, then for almost every $\omega \in \mathbb{R}$, the limit

$$f^*(i\omega) = \lim_{\alpha \to 0,\ \alpha>0} f(\alpha + i\omega)$$

exists, and it defines a function $f^* \in L^2(i\mathbb{R})$, called the *boundary trace* of f. Using boundary traces, an inner product can be defined on $\mathcal{H}^2(\mathbb{C}_0)$ as follows:

$$\langle f, g\rangle_{\mathcal{H}^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(i\omega)\overline{g}^*(i\omega)\,\mathrm{d}\omega.$$

This inner product induces the norm on $\mathcal{H}^2(\mathbb{C}_0)$ that was mentioned earlier. With this norm, $\mathcal{H}^2(\mathbb{C}_0)$ is a Hilbert space.

Let Ω be a non-empty open subset of $\mathbb{C}$. An analytic function $f : \Omega \to \mathbb{C}$ is called *rational* if it has the structure $f(s) = N(s)/D(s)$, where N and D are polynomials. If this fractional representation is minimal, i.e., the order of D is the smallest possible, then the zeros of D are called the *poles* of f. Obviously, f has a unique analytic extension to the complement of the finite set of its poles. f is called *proper* if it has a finite limit as $s \to \infty$ (equivalently, the order of N is at most equal to the order of D). Such an f is called *strictly proper* if its limit at infinity is zero (equivalently, the order of N is less than the order of D). A rational function f with values in U belongs to $\mathcal{H}^2(\mathbb{C}_0, U)$ iff it is strictly proper and all its poles are in the open left half-plane where $\operatorname{Re} s < 0$. In this case, the boundary trace f^* is simply the restriction of f to $i\mathbb{R}$.

The Laplace transformation. For $u \in L^1_{\mathrm{loc}}[0,\infty)$, its *Laplace transform* $\hat{u}$ is defined by

$$\hat{u}(s) = \int_0^\infty e^{-st} u(t)\, \mathrm{d}t \tag{12.4.5}$$

for all $s \in \mathbb{C}$ for which the integral converges absolutely, i.e.,

$$\int_0^\infty e^{-t \operatorname{Re} s} |u(t)| \mathrm{d}t < \infty .$$

This set of numbers s may be the whole complex plane $\mathbb{C}$, it may be an open or a closed right half-plane, or it may be empty. For details about the Laplace transformation we refer the reader to Arendt et al. [8], Doetsch [50] and Widder [236].

It is useful to note that if $u \in \mathcal{H}^1_{\mathrm{loc}}(0,\infty)$ is such that $\widehat{\dot{u}}$ is defined on some right half-plane $\mathbb{C}_\alpha$, with $\alpha \geqslant 0$, then also $\hat{u}$ is defined on $\mathbb{C}_\alpha$ and

$$\widehat{\dot{u}}(s) = s\hat{u}(s) - u(0) .$$

If $u \in L^1_{\mathrm{loc}}[0,\infty)$ is such that $\hat{u}$ is defined on $\mathbb{C}_\alpha$ (where $\alpha \in \mathbb{R}$), and if $y \in L^1_{\mathrm{loc}}[0,\infty)$ is defined by $y(t) = -tu(t)$, then $\hat{y}$ is also defined on $\mathbb{C}_\alpha$ and $\hat{y}(s) = \frac{\mathrm{d}}{\mathrm{d}s}\hat{u}(s)$.

The following theorem is due to R.E.A.C. Paley and N. Wiener.

Theorem 12.4.2. *The Laplace transformation $\mathcal{L} : L^2[0,\infty) \to \mathcal{H}^2(\mathbb{C}_0)$ is unitary.*

The proof of the fact that $\mathcal{L}$ is isometric is easy: Take $u \in L^2[0,\infty)$ and for all $a > 0$ define $u_a(t) = e^{-at}u(t)$. It follows from Theorem 12.4.1 that (12.4.4) holds for $\hat{u}$, and taking limits as $a \to 0$ we obtain (by the dominated convergence theorem applied in $L^1[0,\infty)$) that $\mathcal{L}$ is isometric. To show that $\mathcal{L}$ is onto, we take $f \in \mathcal{H}^2(\mathbb{C}_0)$ and define $u(t) = \frac{1}{\sqrt{2\pi}} e^{at} (\mathcal{F}^* f_a)(t)$, where $a > 0$ and $f_a(\omega) = f(a + i\omega)$. It can be shown that $u \in L^2(\mathbb{R})$ and it is independent of a (this is the easy part). Finally, a more subtle argument shows that $u(t) = 0$ for $t < 0$. Then it is easy to see that $f = \hat{u}$. For the detailed proof see, for instance, Rudin [194, Chapter 19].

In particular, it follows from the last theorem that if $u \in L^2[0,\infty)$, then

$$\|\hat{u}\|_{\mathcal{H}^2} = \|u\|_2 .$$

This last conclusion can be derived also from the Plancherel theorem.

We shall refer to Theorem 12.4.2 as the Paley–Wiener theorem. We also need the following result, called the Paley–Wiener theorem on entire functions.

Theorem 12.4.3. *Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that the restriction of f to $\mathbb{R}$ is in $L^2(\mathbb{R})$. Suppose that there exist positive constants K and T such that*

$$|f(z)| \leqslant Ke^{T|z|} \qquad \forall\, z \in \mathbb{C}.$$

Then there exists $f \in L^2[-T,T]$ such that

$$f(s) = \int_{-T}^{T} F(t)e^{-its}\,\mathrm{d}t \qquad \forall\, s \in \mathbb{C}.$$

For a proof of this theorem we refer the reader to Rudin [194, p. 375].

The inverse Laplace transformation on $\mathcal{H}^2(\mathbb{C}_0)$ is given by

$$(\mathcal{L}^{-1}f)(t) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} e^{(a+i\omega)t} f(a+i\omega)\,\mathrm{d}\omega,$$

where $a > 0$ is arbitrary and the limit is taken in the norm of $L^2[0,\infty)$. Another way of inverting the Laplace transformation is the *Post–Widder formula.*

Theorem 12.4.4. *If $u \in L^1_{\mathrm{loc}}[0,\infty)$ is such that $\hat{u}$ exists on some right half-plane and u is continuous at a point $\tau > 0$, then (denoting $\hat{u}^{(n)} = \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^n \hat{u}$)*

$$u(\tau) = \lim_{n\to\infty} \frac{(-1)^n}{n!} \left(\frac{n}{\tau}\right)^{n+1} \hat{u}^{(n)}\left(\frac{n}{\tau}\right).$$

The proof uses the fact that the sequence of functions $\rho_n(t) = \frac{n^{n+1}}{n!}(te^{-t})^n$ converges to δ_1, the unit pulse ("delta function" or "Dirac mass") at $t = 1$. This implies that

$$u(\tau) = \lim_{n\to\infty} \int_0^\infty u(\tau t)\rho_n(t)\,\mathrm{d}t = \lim_{n\to\infty} \frac{(-1)^n}{n!} \left(\frac{n}{\tau}\right)^{n+1} \int_0^\infty u(\sigma)(-\sigma)^n e^{-\sigma\frac{n}{\tau}}\,\mathrm{d}\sigma.$$

Using the property of the Laplace transformation mentioned just before Theorem 12.4.2, we get the desired formula. For more details see, for instance, Arendt et al. [8, p. 43] or Doetsch [50, Band I, p. 290] or Widder [236, Chapter 6].

Proposition 12.4.5. *If $u \in L^1_{\mathrm{loc}}[0,\infty)$ has a Laplace transform $\hat{u}$ defined on $\mathbb{C}_\alpha$ (for some $\alpha \in \mathbb{R}$), then u is uniquely determined by $\hat{u}$.*

If u is continuous, then this statement follows from the Post–Widder formula. Now suppose that $u \in L^1_{\mathrm{loc}}[0,\infty)$ such that $\hat{u}$ exists on $\mathbb{C}_\alpha$ for some $\alpha \geqslant 0$. From what we said before Theorem 12.4.2, it follows that the function $v(t) = \int_0^t u(\sigma)\,\mathrm{d}\sigma$ is locally absolutely continuous and has a Laplace transform $\hat{v}$ defined on $\mathbb{C}_\alpha$, given by $\hat{v}(s) = \frac{1}{s}\hat{u}(s)$. Thus, $\hat{v}$ is uniquely determined by $\hat{u}$, v is uniquely determined by $\hat{v}$ and u is uniquely determined by v.

The Carleson measure theorem. For $h > 0$ and $\omega \in \mathbb{R}$ we denote

$$R(h,\omega) = \{\, s \in \mathbb{C} \mid 0 < \operatorname{Re} s \leqslant h, \ \ |\operatorname{Im} z - \omega| \leqslant h \,\}.$$

A positive measure μ on the Borel subsets of the right half-plane $\mathbb{C}_0$ is called a *Carleson measure* if there is an $M \geqslant 0$ such that

$$\mu(R(h,\omega)) \leqslant M h \qquad \forall\, h > 0,\ \omega \in \mathbb{R}. \tag{12.4.6}$$

(In some references, the rectangle $R(h,\omega)$ is replaced by a half-disk of radius h centered at $i\omega$, or with a "tent", which is a triangular area with the vertices $i\omega - ih, i\omega + ih, i\omega + h$. This leads to equivalent definitions for a Carleson measure.) For example, if Λ is the part of a straight line lying in $\mathbb{C}_0$ and μ is the one-dimensional Lebesgue measure on Λ, then μ is a Carleson measure.

Theorem 12.4.6. *If μ satisfies* (12.4.6)*, then for some $m_c < 20\sqrt{M}$ we have*

$$\int_{\mathbb{C}_0} |f|^2 \mathrm{d}\mu \leqslant m_c^2 \|f\|_{\mathcal{H}^2}^2 \qquad \forall\, f \in \mathcal{H}^2(\mathbb{C}_0).$$

The above result, obtained by Lennart Carleson in 1962, is called the *Carleson measure theorem*. It has many versions (for the disk or for the half-plane, for a two-dimensional domain as above or for an n-dimensional domain). For a proof of the above version we refer the reader to Koosis [133] or Ho and Russell [100] (whose proof is based on the proof of Duren [54] for the case of the disk). (The constant m_c given above is a bit better than in these references; see the explanations in [88, Proposition 3.2].)

If we apply Theorem 12.4.6 for $f(s) = \frac{1}{s+\lambda}$, with $\lambda \in \mathbb{C}_0$, we obtain that for any Carleson measure μ there exists $k \geqslant 0$ such that

$$\int_{\mathbb{C}_0} \frac{\mathrm{d}\mu}{|s+\lambda|^2} \leqslant \frac{k}{\operatorname{Re}\lambda} \qquad \forall\, \lambda \in \mathbb{C}_0.$$

We mention that, in fact, this estimate is equivalent to μ being a Carleson measure, and it is sometimes used as an alternative definition of a Carleson measure.

12.5 Banach space-valued L^p functions

In this section we introduce spaces of W-valued L^p functions, where W is a Banach space. Most of the results in Section 12.4 remain valid in this more general context. We introduce W-valued Sobolev spaces. Good references for this section are (in alphabetical order) Amann [5], Arendt et al. [8], Cohn [35], Diestel and Uhl [49], Hille and Phillips [97], Rosenblum and Rovnyak [193] and Yosida [239]. In this section we shall assume that the reader knows what a measurable function is, even though now we mean measurability for Banach space-valued functions (this is defined in the same way as for $\mathbb{C}$-valued functions).

Let W be a Banach space. A set $M \subset W$ is called *separable* if there is a finite or countable set $M_0 \subset W$ such that $M \subset \operatorname{clos} M_0$. Let J be an interval of non-zero length. A measurable function $f : J \to W$ is called *strongly measurable* if its range Ran $f = \{f(t) \mid t \in J\}$ is separable. A measurable $f : J \to W$ is called *simple* if Ran f is finite. It can be shown that f is strongly measurable iff there exists a sequence (f_n) of simple functions from J to W such that $\lim f_n(t) = f(t)$ for every $t \in J$. Most Banach spaces of interest to us are separable, and in this case there is no distinction between measurable and strongly measurable functions.

We denote by $\mathcal{M}(J;W)$ the space of all strongly measurable functions from J to W. A function $f \in \mathcal{M}(J,W)$ is called *Bochner integrable* if the function $t \to \|f(t)\|$ is in $\mathcal{L}^1(J)$. In this case, we define its *Bochner integral* by

$$\int_J f(t)\,\mathrm{d}t = \lim \int_J f_n(t)\,\mathrm{d}t,$$

where (f_n) is a sequence of simple functions converging to f at every point in J. The integral of a simple function is easy to define, and it can be shown that the above limit of integrals exists and it is independent of the choice of (f_n). We denote by $\mathcal{L}^1(J;W)$ the space of all Bochner integrable functions $f : J \to W$.

We state below two important theorems on the Lebesgue integral which remain valid for the Bochner integral.

Theorem 12.5.1 (dominated convergence). *Let J be an interval of non-zero length and let (f_n) be a sequence of Bochner integrable functions from J to the Banach space W. Assume that $f(t) := \lim_{n\to\infty} f_n(t)$ exists almost everywhere and that there exists an integrable function $g : J \to [0,\infty)$ such that for every $n \in \mathbb{N}$ and for almost all $t \in J$ we have $\|f_n(t)\| \leqslant g(t)$. Then f is Bochner integrable and*

$$\int_J f(t)\,\mathrm{d}t = \lim_{n\to\infty} \int_J f_n(t)\,\mathrm{d}t, \qquad \lim_{n\to\infty} \int_J \|f(t) - f_n(t)\|\,\mathrm{d}t = 0.$$

Theorem 12.5.2 (Fubini's theorem). *Let J_1 and J_2 be two intervals of non-zero lengths, let W be a Banach space and let $f : J_1 \times J_1 \to W$ be strongly measurable. Suppose that*

$$\int_{J_1}\int_{J_2} \|f(x,y)\|\,\mathrm{d}y\,\mathrm{d}x < \infty.$$

Then the repeated Bochner integrals

$$\int_{J_1}\int_{J_2} f(x,y)\,\mathrm{d}y\,\mathrm{d}x, \qquad \int_{J_2}\int_{J_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

exist and they are equal.

For the proofs of the above two theorems we refer the reader to [8, pp. 11–13].

Let J be an interval of non-zero length and let W be a Banach space. The *dual space* of W is $W' = \mathcal{L}(W,\mathbb{C})$ and its elements are called *functionals* on W. A function $f : J \to W$ is called *weakly measurable* if for every $\psi \in W'$ the function $t \mapsto \psi f(t)$ is measurable. *Pettis' theorem* states that $f : J \to W$ is strongly measurable iff it is weakly measurable and Ran f is separable. An important consequence is that if J is compact and f is continuous, then f is Bochner integrable. (This fact can be obtained also from the approximation with simple functions.) If f is Bochner integrable, then for every $\psi \in W'$ we have $\psi f \in \mathcal{L}^1(J)$ and

$$\psi \int_J f(t)\,\mathrm{d}t = \int_J \psi f(t)\,\mathrm{d}t \qquad \forall\, \psi \in W'.$$

Moreover, $\int_J f(t)\,\mathrm{d}t$ is uniquely determined by the above formula.

For J a real interval, $1 \leqslant p \leqslant \infty$ and W a Banach space, $\mathcal{L}^p(J;W)$ will denote the space of strongly measurable functions $h : J \to X$ for which $t \to \|h(t)\|$ is in $\mathcal{L}^p(J)$. For $p < \infty$ we denote $\|h\|_p = \left(\int_J \|h(t)\|^p \mathrm{d}t\right)^{\frac{1}{p}}$ (this is a seminorm). For $p = \infty$ we denote $\|h\|_\infty = \sup_{t\in J}\|h(t)\|$ (this is a norm). If $h, g \in \mathcal{L}^p(J;W)$, we declare them to be equivalent if $\int_J \|h(t) - g(t)\|\,\mathrm{d}t = 0$. As in the scalar case, $L^p(J;W)$ is defined as the resulting space of equivalence classes. For $p < \infty$ the inherited seminorm on $L^p(J;W)$ becomes a norm and the space is complete. For $L^\infty(J;W)$ we take $\|h\|_\infty$ to be the infimum of the norms of the functions from the equivalence class of h. Then $L^\infty(J;W)$ is also a Banach space. The step functions are dense in $L^p(J;W)$ for $p < \infty$. The spaces $L^p_{\mathrm{loc}}(J;W)$ are defined as in the scalar case. The Fourier transformation on $L^1(\mathbb{R};W)$, $L^2(\mathbb{R};W)$ and the Laplace transformation on $L^1_{\mathrm{loc}}([0,\infty);W)$ are also defined as in the scalar case.

Proposition 12.5.3. *Let U, Y be Banach spaces, $g \in L^1_{\mathrm{loc}}([0,\infty);\mathcal{L}(U,Y))$ and $u \in L^p_{\mathrm{loc}}([0,\infty);U)$. Define*

$$y(t) = \int_0^t g(t-\sigma)u(\sigma)\,\mathrm{d}\sigma$$

for all t for which the integral exists. Then $y \in L^p_{\mathrm{loc}}([0,\infty);Y)$. If $\alpha \in \mathbb{R}$ is such that both Laplace transforms $\hat{u}$ and $\hat{g}$ are defined on $\mathbb{C}_\alpha$, then also $\hat{y}$ is defined on $\mathbb{C}_\alpha$ and

$$\hat{y}(s) = \hat{g}(s)\cdot\hat{u}(s) \qquad \forall\, s \in \mathbb{C}_\alpha.$$

This follows from Theorems 1.9.9 and 1.10.11 in Amann [5]. However, note that in general we cannot take $g(t) = \mathbb{T}_t$, where $\mathbb{T}$ is an operator semigroup, because this g would not be strongly measurable in most cases.

Let Y be a Hilbert space. The space $\mathcal{H}^2(\mathbb{C}_0;Y)$ consists of all the analytic functions $f : \mathbb{C}_0 \to Y$ that satisfy

$$\sup_{\alpha>0} \int_{-\infty}^{\infty} \|f(\alpha+i\omega)\|^2\,\mathrm{d}\omega < \infty,$$

which is similar to (12.4.4). The norm and the boundary trace of f can be defined similarly as in $\mathcal{H}^2(\mathbb{C}_0)$. The boundary trace f^* belongs to $L^2(i\mathbb{R};Y)$. The inner product of two functions in $\mathcal{H}^2(\mathbb{C}_0;Y)$ can be defined using their boundary traces, as in the case of $\mathcal{H}^2(\mathbb{C}_0)$. With this inner product, $\mathcal{H}^2(\mathbb{C}_0;Y)$ is a Hilbert space.

We need the following proposition, which is the Paley–Wiener theorem (Theorem 12.4.2) rewritten for Hilbert space-valued functions; see also [8, p. 48].

Proposition 12.5.4. *The Laplace transformation is a unitary operator from* $L^2([0,\infty);Y)$ *to* $\mathcal{H}^2(\mathbb{C}_0;Y)$.

It follows that if $f \in \mathcal{H}^2(\mathbb{C}_0;Y)$, then f is the Laplace transform of a function $y \in L^2([0,\infty);Y)$; i.e.,

$$f(s) = \int_0^\infty e^{-st} y(t)\,\mathrm{d}t.$$

Moreover, we have

$$\int_0^\infty \|y(t)\|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^\infty \|f^*(i\omega)\|^2.$$

Sketch of the proof. The range of f is separable, hence it is contained in a subspace Y_0 with a countable orthonormal basis $\{b_k \,|\, k \in \mathbb{N}\}$. If f_k are the coordinates of f in this basis, i.e., $f_k(s) = \langle f(s), b_k\rangle$, then according to the classical Paley–Wiener theorem (Theorem 12.4.2), each f_k is the Laplace transform of a function $y_k \in L^2[0,\infty)$. The series $\sum_{k\in\mathbb{N}} y_k\, b_k$ is convergent in $L^2([0,\infty);Y_0)$, because its terms are orthogonal and the norms of its terms are square summable. The sum y of the series has f as its Laplace transform. $\square$

The space $\mathcal{H}^\infty(\mathbb{C}_0;W)$ consists of all the analytic functions $\mathbf{G} : \mathbb{C}_0 \to Z$ for which

$$\sup_{s\in\mathbb{C}_0} \|\mathbf{G}(s)\|_W < \infty.$$

The norm of $\mathbf{G}$ in this space is defined as the above expression. It is easy to see that if $f \in \mathcal{H}^2(\mathbb{C}_0;U)$ and $\mathbf{G} \in \mathcal{H}^\infty(\mathbb{C}_0;\mathcal{L}(U,Y))$, then $\mathbf{G}f \in \mathcal{H}^2(\mathbb{C}_0;Y)$. Denoting $g = \mathbf{G}f$, we have $\|g\|_{\mathcal{H}^2} \leqslant \|\mathbf{G}\|_{\mathcal{H}^\infty}\|f\|_{\mathcal{H}^2}$. This fact is often used in systems theory, where normally $f = \hat{u}$, the Laplace transform of the input signal u of a system, $\mathbf{G}$ is the transfer function of the system, and $g = \hat{y}$ where y is the output signal. The condition $\mathbf{G} \in \mathcal{H}^\infty(\mathbb{C}_0;\mathcal{L}(U,Y))$ is equivalent to the fact that if $u \in L^2([0,\infty);U)$, then $y \in L^2([0,\infty);Y)$. This property is also called input-output stability. A rational function $\mathbf{G}$ with values in $\mathbb{C}^{p\times m}$ belongs to $\mathcal{H}^\infty(\mathbb{C}_0;\mathbb{C}^{p\times m})$ iff it is proper and all its poles p satisfy $\operatorname{Re} p < 0$.

Everything we said about the inverse Laplace transformation in the previous section remains valid for Hilbert space-valued functions. In particular, the integral formula for $\mathcal{L}^{-1}$ and the Post–Widder formula remain true in this context. For a proof of the Banach space-valued version of the Post–Widder formula see [8, p. 43]. In particular, it follows that Proposition 12.4.5 can be generalized for Banach space-valued functions.

Chapter 13

Appendix II: Some Background on Sobolev Spaces

In this chapter we introduce some concepts about distributions, Sobolev spaces and differential operators acting on such spaces. For a more solid grounding the reader should consult Adams [1], Brezis [22], Dautray and Lions [42, 43], Grisvard [77], Hörmander [101], Lions and Magenes [157], Neças [176] and Zuily [246]. Starting from Section 13.5 we assume that the reader knows some basic concepts about differentiable manifolds, as can be found, for instance, in Spivak [208].

Notation. Throughout this chapter, we use the multi-index notation of Laurent Schwartz. We denote $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we set $|x| = ((x_1)^2 + (x_2)^2 + \cdots + (x_n)^2)^{\frac{1}{2}}$,

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad \alpha! = (\alpha_1!) \cdots (\alpha_n!), \qquad |\alpha| = \sum_{i=1}^{n} \alpha_j\,.$$

For $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, we set

$$\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n).$$

As in earlier chapters, we use the following notation for the bilinear product of two vectors in $\mathbb{C}^n$:

$$v \cdot w = v_1 w_1 + \cdots + v_n w_n\,.$$

In this chapter, $\Omega \subset \mathbb{R}^n$ is an open set. For any $m \in \mathbb{Z}_+$, $C^m(\Omega)$ is the space of all the functions $\varphi : \Omega \to \mathbb{C}$ for which all the partial derivatives of order $\leqslant m$ exist and are continuous. $C^0(\Omega)$ is also denoted by $C(\Omega)$. We denote by $C^\infty(\Omega)$ the intersection of all the spaces $C^m(\Omega)$ $(m \in \mathbb{N})$. There is no boundedness requirement for functions in $C^m(\Omega)$. If $\alpha \in \mathbb{Z}_+^n$ and $f \in C^m(\Omega)$ with $|\alpha| \leqslant m$, we denote

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f\,.$$

13.1 Test functions

If K is the closure of an open subset of $\mathbb{R}^n$, $m \in \mathbb{Z}_+$ or $m = \infty$, we denote by $C^m(K)$ the space of all the restrictions to K of functions in $C^m(\mathbb{R}^n)$. We denote by $C^\infty(K)$ the intersection of all the spaces $C^m(K)$ $(m \in \mathbb{N})$. If K, as above, is compact and $m < \infty$, then for any $\varphi \in C^m(K)$ we can define

$$\|\varphi\|_{C^m(K)} = \sup_{x\in K,\, |\alpha|\leqslant m} |(\partial^\alpha \varphi)(x)|. \tag{13.1.1}$$

With this norm $C^m(K)$ is a Banach space. For $\varphi \in C^m(\mathbb{R}^n)$ and K an arbitrary compact subset of $\mathbb{R}^n$, we still use the notation (13.1.1), even though for such (arbitrary compact) K, (13.1.1) usually does not define a norm on $C^m(K)$ (because $\partial^\alpha\varphi$ is not determined by the restriction $\varphi|_K$).

If $\varphi \in C(\Omega)$, the *support* of φ is the closure (in $\mathbb{R}^n$) of $\{x \in \Omega \mid \varphi(x) \neq 0\}$. The support of φ is denoted by supp φ. We denote by $\mathcal{D}(\Omega)$ the set of all $\varphi \in C^\infty(\Omega)$ which have compact support contained in Ω. These functions are called *test functions*. For a compact $K \subset \Omega$, we denote by $\mathcal{D}_K(\Omega)$ the set of all $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}\varphi \subset K$. For $p \in [1,\infty)$, we denote by $L^p(\Omega)$ the space of all the measurable functions $f : \Omega \to \mathbb{C}$ such that $\int_\Omega |f(x)|^p\,\mathrm{d}x < \infty$. We denote by $L^\infty(\Omega)$ the space of all the measurable and essentially bounded functions from Ω to $\mathbb{C}$ and by $L^1_{\mathrm{loc}}(\Omega)$ the space of all the measurable functions $f : \Omega \to \mathbb{C}$ such that $\int_K |f(x)|\mathrm{d}x < \infty$ for every compact $K \subset \Omega$. In the last three spaces, we do not distinguish between functions that are equal almost everywhere. (Two functions $f, g \in L^1_{\mathrm{loc}}(\Omega)$ are equal almost everywhere iff $\int_K |f(x) - g(x)|\mathrm{d}x = 0$ for every compact $K \subset \Omega$.) The essential supremum norm on $L^\infty(\Omega)$ is denoted by $\|\cdot\|_\infty$. The concepts used above are supposed to be known from analysis, here we are only clarifying our notation.

We have used several times in this book the existence of test functions with special properties. We give below a detailed construction of these functions. First we note that there are test functions other than the zero function.

Lemma 13.1.1. *There exists $\varphi \in \mathcal{D}(\mathbb{R}^n)$ such that*

$$\varphi(0) > 0 \qquad \textit{and} \qquad \varphi(x) \geqslant 0 \qquad \forall\, x \in \mathbb{R}^n.$$

Proof. It is not difficult to check that the function

$$f(t) = \begin{cases} 0 & \text{if} \quad t \leqslant 0, \\ e^{-\frac{1}{t}} & \text{if} \quad t > 0 \end{cases} \tag{13.1.2}$$

is of class C^∞ on $\mathbb{R}$. It follows that the function

$$\varphi(x) = f\left(1 - |x|^2\right)$$

has the required properties. $\square$

By a simple change of variables we see that, for every $\delta > 0$, the function

$$x \mapsto \varphi\left(\frac{x - x_0}{\delta}\right)$$

is non-negative, positive at x_0 and supported in the ball of radius δ centered at x_0.

Lemma 13.1.2. *There exists a non-decreasing function $\theta \in C^\infty(\mathbb{R})$ such that*

$$\theta(x) = \begin{cases} 0 & \textit{if} \quad x \leqslant 0, \\ 1 & \textit{if} \quad x \geqslant 1. \end{cases}$$

Proof. We consider again the function in $C^\infty(\mathbb{R})$ defined by (13.1.2). We clearly have that $\operatorname{supp}(f) = [0, \infty)$ and $0 \leqslant f(x) \leqslant 1$ for every $x \in \mathbb{R}$. Define $g(x) = f(x)f(1-x)$ and $G(x) = \int_0^x g(t)dt$. Then $0 \leqslant g(x) \leqslant 1$ for $x \in \mathbb{R}$ and $\operatorname{supp}(g) \subset [0,1]$. Moreover, $g \neq 0$ since $g(\frac{1}{2}) = \left[f\left(\frac{1}{2}\right)\right]^2 \neq 0$. The function $\theta(x) = \frac{G(x)}{G(1)}$ is thus in $C^\infty(\mathbb{R})$, it is non-decreasing and it satisfies

$$\theta(x) = \begin{cases} 0 & \text{if} \quad x \leqslant 0, \\ 1 & \text{if} \quad x \geqslant 1. \end{cases}$$ □

Proposition 13.1.3. *Let $a < c < d < b$ be real numbers. Then there exists $\rho \in \mathcal{D}(\mathbb{R})$ such that*

(1) $\rho(x) = 1$ *for every* $x \in [c, d]$;
(2) $\operatorname{supp} \rho \subset (a, b)$;
(3) $0 \leqslant \rho(x) \leqslant 1$ *for every* $x \in \mathbb{R}$.

Proof. Set $\rho(x) = \theta\left(\frac{x-a}{c-a}\right)\theta\left(\frac{b-x}{b-d}\right)$, where θ is the function constructed in Lemma 13.1.2. It can be easily checked that ρ has the required properties. □

Corollary 13.1.4. *Let $0 < r < R$ and let $n \in \mathbb{N}$. Then there exists $\widetilde{\rho} \in C^\infty(\mathbb{R}^n)$ such that $\widetilde{\rho}(x) = 1$ if $\|x\| < r$ and $\widetilde{\rho}(x) = 0$ if $\|x\| > R$.*

Proof. We take $\widetilde{\rho}(x) = \rho\left(\|x\|^2\right)$ where ρ is the function in Proposition 13.1.3, with

$$-a = b = R^2 \quad \text{and} \quad -c = d = r^2.$$ □

For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball centered at x and of radius r. For K a compact subset in $\mathbb{R}^n$ and for $\varepsilon > 0$, we denote

$$K_\varepsilon = K + B(0, \varepsilon) = \bigcup_{x \in K} B(x, \varepsilon).$$

Proposition 13.1.5. *Let K be a compact subset of $\mathbb{R}^n$. Then for every $\varepsilon > 0$, there exists $\varphi \in \mathcal{D}(K_{2\varepsilon})$ such that $\varphi(x) = 1$ for $x \in K_\varepsilon$ and $0 \leqslant \varphi(x) \leqslant 1$ for all $x \in \mathbb{R}^n$.*

Proof. Since clos K_ε is compact, there exist $x_1, \dots, x_p \in K$ such that

$$\text{clos } K_\varepsilon \subset \bigcup_{j=1}^{p} B\left(x_j, \frac{4\varepsilon}{3}\right).$$

According to Corollary 13.1.4, for each $j \in \{1, \dots, p\}$ there exists a function $\varphi_j \in \mathcal{D}\left(B\left(x_j, \frac{5\varepsilon}{3}\right)\right)$ such that $\varphi_j(x) \geqslant 0$ for every $x \in \mathbb{R}^n$ and $\phi_j(x) = 1$ for $x \in B\left(x_j, \frac{4\varepsilon}{3}\right)$. Let $\phi(x) = \sum_{j=1}^N \varphi_j(x)$. We have $\phi(x) \geqslant 1$ for all $x \in \bigcup_{j=1}^p B(x_j, \varepsilon)$. On the other hand, since

$$\text{clos } \bigcup_{j=1}^{p} B\left(x_j, \frac{5\varepsilon}{3}\right) \subset K_{2\varepsilon},$$

we have that $\phi \in \mathcal{D}(K_{2\varepsilon})$. Let $\theta \in C^\infty(\mathbb{R})$ be the function from Lemma 13.1.2. It is easy to see that the function $\varphi(x) = \theta(\phi(x))$ satisfies the required conditions. □

Proposition 13.1.6. *Suppose that $K \subset \mathbb{R}^n$ is compact and let $D_1, \dots, D_N$ be open sets such that $K \subset \bigcup_{k=1}^N D_k$. Then there exist functions $\varphi_k \subset \mathcal{D}(D_k)$ ($k = 1, \dots, N$) such that $\varphi_k \geqslant 0$ and $\sum_{k=1}^N \varphi_k(x) = 1$ for every x in an open neighborhood of K.*

The family of functions $\varphi_1, \dots, \varphi_N$ in the above proposition is called a *partition of unity* subordinated to the compact K and to its covering $D_1, \dots, D_N$.

In order to prove Proposition 13.1.6, we need the following lemma.

Lemma 13.1.7. *Let $K \subset \mathbb{R}^n$ be compact and let $(U_j)_{j\in\{1,\dots,N\}}$ be open sets covering K. Then there exist compact sets $(K_j)_{j\in\{1,\dots,N\}}$ such that $K_j \subset U_j$ for all $j \in \{1, \dots, N\}$ and*

$$K = \bigcup_{j=1}^{N} K_j. \tag{13.1.3}$$

Proof. For $x \in K$ let $r_x > 0$ be such that clos $B(x, r_x) \subset \bigcap_{x\in D_j} D_j$. Then

$$K \subset \bigcup_{x\in K} B(x, r_x),$$

so that there exist $x_1, \dots, x_M \in K$ such that

$$K \subset \bigcup_{i=1}^{M} B(x_i, r_{x_i}).$$

Denote

$$K_j = K \bigcap \left(\bigcup_{\text{clos } B\left(x_i, r_{x_i}\right) \subset D_j} \text{clos } B(x_i, r_{x_i}) \right).$$

Then clearly K_j is a compact set contained in K and $K_j \subset D_j$. We still have to check (13.1.3). Let $x \in K$, then there exists $i \in \{1, \dots, M\}$ such that $x \in B(x_i, r_{x_i})$. On the other hand, there exists $j_0 \in \{1, \dots, N\}$ such that $x_i \in D_{j_0}$, so that clos $B(x_i, r_{x_i}) \subset D_{j_0}$. It follows that $x \in K_{j_0}$. $\square$

We are now in a position to prove the existence of the partition of unity.

Proof of Proposition 13.1.6. According to Lemma 13.1.7, there exist compacts $(K_j)_{j\in\{1,\dots,N\}}$ such that $K_j \subset D_j$, for all $j \in \{1, \dots, N\}$, and $K = \bigcup_{i=1}^N K_j$. Moreover, by applying Proposition 13.1.5, it follows that for $j \in \{1, \dots, N\}$ there exists $\psi_j \in \mathcal{D}(D_j)$ with $\psi_j(x) \in [0, 1]$, for all $x \in \mathbb{R}^n$ and $\psi_j(x) = 1$ for $x \in K_j$. Let

$$V = \left\{ x \in \bigcup_{j=1}^N D_j \;\middle|\; \sum_{j=1}^N \psi_j(x) > 0 \right\}.$$

Then $K \subset V$ and V is an open set. According to Proposition 13.1.5, there exists $\eta \in \mathcal{D}(V)$ such that $\eta(x) \in [0, 1]$ for all $x \in \mathbb{R}^n$ and $\eta = 1$ on an open set W such that $K \subset W \subset V$. Define

$$\varphi_j = \frac{\psi_j}{(1-\eta) + \sum_{k=1}^N \psi_k}. \tag{13.1.4}$$

Then $\varphi_j \in \mathcal{D}(U_j)$, since the denominator of the expression on the right-hand side of (13.1.4) is positive on V and it equals 1 outside V. Since $\eta = 1$ on W, (13.1.4) implies that $\sum_{j=1}^N \varphi_j(x) = 1$ for all $x \in W$. $\square$

Corollary 13.1.8. *Let K_1 and K_2 be two compact disjoint subsets of the open set $\Omega \subset \mathbb{R}^n$. Then there exists a function $\varphi \in \mathcal{D}(\Omega)$ such that*

$$\varphi(x) = \begin{cases} 1 & \textit{if} \quad x \in K_1, \\ -1 & \textit{if} \quad x \in K_2, \end{cases}$$

and $|\varphi(x)| \leqslant 1$ for all $x \in \Omega$.

Proof. Let U_1 and U_2 be two open subsets of Ω such that $K_1 \subset U_1$, $K_2 \subset U_2$, $U_1 \cap U_2 = \emptyset$. According to Proposition 13.1.5, there exist $\varphi_1, \varphi_2 \in \mathcal{D}(\Omega)$ such that $\varphi_i(x) = 1$ for $x \in K_i$, $\varphi_i \in \mathcal{D}(U_i)$, $i \in \{1, 2\}$, and $0 \leqslant \varphi_i(x) \leqslant 1$, $i \in \{1, 2\}$. The function φ, defined by

$$\varphi(x) = \varphi_1(x) - \varphi_2(x) \qquad \forall\, x \in \Omega,$$

clearly has the required properties. $\square$

We end this section by a result showing that there are "a lot" of test functions.

Proposition 13.1.9. *For every $p \in [1, \infty)$ we have that $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.*

For the proof of the above result we refer the reader to Adams [1, p. 31].

13.2 Distributions on a domain

If $u : \mathcal{D}(\Omega) \to \mathbb{C}$ is linear, then the action of u on a test function $\varphi \in \mathcal{D}(\Omega)$ is denoted by $\langle u, \varphi\rangle$. We adopt a bilinear convention: $\langle u, \varphi\rangle$ is linear in both components (unlike the pairing of a Hilbert space with its dual).

Definition 13.2.1. A *distribution* on Ω is a linear map $u : \mathcal{D}(\Omega) \to \mathbb{C}$ which satisfies the following continuity assumption: For every compact $K \subset \Omega$ there are an $m \in \mathbb{Z}_+$ and a constant $C \geqslant 0$ (both may depend on K) such that

$$|\langle u, \varphi\rangle| \leqslant C\|\varphi\|_{C^m(K)} \qquad \forall\, \varphi \in \mathcal{D}_K(\Omega). \tag{13.2.1}$$

The set of all distributions on Ω is denoted by $\mathcal{D}'(\Omega)$ and clearly this is a vector space. If, for some $u \in \mathcal{D}'(\Omega)$, the constant m in (13.2.1) can be chosen independently of K, then the smallest such integer m is called the *order* of u.

If $f \in L^1_{\rm loc}(\Omega)$, then we can define $u_f : \mathcal{D}(\Omega) \to \mathbb{C}$ by

$$\langle u_f, \varphi\rangle = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x \qquad \forall\, \varphi \in \mathcal{D}(\Omega).$$

Then $u_f \in \mathcal{D}'(\Omega)$ and it is of order zero. Indeed, for every compact $K \subset \Omega$ we have

$$|\langle u_f, \varphi\rangle| \leqslant \left[\int_K |f(x)|\,\mathrm{d}x\right] \|\varphi\|_{C(K)} \qquad \forall\, \varphi \in \mathcal{D}(\Omega).$$

Such distributions are called *regular*.

Proposition 13.2.2. *If $f \in L^1_{\rm loc}(\Omega)$ is such that $u_f = 0$, then $f(x) = 0$ for almost every $x \in \Omega$.*

Proof. We have to show that if $f \in L^1_{\rm loc}(\Omega)$ is such that

$$\int_\Omega f\varphi\,\mathrm{d}x = 0 \qquad \forall\, \varphi \in \mathcal{D}(\Omega), \tag{13.2.2}$$

then $f(x) = 0$ almost everywhere in Ω. First we assume that $f \in L^1(\Omega)$ and that Ω is bounded. According to Proposition 13.1.9, for every $\varepsilon > 0$, there exists $f_1 \in \mathcal{D}(\Omega)$ such that $\|f - f_1\|_{L^1(\Omega)} < \varepsilon$. Using (13.2.2) we have

$$\left|\int_\Omega f_1\varphi\,\mathrm{d}x\right| \leqslant \varepsilon\|\varphi\|_{L^\infty(\Omega)} \qquad \forall\, \varphi \in \mathcal{D}(\Omega). \tag{13.2.3}$$

Let

$$K_1 = \{x \in \Omega \mid h_1(x) \geqslant \varepsilon\}, \quad K_2 = \{x \in \Omega \mid h_1(x) \leqslant -\varepsilon\}.$$

Since K_1 and K_2 are compact sets and $K_1 \cap K_2 = \emptyset$, by applying Corollary 13.1.8 we obtain the existence of a function $\varphi_0 \in \mathcal{D}(\Omega)$ such that

$$\varphi_0(x) = \begin{cases} 1 & \text{if} \quad x \in K_1, \\ -1 & \text{if} \quad x \in K_2, \end{cases}$$

and $|\varphi_0(x)| \leqslant 1$ for all $x \in \Omega$. Putting $K = K_1 \cup K_2$ it follows that

$$\int_\Omega f_1\varphi_0 \mathrm{d}x = \int_{\Omega\setminus K} f_1\varphi_0 + \int_K f_1\varphi_0\,,$$

so that, thanks to (13.2.3), we have

$$\int_K |f_1|\mathrm{d}x = \int_K f_1\varphi_0\mathrm{d}x \leqslant \varepsilon + \int_{\Omega\setminus K} |f_1|\mathrm{d}x\,.$$

Consequently, denoting the Lebesgue measure of Ω by $\mu(\Omega)$, we see that

$$\int_\Omega |f_1|\mathrm{d}x = \int_K |f_1| + \int_{\Omega\setminus K} |f_1|\mathrm{d}x \leqslant \varepsilon + 2\int_{\Omega\setminus K} |f_1|\mathrm{d}x \leqslant \varepsilon + 2\varepsilon\mu(\Omega)\,,$$

since $|f_1| \leqslant \varepsilon$ on $\Omega \setminus K$. Thus

$$\|f\|_{L^1(\Omega)} \leqslant \|f - f_1\|_{L^1(\Omega)} + \|f_1\|_{L^1(\Omega)} \leqslant 2\varepsilon + 2\varepsilon\mu(\Omega)\,.$$

Since this holds for all $\varepsilon > 0$, we conclude that $f = 0$ almost everywhere on Ω.

Let Ω be an arbitrary open set in $\mathbb{R}^n$. Then $\Omega = \bigcup_{k\in\mathbb{N}} \Omega_k$ with Ω_k open, clos Ω_k compact, clos $\Omega_k \subset \Omega$. Indeed, we may take, for instance,

$$\Omega_k = \left\{ x \in \Omega \;\middle|\; d(x, \mathbb{R}^n \setminus \Omega) > \frac{1}{k} \text{ and } |x| < k \right\}.$$

Here, $d(x, M)$ denotes the distance from the point $x \in \mathbb{R}^n$ to the set $M \subset \mathbb{R}^n$. By applying the result for bounded Ω proved earlier, with Ω_k in place of Ω and with the corresponding restriction of f, we obtain that $f = 0$ almost everywhere on Ω_k, so that $f = 0$ almost everywhere on Ω. $\square$

Due to the above proposition, we may regard $L^1_{\mathrm{loc}}(\Omega)$ as a subspace of $\mathcal{D}'(\Omega)$. In this sense, distributions are generalizations of L^1_{loc} functions and are sometimes called *generalized functions*. When u is a distribution on Ω then by $u \in L^2(\Omega)$ we mean that u is regular and it is represented by a function in $L^2(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$. The meaning of $u \in L^\infty(\Omega)$, $u \in C^m(\Omega)$, etc. is similar.

Example 13.2.3. For $a \in \Omega$ we consider the linear map $\delta_a : \mathcal{D}(\Omega) \to \mathbb{C}$ defined by

$$\langle \delta_a, \varphi\rangle = \varphi(a) \qquad \forall\, \varphi \in \mathcal{D}(\Omega)\,.$$

Then for every compact $K \subset \Omega$ we have

$$|\langle \delta_a, \varphi\rangle| \leqslant \|\varphi\|_{C(K)} \qquad \forall\, \varphi \in \mathcal{D}(\Omega)\,.$$

Thus, δ_a is a distribution of order zero on Ω, called the *Dirac mass* at a. This distribution is not regular. Indeed, suppose that there exists $f \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\langle \delta_a, \varphi\rangle = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x = \varphi(a) \qquad \forall\, \varphi \in \mathcal{D}(\Omega)\,. \tag{13.2.4}$$

Denote $\Omega_a = \Omega \setminus \{a\}$. Then $\langle \delta_a, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega_a)$. As remarked a little earlier, this implies that $f(x) = 0$ almost everywhere in Ω_a and thus almost everywhere in Ω. Consequently, $\int_\Omega f(x)\varphi(x)\,\mathrm{d}x = 0$ for all $\varphi \in \mathcal{D}(\Omega)$, which contradicts (13.2.4).

There is no good way to define a norm, or even a distance, on the spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$. However, convergent sequences can be defined as follows.

Definition 13.2.4. The sequence (φ_k) with terms in $\mathcal{D}(\Omega)$ *converges* to $\varphi \in \mathcal{D}(\Omega)$ if there exists a compact $K \subset \Omega$ such that

1. $\operatorname{supp} \varphi_k \subset K$ for all $k \in \mathbb{N}$ and $\operatorname{supp} \varphi \subset K$,
2. for all $m \in \mathbb{Z}_+$ we have $\lim_{k \to \infty} \|\varphi_k - \varphi\|_{C^m(K)} = 0$.

The sequence (u_k) in $\mathcal{D}'(\Omega)$ *converges* to $u \in \mathcal{D}'(\Omega)$ if

$$\lim_{k \to \infty} \langle u_k, \varphi \rangle = \langle u, \varphi \rangle \qquad \forall\, \varphi \in \mathcal{D}(\Omega).$$

It is easy to see that a sequence in $\mathcal{D}(\Omega)$ or in $\mathcal{D}'(\Omega)$ cannot converge to two different limits. It is also easy to see that the sum of two convergent sequences (in one of the above spaces) is convergent to the sum of their limits.

Remark 13.2.5. Let $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$. If the sequence (u_k) in $L^p(\Omega)$ is such that $u_k \to u_0$ in $L^p(\Omega)$, then $u_k \to u_0$ also in $\mathcal{D}'(\Omega)$. Indeed,

$$\left| \int_\Omega u_0(x)\varphi(x)\,\mathrm{d}x - \int_\Omega u_k(x)\varphi(x)\,\mathrm{d}x \right| \leqslant \|u_0 - u_k\|_{L^p(\Omega)} \|\varphi\|_{L^q(\Omega)}$$

for all $\varphi \in \mathcal{D}(\Omega)$, which clearly implies that $u_k \to u_0$ in $\mathcal{D}'(\Omega)$.

Definition 13.2.6. Let $u \in \mathcal{D}'(\Omega)$ and let $j \in \{1, \dots, n\}$. The *partial derivative* of u with respect to x_j, denoted $\frac{\partial u}{\partial x_j}$, is the distribution defined by

$$\left\langle \frac{\partial u}{\partial x_j}, \varphi \right\rangle = -\left\langle u, \frac{\partial \varphi}{\partial x_j} \right\rangle \qquad \forall\, \varphi \in \mathcal{D}(\Omega).$$

It is easy to check that indeed the above formula defines a new distribution in $\mathcal{D}'(\Omega)$. Moreover, if $u \in C^1(\Omega)$, then its partial derivatives in $\mathcal{D}'(\Omega)$ coincide with its usual partial derivatives. Higher order partial derivatives are defined recursively in the obvious way. It is easy to check that, for all $\alpha \in \mathbb{Z}_+^n$,

$$\langle \partial^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle \qquad \forall\, \varphi \in \mathcal{D}(\Omega). \tag{13.2.5}$$

Example 13.2.7. Let $H \in L^\infty(\mathbb{R})$ be the *Heaviside function*, which is the characteristic function of the interval $[0, \infty)$. Then $\partial^1 H = \frac{\mathrm{d}H}{\mathrm{d}x} = \delta_0$ in $\mathcal{D}'(\mathbb{R})$, since

$$\left\langle \frac{\mathrm{d}H}{\mathrm{d}x}, \varphi \right\rangle = -\left\langle H, \frac{\mathrm{d}\varphi}{\mathrm{d}x} \right\rangle = -\int_0^\infty \frac{\mathrm{d}\varphi}{\mathrm{d}x}\,\mathrm{d}x = \varphi(0) \qquad \forall\, \varphi \in \mathcal{D}(\mathbb{R}).$$

Example 13.2.8. Let $u \in L^1(\mathbb{R})$ be given by $u(x) = \log|x|$. The derivative $\partial^1 u$ of this (regular) distribution is denoted by PV $\frac{1}{x}$ and it is given for all $\varphi \in \mathcal{D}(\mathbb{R})$ by

$$\left\langle \mathrm{PV}\,\frac{1}{x}, \varphi \right\rangle = \lim_{\varepsilon \to 0,\ \varepsilon > 0} \int_{|x|>\varepsilon} \varphi(x)\,\frac{\mathrm{d}x}{x} = \int_{-R}^{R} [\varphi(x) - \varphi(0)]\,\frac{\mathrm{d}x}{x},$$

where $R > 0$ is such that $\operatorname{supp}\varphi \subset [-R, R]$. (PV stands for "principal value".) Note that in the last integral we are integrating a continuous function on $[-R, R]$. The distribution PV $\frac{1}{x}$ is not regular. However, its restriction to $\Omega_0 = \{x \in \mathbb{R} \mid x \neq 0\}$ is regular and it is represented by the function $\frac{\mathrm{d}}{\mathrm{d}x}u(x) = \frac{1}{x}$.

Proposition 13.2.9. *Let (u_k) be a sequence in $\mathcal{D}'(\Omega)$ such that $u_k \to u$ in $\mathcal{D}'(\Omega)$. Then for every multi-index $\alpha \in \mathbb{Z}_+^n$ we have that $\partial^\alpha u_n \to \partial^\alpha u$ in $\mathcal{D}'(\Omega)$.*

Proof. For any $\varphi \in \mathcal{D}(\Omega)$ we have

$$\lim_{n\to\infty} \langle \partial^\alpha u_n, \varphi\rangle = (-1)^{|\alpha|} \lim_{n\to\infty} \langle u_n, \partial^\alpha\varphi\rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha\varphi\rangle = \langle \partial^\alpha u, \varphi\rangle . \qquad \square$$

For $f \in C^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$, the product $fu \in \mathcal{D}'(\Omega)$ is defined by

$$\langle fu, \varphi\rangle = \langle u, f\varphi\rangle \qquad \forall\, \varphi \in \mathcal{D}(\Omega).$$

It is easy to check that this formula defines indeed a distribution. The following version of *Leibniz' formula* holds (as it is easy to verify):

$$\frac{\partial}{\partial x_k}(fu) = \frac{\partial f}{\partial x_k}\,u + f\,\frac{\partial u}{\partial x_k} \qquad \forall\, k \in \{1, \dots, n\}.$$

For $u \in \mathcal{D}'(\Omega)$ and $\mathcal{O}$ an open subset of Ω, the *restriction* of u to $\mathcal{O}$, denoted by $u|_{\mathcal{O}} \in \mathcal{D}'(\mathcal{O})$, is defined by

$$\langle u|_{\mathcal{O}}, \varphi\rangle_{\mathcal{D}'(\mathcal{O}),\mathcal{D}(\mathcal{O})} = \langle u, \varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \qquad \forall\, \varphi \in \mathcal{D}(\mathcal{O}).$$

Proposition 13.2.10. *Let $\mathcal{I}$ be an arbitrary index set and suppose that $\Omega = \bigcup_{j\in\mathcal{I}} D_j$, where each D_j is open. If $u \in \mathcal{D}'(\Omega)$ is such that $u|_{D_j} = 0$ for all $j \in \mathcal{I}$, then $u = 0$.*

Proof. Let $\eta \in \mathcal{D}(\Omega)$. Since $\operatorname{supp}\eta$ is compact, there exists a finite index set $\mathcal{F} \subset \mathcal{I}$ such that $\operatorname{supp}\eta \subset \bigcup_{j\in\mathcal{F}} D_j$. Let ϕ_j $(j \in \mathcal{F})$ be a partition of unity subordinated to $\operatorname{supp}\eta$ (see Proposition 13.1.6) and to its covering D_j $(j \in \mathcal{F})$. We have that $\eta = \sum_{j\in\mathcal{F}} \phi_j\eta$, with $\phi_j\eta \in \mathcal{D}(D_j)$. Thus,

$$\langle u, \eta\rangle = \sum_{j\in\mathcal{F}} \langle u, \phi_j\eta\rangle = \sum_{j\in\mathcal{F}} \left\langle u|_{D_j}, \phi_j\eta\right\rangle = 0. \qquad \square$$

It follows from the last proposition that for any $u \in \mathcal{D}'(\Omega)$, the union $\mathcal{O}$ of all the open sets $D \subset \Omega$, such that $u|_D = 0$, has again the property $u|_{\mathcal{O}} = 0$. The complement of $\mathcal{O}$ (in Ω) is called the *support* of u and it is denoted by $\operatorname{supp} u$.

13.3 The operators div, grad, rot and Δ

Let Ω be an open connected set in $\mathbb{R}^n$. Distributions with values in $\mathbb{C}^m$ ($m \in \mathbb{N}$) and spaces of such distributions are defined componentwise in the obvious way. The notation $\mathcal{D}'(\Omega, \mathbb{C}^m)$ is used for such distributions. The differential operators

$$\operatorname{div} : \mathcal{D}'(\Omega; \mathbb{R}^n) \to \mathcal{D}'(\Omega), \qquad \operatorname{grad} : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega; \mathbb{R}^n)$$

are defined by

$$\operatorname{div} v = \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n}, \qquad \operatorname{grad} \psi = \left(\frac{\partial \psi}{\partial x_1}, \ldots, \frac{\partial \psi}{\partial x_n} \right).$$

For $n = 3$, we also introduce the operator

$$\operatorname{rot} : \mathcal{D}'(\Omega; \mathbb{C}^3) \to \mathcal{D}'(\Omega; \mathbb{C}^3)$$

by

$$(\operatorname{rot} v)_j = \frac{\partial v_l}{\partial x_k} - \frac{\partial v_k}{\partial x_l}, \qquad (j, k, l) \in \{ (1, 2, 3), (2, 3, 1), (3, 1, 2) \}.$$

A non-rigorous but useful way of thinking of these operators is to introduce the "vector"

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)$$

and do computations with it as if it were a vector in $\mathbb{R}^n$. Then formally $\operatorname{grad} \psi = \nabla \psi$ (as if we would multiply a vector with a scalar), $\operatorname{div} v = \nabla \cdot v$ (as if we would take the bilinear product of two vectors). For $n = 3$ we have $\operatorname{rot} v = \nabla \times v$ (as if we would take the vector product of two vectors).

The following identities are easily verified by direct computation:

$$\operatorname{rot} \operatorname{grad} = 0, \qquad \operatorname{div} \operatorname{rot} = 0.$$

According to Leibniz' formula, for $\varphi \in C^\infty(\Omega)$, $\psi \in \mathcal{D}'(\Omega)$ and $v \in \mathcal{D}'(\Omega; \mathbb{C}^n)$,

$$\operatorname{div} (\varphi v) = (\operatorname{grad} \varphi) \cdot v + \varphi \operatorname{div} v, \tag{13.3.1}$$

$$\operatorname{grad} (\varphi \psi) = \psi (\operatorname{grad} \varphi) + \varphi (\operatorname{grad} \psi). \tag{13.3.2}$$

If $n = 3$, $q \in \mathcal{D}(\Omega; \mathbb{C}^3)$ and $v \in \mathcal{D}'(\Omega; \mathbb{C}^3)$, then

$$\operatorname{div} (q \times v) = \operatorname{rot} q \cdot v - q \cdot \operatorname{rot} v.$$

We denote $\Delta = \operatorname{div} \operatorname{grad}$, which is called the *Laplacian*. (In the formal calculus mentioned earlier, $\Delta = \nabla \cdot \nabla$.) Thus, according to Definition 13.2.6, for every distribution $\psi \in \mathcal{D}'(\Omega)$ we have

$$\langle \Delta \psi, \varphi \rangle = - \langle \nabla \psi, \nabla \varphi \rangle = \langle \psi, \Delta \varphi \rangle \qquad \forall\, \varphi \in \mathcal{D}(\Omega). \tag{13.3.3}$$

The operator Δ can be applied also to vector-valued distributions, acting componentwise. It is easy to check that

$$\operatorname{rot}\operatorname{rot} = \operatorname{grad}\operatorname{div} - \Delta .$$

If $v \in \mathcal{D}'(\Omega;\mathbb{C}^n)$ and $\psi \in \mathcal{D}(\Omega;\mathbb{C}^n)$ then we denote $\langle v, \psi\rangle = \sum_{k=1}^n \langle v_k, \psi_k\rangle = \langle \psi, v\rangle$. It is easy to verify that

$$\langle \operatorname{div} v, \varphi\rangle = -\langle v, \operatorname{grad}\varphi\rangle \quad \forall\, v \in \mathcal{D}'(\Omega;\mathbb{C}^n), \quad \varphi \in \mathcal{D}(\Omega), \tag{13.3.4}$$

$$\langle \operatorname{rot} v, \psi\rangle = \langle v, \operatorname{rot}\psi\rangle \qquad \forall\, v \in \mathcal{D}'(\Omega;\mathbb{C}^3), \quad \varphi \in \mathcal{D}(\Omega;\mathbb{C}^3),$$

$$\Delta(\varphi\psi) = (\Delta\varphi)\psi + 2\langle\nabla\varphi, \nabla\psi\rangle + \varphi(\Delta\psi) \qquad \forall\, \psi \in \mathcal{D}'(\Omega), \qquad \varphi \in \mathcal{D}(\Omega). \tag{13.3.5}$$

Remark 13.3.1. If we take $\Omega = \mathbb{R}^n \setminus \{0\}$, then for every $q \in \mathbb{R}$, the function $f(x) = |x|^q$ defines a regular distribution on Ω and $\operatorname{grad} f = q|x|^{q-2}x$. Using (13.3.1) we obtain $\Delta f = q\operatorname{grad}(|x|^{q-2}) \cdot x + q|x|^{q-2}\operatorname{div} x$, whence

$$\Delta|x|^q = q\operatorname{div}(|x|^{q-2}x) = q(q+n-2)|x|^{q-2} . \tag{13.3.6}$$

If we include also the point zero, i.e., if $\Omega = \mathbb{R}^n$, then the computation becomes more interesting. We compute $\Delta|x|^q$ for $q = 2-n$ and $\Omega = \mathbb{R}^n$ in Example 13.7.5.

In the remaining part of this section we give a result showing that partial derivatives in $\mathcal{D}'(\Omega)$ preserve an important property of classical partial derivatives.

Theorem 13.3.2. *Suppose that Ω is connected and that $u \in \mathcal{D}'(\Omega)$ is such that* $\operatorname{grad} u = 0$*. Then u is a constant function.*

For the proof of this theorem we need the following lemma.

Lemma 13.3.3. *Let $\eta \in \mathcal{D}(\mathcal{R})$, where $\mathcal{R}$ is an n-dimensional open hypercube. Then the following conditions are equivalent:*

(1) $\int_{\mathcal{R}} \eta(x)\mathrm{d}x = 0$.
(2) *There exists $\psi \in \mathcal{D}(\mathcal{R};\mathbb{C}^n)$ such that* $\eta = \operatorname{div}\psi$*.*

Proof. The fact that (2) implies (1) can be checked by simple integration by parts.

We show by induction that (1) implies (2). Without loss of generality, we may assume that $\mathcal{R} = \mathcal{R}_n = (-R, R)^n$ for some $R > 0$. It is easy to check that the implication (1) $\Rightarrow$ (2) holds for $n = 1$. Assume that $k \geqslant 2$ and that the implication holds for all $n \leqslant k-1$. Consider the function f defined by

$$f(x_1, \ldots, x_{k-1}) = \int_{-R}^{R} \eta(x_1, \ldots, x_{k-1}, y)\,\mathrm{d}y . \tag{13.3.7}$$

Then $\operatorname{supp} f \in \mathcal{D}\left(\mathcal{R}_{k-1}\right)$. Moreover, by applying Fubini's theorem, we obtain that $\int_{\mathcal{R}_{k-1}} f(x)\mathrm{d}x = 0$ so that there exist $g_1, \ldots, g_{k-1} \in \mathcal{D}(\mathcal{R}_{k-1})$ with

$$f = \sum_{j=1}^{k-1} \frac{\partial g_j}{\partial x_j}. \tag{13.3.8}$$

Let $\rho \in \mathcal{D}(\mathcal{R}_1)$ satisfying $\int_{-R}^{R} \rho(t)\mathrm{d}t = 1$ and consider the function $\widetilde{\eta}$ defined by

$$\widetilde{\eta}(x) = \eta(x) - \sum_{j=1}^{k-1} \frac{\partial g_j}{\partial x_j}\left(x_1, \ldots, x_{k-1}\right) \rho(x_k) \qquad \forall\, x \in \mathcal{R}_k. \tag{13.3.9}$$

The above relation and (13.3.8) imply that

$$\widetilde{\eta}(x) = \eta(x) - f(x_1, \ldots, x_{k-1})\rho(x_k).$$

This, combined with (13.3.7) and with the fact that $\int_{-R}^{R} \rho(t)\,\mathrm{d}t = 1$, implies that

$$\operatorname{supp} \eta \subset R_k, \quad \int_{-R}^{R} \widetilde{\eta}(x_1, \ldots, x_{k-1}, t)\,\mathrm{d}t = 0 \qquad \forall\, (x_1, \ldots, x_{k-1}) \in \mathcal{R}_{k-1}. \tag{13.3.10}$$

Denote

$$\psi_j\left(x_1, \ldots, x_k\right) = g_j\left(x_1, \ldots, x_{k-1}\right) \rho\left(x_k\right), \qquad \forall\, j \in \{1, \ldots, k-1\}.$$

We have that $\psi_j \in \mathcal{D}(\mathcal{R}_k)$ and from (13.3.9) and the last formula it follows that, for all $j \in \{1, \ldots, k-1\}$,

$$\widetilde{\eta} = \eta - \sum_{j=1}^{k-1} \frac{\partial \psi_j}{\partial x_j}. \tag{13.3.11}$$

This, combined with (13.3.10), implies that the function

$$\psi_k(x) = \int_{-R}^{x_k} \widetilde{\eta}(x_1, \ldots, x_{k-1}, t)\,\mathrm{d}t$$

satisfies the conditions $\psi_k \in \mathcal{D}(\mathcal{R}_k)$ and $\frac{\partial \psi_k}{\partial x_k} = \widetilde{\eta}$. These facts, combined with (13.3.11), imply that $\eta = \operatorname{div} \psi$. $\square$

Proof of Theorem 13.3.2. As a first step we suppose that $\mathcal{R} \subset \Omega$ is an open hypercube and we show that the restriction of u to $\mathcal{R}$ is a constant function. Let $\theta \in \mathcal{D}(\mathcal{R})$ be such that $\int_{\mathcal{R}} \theta(x)\mathrm{d}x = 1$ and let $\varphi \in \mathcal{D}(\mathcal{R})$. The function $\eta(x) = \varphi(x) - \left[\int_{\mathcal{R}} \varphi(x)\mathrm{d}x\right] \theta(x)$ is in $\mathcal{D}(\mathcal{R})$ and $\int_{\mathcal{R}} \eta(x)\mathrm{d}x = 0$. According to Lemma 13.3.3, there exists $\psi \in \mathcal{D}(\mathcal{R}; \mathbb{C}^n)$ such that

$$\operatorname{div} \psi = \varphi - \left[\int_{\mathcal{O}} \varphi(x)\mathrm{d}x\right] \theta.$$

By applying u to the above formula it follows that

$$\langle u, \varphi \rangle = \left[\int_{\mathcal{R}} \varphi(x) \mathrm{d}x \right] \langle u, \theta \rangle + \langle u, \operatorname{div} \psi \rangle \,.$$

Using (13.3.4) and the fact that $\operatorname{grad} u = 0$, it follows that the last term on the right-hand side above vanishes. Denoting by C the constant $\langle u, \theta \rangle$, it follows that

$$\langle u, \varphi \rangle = C \int_{\mathcal{R}} \varphi(x) \mathrm{d}x \qquad \forall\, \varphi \in \mathcal{D}(\mathcal{R}).$$

Thus, $u|_{\mathcal{R}} = C$ (a constant function).

The second step is to show that the constant C from the first step is the same for all the hypercubes contained in Ω. Since Ω is connected, for any two open hypercubes $\mathcal{R}_\alpha$, $\mathcal{R}_\omega \subset \Omega$ there exists a chain of open hypercubes $(\mathcal{R}_1, \ldots, \mathcal{R}_p)$ such that $\mathcal{R}_1 = \mathcal{R}_\alpha$, $\mathcal{R}_\omega = \mathcal{R}_p$, $\mathcal{R}_k \subset \Omega$, $\mathcal{R}_k \cap \mathcal{R}_{k+1} \neq \emptyset$ for all $k \in \{1, \ldots, p-1\}$. Thus, it suffices to show that the constant C from the first step is the same for any two open hypercubes with non-empty intersection. This follows by considering the restriction of u to the intersection of these hypercubes.

The third step is to show that $u = C$ where C is the constant from the second step. It follows from the result of the second step that $(u - C)|_{\mathcal{R}} = 0$ for every open hypercube $\mathcal{R} \subset \Omega$. The union of all the open hypercubes contained in Ω is Ω. Thus, by Proposition 13.2.10 we have $u - C = 0$. $\square$

13.4 Definition and first properties of Sobolev spaces

In this section we gather, for easy reference, several basic definitions and results on Sobolev spaces. For more information and for detailed proofs we refer the reader to Adams [1], Grisvard [77], Evans [59], Lions and Magenes [157], Neças and [176].

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $m \in \mathbb{N}$.

Definition 13.4.1. The *Sobolev space* $\mathcal{H}^m(\Omega)$ is formed by the distributions $f \in \mathcal{D}'(\Omega)$ having the property that $\partial^\alpha f \in L^2(\Omega)$ for every $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leqslant m$.

From the above definition it clearly follows that $\mathcal{H}^0(\Omega) = L^2(\Omega)$.

Proposition 13.4.2. *$\mathcal{H}^m(\Omega)$ is a Hilbert space with the scalar product*

$$\langle f, g \rangle_m = \sum_{|\alpha| \leqslant m} \int_\Omega (\partial^\alpha f)\, \overline{(\partial^\alpha g)}\, \mathrm{d}x \qquad \forall\, f, g \in \mathcal{H}^m(\Omega). \tag{13.4.1}$$

Proof. It can be easily checked that (13.4.1) defines an inner product on $\mathcal{H}^m(\Omega)$. Therefore, we only have to show that $\mathcal{H}^m(\Omega)$ is complete with respect to the associated norm $\|\cdot\|_m$. Let (f_j) be a Cauchy sequence with respect to the norm $\|\cdot\|_m$. Then, for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leqslant m$ we have

$$\lim_{j,k \to \infty} \|\partial^\alpha f_j - \partial^\alpha f_k\|_{L^2}^2 = 0 \,.$$

Consequently, if $|\alpha| \leqslant m$, then $(\partial^\alpha f_j)$ is a Cauchy sequence in $L^2(\Omega)$, which is a Hilbert space. We can thus conclude that, for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leqslant m$, there exists $g_\alpha \in L^2(\Omega)$ such that $\partial^\alpha f_j \to g_\alpha$ in $L^2(\Omega)$. Since the convergence in $L^2(\Omega)$ implies the convergence in $\mathcal{D}'(\Omega)$ (see Remark 13.2.5), we obtain that $f_j \to g_0$ in $\mathcal{D}'(\Omega)$. By applying Proposition 13.2.9 we obtain that $\partial^\alpha f_j \to \partial^\alpha g_0$ in $\mathcal{D}'(\Omega)$. We have thus shown that $\partial^\alpha g_0 = g_\alpha \in L^2(\Omega)$ which implies that $g_0 \in \mathcal{H}^m(\Omega)$. Moreover, by the definition of g_α, we have that

$$\|g_0 - f_j\|_m^2 = \sum_{|\alpha| \leqslant m} \|g_\alpha - \partial^\alpha f_j\|_{L^2}^2 \to 0,$$

so we obtain that $f_j \to g_0$ in the norm of $\mathcal{H}^m(\Omega)$. □

Remark 13.4.3. Let $\Omega \subset \mathbb{R}^n$ be open, $X = L^2(\Omega)$, $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$ and let A be defined by

$$A\varphi = \partial^\alpha \varphi, \quad \mathcal{D}(A) = \left\{\varphi \in L^2(\Omega) \;\middle|\; \partial^\alpha \varphi \in L^2(\Omega)\right\}.$$

Then A is a closed operator on X. Indeed, let (φ_k) be a sequence in $\mathcal{D}(A)$ such that

$$\varphi_k \to \varphi, \quad A\varphi_k \to \psi \quad \text{in} \quad X.$$

From $\varphi_k \to \varphi$ we get, by Proposition 13.2.9, that $A\varphi_k \to \partial^\alpha \varphi$ in $\mathcal{D}'(\Omega)$. Thus $\partial^\alpha \varphi = \psi$ in $\mathcal{D}'(\Omega)$. Consequently, $\varphi \in \mathcal{D}(A)$ and $A\varphi = \psi$, so that A is closed.

Sobolev spaces of positive non-integer order are defined as follows.

Definition 13.4.4. *For $m \in \mathbb{N}$ and $s = m + \sigma$ with $\sigma \in (0,1)$, the Sobolev space $\mathcal{H}^s(\Omega)$ is formed by the functions $f \in \mathcal{H}^m(\Omega)$ such that*

$$\int_\Omega \int_\Omega \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|^2}{|x-y|^{n+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y < \infty$$

for every multi-index α such that $|\alpha| = m$.

For s, m and σ as above, $\mathcal{H}^s(\Omega)$ is a Hilbert space with the norm

$$\|\varphi\|_s^2 = \|\varphi\|_m^2 + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|\partial^\alpha \varphi(x) - \partial^\alpha \varphi(y)|^2}{|x-y|^{n+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y.$$

It is clear that $\mathcal{H}^s(\Omega) \subset \mathcal{H}^m(\Omega)$, with continuous embedding. If $\partial\Omega$ is of class C^1 (as defined in the next section), then we also have

$$\mathcal{H}^{m+1}(\Omega) \subset \mathcal{H}^s(\Omega)$$

with continuous embedding. This fact is not easy to check, the proof and other details can be found, for instance, in Adams [1, p. 214]. For bounded Ω, a much stronger result is contained in Theorem 13.5.3 below. For alternative definitions of $\mathcal{H}^s(\Omega)$ and its norm (assuming smooth $\partial\Omega$) see also [157, Section 9.1].

Remark 13.4.5. If $f : \Omega \to \mathbb{C}$ and $s \geqslant 0$, we say that $f \in \mathcal{H}^s_{\rm loc}(\Omega)$ if $f \in \mathcal{H}^s(\mathcal{O})$ for every bounded open set $\mathcal{O}$ with clos $\mathcal{O} \subset \Omega$.

If Ω is an open subset of $\mathbb{R}^n$ and $s > \frac{n}{2}$, then any function $f \in \mathcal{H}^s_{\rm loc}(\Omega)$ is continuous on Ω. Indeed, for every $\varphi \in \mathcal{D}(\Omega)$, the product φf may be regarded as a function in $\mathcal{H}^s(\mathbb{R}^n)$. Using Fourier transforms it follows that φf is continuous; see Taylor [217, p. 272]. Clearly this implies the continuity of f on Ω. It follows from here that for every $m \in \mathbb{Z}_+$,

$$s > \frac{n}{2} + m \;\Rightarrow\; \mathcal{H}^s_{\rm loc}(\Omega) \subset C^m(\Omega).$$

If Ω is bounded, $\partial\Omega$ is Lipschitz and $s > \frac{n}{2}$, then the functions in $\mathcal{H}^s(\Omega)$ are continuous on clos Ω. This follows easily by combining Theorems 1.4.3.1 and 1.4.4.1 from Grisvard [77] (see also [157, Theorem 9.8] for the case of smooth boundary). It follows from here that for such Ω and every $m \in \mathbb{Z}_+$,

$$s > \frac{n}{2} + m \;\Rightarrow\; \mathcal{H}^s(\Omega) \subset C^m(\text{clos}\,\Omega).$$

We define below a space which is very useful in the study of boundary value problems for elliptic PDEs.

Definition 13.4.6. For $s > 0$, the space $\mathcal{H}^s_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $\mathcal{H}^s(\Omega)$.

We mention that if Ω is bounded, with Lipschitz boundary and $s < \frac{1}{2}$, then we have $\mathcal{H}^s_0(\Omega) = \mathcal{H}^s(\Omega)$; see Grisvard [77, Corollary 1.4.4.5] (see also [157, Theorem 11.1] for the case when the boundary is smooth).

Sobolev spaces of negative order are defined as follows.

Definition 13.4.7. For any $s > 0$ the Sobolev space $\mathcal{H}^{-s}(\Omega)$ is defined as the dual of $\mathcal{H}^s_0(\Omega)$ with respect to the pivot space $L^2(\Omega)$ (duality with respect to a pivot space has been explained in Section 2.9).

Remark 13.4.8. Let $s = m + \sigma$, where $m \in \mathbb{Z}_+$ and $\sigma \in [0, 1)$. From the above definition we see that any $u \in \mathcal{H}^{-s}(\Omega)$ is a continuous linear functional on $\mathcal{H}^s_0(\Omega)$, hence also on $\mathcal{H}^{m+1}_0(\Omega)$. This implies that, when applied to $\varphi \in \mathcal{D}(\Omega)$, u satisfies condition (13.2.1) (with $m + 1$ in place of m). Since $\mathcal{D}(\Omega)$ is dense in $\mathcal{H}^s_0(\Omega)$, u is completely determined by its restriction to $\mathcal{D}(\Omega)$. Thus, we may regard u as a distribution:

$$\mathcal{H}^{-s}(\Omega) \subset \mathcal{D}'(\Omega).$$

This embedding is continuous, in the following sense: the convergence of a sequence in $\mathcal{H}^{-s}(\Omega)$ implies its convergence in $\mathcal{D}'(\Omega)$ (this is easy to see).

There is a little annoyance with the embedding described above: when we defined duality with respect to a pivot space, we used a pairing that is antilinear in the second argument, while the pairing of distributions with test functions is linear in both arguments. Thus, for $u \in \mathcal{H}^{-s}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$,

$$\langle u, \varphi \rangle_{\mathcal{H}^{-s}(\Omega), \mathcal{H}^s_0(\Omega)} = \langle u, \overline{\varphi} \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}.$$

Proposition 13.4.9. *Let α be a multi-index with $|\alpha| = m$. Then for every $p \in \mathbb{Z}$ we have $\partial^\alpha \in \mathcal{L}(\mathcal{H}^p(\Omega), \mathcal{H}^{p-m}(\Omega))$, with $\|\partial^\alpha\| \leqslant 1$.*

Proof. If $p \geqslant m$, then this is clear from the definition of $\mathcal{H}^p(\Omega)$. For $p = 0$ we argue as follows: Let $u \in L^2(\Omega)$. It is clear from (13.2.5) that

$$\langle \partial^\alpha u, \varphi \rangle_{\mathcal{D}', \mathcal{D}} \leqslant \|u\|_{L^2} \cdot \|\partial^\alpha \varphi\|_{L^2} \leqslant \|u\|_{L^2} \cdot \|\varphi\|_m \qquad \forall\, \varphi \in \mathcal{D}(\Omega).$$

Since $\mathcal{D}(\Omega)$ is dense in $\mathcal{H}_0^m(\Omega)$, it follows that $\partial^\alpha u$ has a continuous extension to $\mathcal{H}_0^m(\Omega)$, so that $\partial^\alpha u \in \mathcal{H}^{-m}(\Omega)$ and $\|\partial^\alpha u\|_{\mathcal{H}^{-m}} \leqslant \|u\|_{L^2}$. For $0 < p < m$ we decompose $\alpha = \alpha_1 + \alpha_2$ such that $|\alpha_1| = p$ and $|\alpha_2| = m - p$. Now the statement follows from $\partial^\alpha = \partial^{\alpha_2} \partial^{\alpha_1}$ by combining the cases $p \geqslant m$ and $p = 0$ discussed earlier.

It remains to consider the case $p < 0$. If $u \in \mathcal{H}^p(\Omega)$, then

$$|\langle u, \psi \rangle_{\mathcal{D}', \mathcal{D}}| \leqslant \|u\|_{\mathcal{H}^p} \cdot \|\psi\|_{-p} \qquad \forall\, \psi \in \mathcal{D}(\Omega).$$

Using this and (13.2.5), we obtain

$$|\langle \partial^\alpha u, \varphi \rangle_{\mathcal{D}', \mathcal{D}}| \leqslant \|u\|_{\mathcal{H}^p} \cdot \|\partial^\alpha \varphi\|_{-p} \leqslant \|u\|_{\mathcal{H}^p} \cdot \|\varphi\|_{m-p}.$$

This shows that $\partial^\alpha u \in \mathcal{H}^{p-m}(\Omega)$ and $\|\partial^\alpha u\|_{\mathcal{H}^{p-m}} \leqslant \|u\|_{\mathcal{H}^p}$. □

In the remaining part of this section we take a closer look at the spaces $\mathcal{H}_0^1(\Omega)$. Under a simple geometric assumption, functions in such a space satisfy the following remarkable inequality, called the *Poincaré inequality*.

Proposition 13.4.10 (Poincaré inequality). *Suppose that Ω is contained between a pair of parallel hyperplanes situated at a distance $\delta > 0$. Then*

$$\|f\|_{L^2} \leqslant \delta \|\nabla f\|_{L^2} \qquad \forall\, f \in \mathcal{H}_0^1(\Omega).$$

Proof. First notice that it suffices to prove the proposition for real-valued f since the complex case follows easily. Using that $\mathcal{D}(\Omega)$ is dense in $\mathcal{H}_0^1(\Omega)$ we see that it suffices to prove the inequality for $f \in \mathcal{D}(\Omega)$. Consider Cartesian coordinates such that $\Omega \subset \left\{x \in \mathbb{R}^n \mid -\frac{\delta}{2} < x_1 < \frac{\delta}{2}\right\}$ and extend f to vanish outside Ω. Then for any $x' \in \mathbb{R}^{n-1}$ and $x_1 \in \left(-\frac{\delta}{2}, 0\right)$ we have

$$f^2(x_1, x') = \int_{-\frac{\delta}{2}}^{x_1} \frac{\partial}{\partial x_1}(f^2(\xi, x'))\,\mathrm{d}\xi = 2\int_{-\frac{\delta}{2}}^{x_1} f(\xi, x') \frac{\partial f}{\partial x_1}(\xi, x')\,\mathrm{d}\xi,$$

which implies (using the Cauchy–Schwarz inequality) that

$$f^2(x_1, x') \leqslant 2\left(\int_{-\frac{\delta}{2}}^{0} f^2(\xi, x')\mathrm{d}\xi\right)^{\frac{1}{2}} \left(\int_{-\frac{\delta}{2}}^{0} \left[\frac{\partial f}{\partial x_1}(\xi, x')\right]^2 \mathrm{d}\xi\right)^{\frac{1}{2}}.$$

Integrating the above relation with respect to x_1, we get

$$\int_{-\frac{\delta}{2}}^{0} f^2(x_1,x')\,\mathrm{d}x_1 \leqslant \delta \left(\int_{-\frac{\delta}{2}}^{0} f^2(x_1,x')\mathrm{d}x_1\right)^{\frac{1}{2}} \left(\int_{-\frac{\delta}{2}}^{0} \left[\frac{\partial f}{\partial x_1}(x_1,x')\right]^2 \mathrm{d}x_1\right)^{\frac{1}{2}}.$$

From the above relation we obtain that

$$\int_{-\frac{\delta}{2}}^{0} f^2(x_1,x')\,\mathrm{d}x_1 \leqslant \delta^2 \int_{-\frac{\delta}{2}}^{0} \left[\frac{\partial f}{\partial x_1}(x_1,x')\right]^2 \mathrm{d}x_1\,.$$

Integration with respect to x' yields

$$\begin{aligned}\int_{(-\frac{\delta}{2},0)\times\mathbb{R}^{n-1}} f^2(x)\,\mathrm{d}x &\leqslant \delta^2\!\int_{(-\frac{\delta}{2},0)\times\mathbb{R}^{n-1}} \left[\frac{\partial f}{\partial x_1}(x)\right]^2 \mathrm{d}x\\ &\leqslant \delta^2\!\int_{(-\frac{\delta}{2},0)\times\mathbb{R}^{n-1}} |(\nabla f)(x)|^2\mathrm{d}x\,.\end{aligned}$$

Adding this to the corresponding result for $x_1 \in (0,\frac{\delta}{2})$ we obtain the desired inequality. (If we had not split the domain into two slices, we would have obtained 2δ in place of δ in the estimate in the proposition.) □

Lemma 13.4.11. *Let Ω_1, Ω_2 be two open subsets of $\mathbb{R}^n$, with* clos $\Omega_1 \subset \Omega_2$. *Then the extension operator E defined by*

$$(Ef)(x) = \begin{cases} f(x) & \text{if} \quad x \in \Omega_1\,, \\ 0 & \text{if} \quad x \notin \Omega_1 \end{cases} \qquad \forall\, f \in \mathcal{H}_0^1(\Omega_1)$$

is isometric from $\mathcal{H}_0^1(\Omega_1)$ to $\mathcal{H}_0^1(\Omega_2)$.

Proof. For every $f \in \mathcal{H}_0^1(\Omega_1)$ there exists a sequence (f_n) in $\mathcal{D}(\Omega_1)$ such that $f_n \to f$ in $\mathcal{H}_0^1(\Omega_1)$. If we denote $g_n = Ef_n$, for every $n \in \mathbb{N}$ then, (g_n) is clearly a Cauchy sequence in $\mathcal{H}_0^1(\Omega_2)$, so that $g_n \to g$ in $\mathcal{H}_0^1(\Omega_2)$. It is easily seen that $g(x) = f(x)$ if $x \in \Omega_1$ and that $g(x) = 0$ if $x \in \Omega_2 \setminus \Omega_1$, so that $Ef = g \in \mathcal{H}_0^1(\Omega_2)$ and $\|Ef\|_{\mathcal{H}_0^1(\Omega_2)} = \|f\|_{\mathcal{H}_0^1(\Omega_1)}$. □

Proposition 13.4.12. *Let $n \in \mathbb{N}$ and let Ω be a bounded open subset of $\mathbb{R}^n$. Then the embedding operator J_Ω of $\mathcal{H}_0^1(\Omega)$ in $L^2(\Omega)$ is compact.*

Proof. Let Q be an open hypercube in $\mathbb{R}^n$, with clos $\Omega \subset Q$ and we denote by $E \in \mathcal{L}(\mathcal{H}_0^1(\Omega), \mathcal{H}_0^1(Q))$ the extension operator in Lemma 13.4.11. Moreover, for $g \in \mathcal{H}_0^1(Q)$ we denote by Rg the restriction of g to Ω. By using the facts that $R \in \mathcal{L}(L^2(Q), L^2(\Omega))$ and $J_\Omega = RJ_QE$, we see that the compactness of J_Ω follows from the compactness of J_Q.

We still have to show that J_Q is compact. For the sake of simplicity, we assume that $Q = (0,\pi)^n$. From the elementary theory of Fourier series we know that the family $(\varphi_\alpha)_{\alpha\in\mathbb{N}^n}$ defined by

$$\varphi_\alpha(x) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \prod_{k=1}^{n} \sin(\alpha_k x_k) \qquad \forall\, \alpha \in \mathbb{N}^n,\ x \in Q,$$

is an orthonormal basis in $L^2(Q)$. In this proof, we need the notation $\|\alpha\|^2 = \sum_{k=1}^n \alpha_k^2$ for a multi-index $\alpha \in \mathbb{N}^n$. Let $f \in \mathcal{H}_0^1(Q)$. Then

$$\|f\|^2_{L^2(Q)} = \sum_{\alpha\in\mathbb{N}^n} \left|\langle f, \varphi_\alpha\rangle_{L^2(Q)}\right|^2,$$

$$\|f\|^2_{H^1(Q)} = \sum_{\alpha\in\mathbb{N}^n} (1+\|\alpha\|^2) \left|\langle f, \varphi_\alpha\rangle_{L^2(Q)}\right|^2.$$

From the above formulas it follows that if $m \in \mathbb{N}$ and $J_{Q,m} \in \mathcal{L}(\mathcal{H}_0^1(Q), L^2(Q))$ is defined by

$$J_{Q,m} f = \sum_{\alpha\in\mathbb{N}^n,\ \|\alpha\|^2 \leqslant m} \langle f, \varphi_\alpha\rangle_{L^2(Q)}\, \varphi_\alpha \qquad \forall\, f \in \mathcal{H}_0^1(\Omega),$$

then

$$\|J_Q f - J_{Q,m} f\|^2_{L^2(Q)} \leqslant \frac{1}{1+m} \|f\|^2_{\mathcal{H}_0^1(\Omega)}.$$

This implies that

$$\lim_{m\to\infty} \|J_Q - J_{Q,m}\|_{\mathcal{L}(\mathcal{H}_0^1(Q), L^2(Q))} = 0.$$

Since $\dim \operatorname{Ran} J_{Q,m} < \infty$, according to Proposition 12.2.2 J_Q is compact. □

13.5 Regularity of the boundary and Sobolev spaces on manifolds

Some of the properties of Sobolev spaces strongly depend on the regularity properties of the boundary $\partial\Omega$ of Ω. For more details on the concepts and results introduced in this section we refer the reader to Grisvard [77] and Nečas [176].

Definition 13.5.1. Let Ω be an open subset of $\mathbb{R}^n$. We say that $\partial\Omega$ *is Lipschitz* if there exists an $L \geqslant 0$ (called the *Lipschitz constant* of $\partial\Omega$) such that the following property holds: for every $x \in \partial\Omega$ There exist a neighborhood V of x in $\mathbb{R}^n$ and a system of orthonormal coordinates denoted by $(y_1, \dots, y_n)$ such that

1. V is a rectangle in the new coordinates, i.e.,

$$V = \{(y_1, \dots y_n) \mid -a_i < y_j < a_j,\ 1 \leqslant j \leqslant n\};$$

2. there exists a Lipschitz function φ with Lipschitz constant $\leqslant L$ defined on

$$V' = \{(y_1, \dots y_{n-1}) \mid -a_i < y_j < a_j,\ 1 \leqslant j \leqslant n-1\},$$

such that $|\varphi(y')| \leqslant \frac{a_n}{2}$ for every $y' = (y_1, \dots y_{n-1}) \in V'$,

$$\Omega \cap V = \{y = (y', y_n) \in V \mid y_n < \varphi(y')\},$$
$$\partial\Omega \cap V = \{y = (y', y_n) \in V \mid y_n = \varphi(y')\}.$$

In other words, in a neighborhood of any point $x \in \partial\Omega$ the set Ω is below the graph of φ and $\partial\Omega$ is the graph of φ. Consequently, if Ω is an open set with Lipschitz boundary, then Ω is not on both sides of $\partial\Omega$ at any point of $\partial\Omega$. For instance, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ does not have a Lipschitz boundary. More generally, a domain with a cut in $\mathbb{R}^n$ does not have a Lipschitz boundary.

If D is an open set in $\mathbb{R}^n$, $f : D \to \mathbb{C}$ and $m \in \mathbb{N}$, we say that f *is of class* $C^{m,1}$ if f is of class C^m and all the partial derivatives of f of order m are Lipschitz continuous. Equivalently, all the derivatives of f of order $\leqslant m+1$ are in $L^\infty(D)$.

Definition 13.5.2. Let Ω be an open subset of $\mathbb{R}^n$ and $m \in \mathbb{Z}_+$. We say that $\partial\Omega$ *is of class* C^m (respectively, of class $C^{m,1}$) if the properties in the previous definition hold but with φ of class C^m (respectively, of class $C^{m,1}$) and the L^∞ norm of all these φ and their first m (respectively, first $m+1$) derivatives are uniformly bounded. We say that $\partial\Omega$ is of class C^∞ if it is of class C^m for every $m \in \mathbb{N}$.

Thus, $\partial\Omega$ is Lipschitz iff it is of class $C^{0,1}$ and the inclusions between the above classes can be written informally as $C^{m,1} \subset C^m \subset C^{m-1,1}$ for all $m \in \mathbb{N}$.

For example, the interior of a convex polygon in $\mathbb{R}^2$ has a Lipschitz boundary but its boundary is not of class C^1. If $\Omega = \{(x, y) \in \mathbb{R}^2 \mid y > \sin x\}$, then $\partial\Omega$ is of class C^m for all m, but if we replace $\sin x$ with $\sin(x^2)$, then $\partial\Omega$ is not Lipschitz. If $\Omega \subset \mathbb{R}$ consists of finitely many open intervals whose closures are disjoint, then (according to the earlier definition), $\partial\Omega$ (which consists of finitely many points) is of class C^∞.

For bounded open sets with Lipschitz boundary, the following theorem is a generalization of Proposition 13.4.12. For a proof see [176, Theorem 6.1].

Theorem 13.5.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Suppose that $0 \leqslant s_1 < s_2$. Then $\mathcal{H}^{s_2}(\Omega) \subset \mathcal{H}^{s_1}(\Omega)$, with compact embedding.*

We quote from Grisvard [77, Theorem 1.4.2.1] a result concerning the density of spaces of smooth functions in Sobolev spaces (for related results and particular cases see also Neças [176, Section 3.2] and Adams [1, Theorems 3.18 and 7.40]).

Theorem 13.5.4. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary and $m \in \mathbb{Z}_+$. Then $C^\infty(\text{clos}\,\Omega)$ is dense in $\mathcal{H}^s(\Omega)$, for all $s \geqslant 0$.*

Moreover, the space $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{H}^m(\mathbb{R}^n)$.

It follows from here that for any Ω as above and for any numbers s_1, s_2 with $0 \leqslant s_1 < s_2$, $\mathcal{H}^{s_2}(\Omega)$ is dense in $\mathcal{H}^{s_1}(\Omega)$.

An important and difficult theory which requires the regularity of the boundary is the so-called *elliptic regularity theory.* We give below, without proof, one of the main results from this theory, and we refer the reader to Brezis [22, Section IX.6] and Evans [59, Section 6.3] for the proof and for more sophisticated versions.

Theorem 13.5.5. *Let Ω be a bounded open set with a boundary $\partial\Omega$ of class C^2 and let $f \in L^2(\Omega)$. If $\varphi \in \mathcal{H}_0^1(\Omega)$ satisfies (in $\mathcal{D}'(\Omega)$) the equation*

$$-\Delta\varphi + \varphi = f,$$

then $\varphi \in \mathcal{H}^2(\Omega)$.

Remark 13.5.6. If f and φ are as in the above theorem, then, without any smoothness assumption on $\partial\Omega$ and without the boundedness assumption on Ω, we have that $\varphi \in \mathcal{H}^2_{\rm loc}(\Omega)$; i.e., $\varphi \in \mathcal{H}^2(\mathcal{O})$ for any bounded open set $\mathcal{O}$ with clos $\mathcal{O} \subset \Omega$. For the proof (which is much easier than the proof of Theorem 13.5.5) we refer the reader, for instance, to [59, p. 309]. More generally, if $f \in \mathcal{H}^m_{\rm loc}(\Omega)$, where $m \in \mathbb{Z}_+$, and φ satisfies the equation in the theorem, then $\varphi \in \mathcal{H}^{m+2}_{\rm loc}(\Omega)$; see the same reference. (The space $\mathcal{H}^m_{\rm loc}(\Omega)$ has been defined in Remark 13.4.5.)

We will need Sobolev spaces on open subsets of $\partial\Omega$, where Ω is a bounded open set with Lipschitz boundary. If Ω is such a set and $x \in \partial\Omega$, then there exist a neighborhood V of x in $\mathbb{R}^n$, a system of orthonormal coordinates $(y_1, \dots, y_n)$ in $\mathbb{R}^n$ satisfying condition 1 in Definition 13.5.1 and a Lipschitz function φ defined on the $(n-1)$-dimensional rectangle V' that corresponds to the coordinates $y_1, \dots, y_{n-1}$, satisfying condition 2 in the same definition, such that

$$\partial\Omega \cap V = \{y = (y', y_n) \in V \mid y_n = \varphi(y')\}.$$

If Ω is an open set with a Lipschitz boundary, then the set $\partial\Omega$ can be seen as an $(n-1)$-dimensional Lipschitz manifold in $\mathbb{R}^n$. Indeed, if we define Φ on V' by

$$\Phi(y_1, \dots, y_{n-1}) = [y_1, \dots, y_{n-1}, \varphi(y_1, \dots, y_{n-1})], \tag{13.5.1}$$

then Φ^{-1} is a chart from $\partial\Omega \cap V$ onto V'. Taking a collection of such charts (Φ_j^{-1}) corresponding to a collection of rectangular sets (V_j) as above that cover $\partial\Omega$, we obtain an atlas of $\partial\Omega$, since the maps $\Phi_j^{-1}\Phi_k$ are Lipschitz on their domains.

Definition 13.5.7. Let Ω be a bounded open subset of $\mathbb{R}^n$ with a boundary $\partial\Omega$ of class $C^{m,1}$, where $m \in \mathbb{Z}_+$. Let Γ be an open subset of $\partial\Omega$ and let $s \in [0, m+1]$. The space $\mathcal{H}^s(\Gamma)$ consists of those $f \in L^2(\Gamma)$ such that, with V and Φ as in (13.5.1),

$$f \circ \Phi \in \mathcal{H}^s(\Phi^{-1}(\Gamma \cap V))$$

for all possible V, V' and φ as in Definition 13.5.1.

It is enough to verify the above condition for one atlas $(\partial\Omega \cap V_j, \Phi_j^{-1})_{j=1}^J$ of $\partial\Omega$, where Φ_j corresponds to φ_j as in (13.5.1). The bound $s \leqslant m+1$ implies that if the condition in Definition 13.5.7 holds for one atlas, then it holds for any other atlas. One possible norm on $\mathcal{H}^s(\Gamma)$ is given by

$$\|f\|^2_{\mathcal{H}^s(\Gamma)} = \sum_{j=1}^{J} \|f \circ \Phi_j\|^2_{\mathcal{H}^s(\Phi_j^{-1}(\Gamma \cap V_j))}, \tag{13.5.2}$$

where $(\partial\Omega \cap V_j, \Phi_j^{-1})_{j=1}^J$ is an atlas of $\partial\Omega$ such that Φ_j corresponds to φ_j as in (13.5.1) and each couple (V_j, φ_j) satisfies the conditions in Definition 13.5.1. The condition $s \leqslant m+1$ ensures that for different atlases we get equivalent norms.

If $s \in (0,1)$, then any norm of the form (13.5.2) is equivalent to the norm given by

$$\|f\|_s^2 = \int_\Gamma \int_\Gamma \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s-1}} \,\mathrm{d}\sigma_x \,\mathrm{d}\sigma_y + \int_\Gamma |f(x)|^2 \,\mathrm{d}\sigma, \tag{13.5.3}$$

where $\mathrm{d}\sigma$ is the surface measure on $\partial\Omega$. It can be shown that, with the norm from (13.5.2), $\mathcal{H}^s(\Gamma)$ is a Hilbert space (for each $s \in [0, m+1]$).

Proposition 13.5.8. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a boundary $\partial\Omega$ of class $C^{m,1}$, where $m \in \mathbb{Z}_+$. Let $s_1, s_2 \in [0, m+1]$ with $s_1 < s_2$. Then we have $\mathcal{H}^{s_2}(\partial\Omega) \subset \mathcal{H}^{s_1}(\partial\Omega)$, with compact embedding.*

Proof. According to the definition of compact operators (Definition 12.2.1), we have to show that if (z_n) is a bounded sequence in $\mathcal{H}^{s_2}(\partial\Omega)$, then this sequence has a convergent subsequence in $\mathcal{H}^{s_1}(\partial\Omega)$. Let $(\partial\Omega \cap V_j, \Phi_j^{-1})_{j=1}^J$ be an atlas of $\partial\Omega$ as in (13.5.2). Then for each $j \in \{1, \dots, J\}$, $(z_n \circ \Phi_j)$ is a bounded sequence in $\mathcal{H}^{s_2}(V_j')$. Here, V_j' is the $(n-1)$-dimensional basis of the rectangle V_j, as in Definition 13.5.1. According to Theorem 13.5.3, the sequence (z_n) contains a subsequence (z_n^1) such that $(z_n^1 \circ \Phi_j)$ is convergent in $\mathcal{H}^{s_1}(V_1')$. By the same argument, the sequence (z_n^1) contains a subsequence (z_n^2) such that $(z_n^2 \circ \Phi_j)$ is convergent in $\mathcal{H}^{s_1}(V_2')$. Continuing the process, after J steps we obtain a subsequence of (z_n) that is a Cauchy sequence with respect to the norm from (13.5.2) (with $s = s_1$). Hence, this subsequence is convergent in $\mathcal{H}^{s_1}(\partial\Omega)$, so that the embedding is compact. $\square$

Definition 13.5.9. With Ω as in the last proposition, let Γ be an open subset of $\partial\Omega$. We say that Γ *has Lipschitz boundary in* $\partial\Omega$ if there exists an atlas $(\partial\Omega \cap V_j, \Phi_j^{-1})_{j=1}^J$ of $\partial\Omega$ as in (13.5.2) such that, for each $k \in \{1, \dots, J\}$,

$$\Phi_k^{-1}(\Gamma \cap V_k) \ \text{ has Lipschitz boundary in } \ V_k'.$$

Proposition 13.5.10. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a boundary $\partial\Omega$ of class $C^{m,1}$, where $m \in \mathbb{Z}_+$. Let $s_1, s_2 \in [0, m+1]$ with $s_1 < s_2$. Let Γ be an open subset of $\partial\Omega$ that has Lipschitz boundary in $\partial\Omega$.*

Then we have $\mathcal{H}^{s_2}(\Gamma) \subset \mathcal{H}^{s_1}(\Gamma)$, with compact embedding.

The proof is similar to the proof of the previous proposition. In some places $\partial\Omega$ has to be replaced with Γ and V_j' has to be replaced with $\Phi_j^{-1}(\Gamma \cap V_j)$.

13.6 Trace operators and the space $\mathcal{H}^1_{\Gamma_0}(\Omega)$

In this section we recall some results giving a weak sense to boundary values of functions defined on a domain $\Omega \subset \mathbb{R}^n$, that belong to certain Sobolev spaces. Such boundary functions or distributions are called (boundary) traces of the functions defined on Ω. We also introduce and investigate the space $\mathcal{H}^1_{\Gamma_0}(\Omega)$, which consists of those $f \in \mathcal{H}^1(\Omega)$ whose trace vanishes on a part Γ_0 of the boundary.

In general a function f in $\mathcal{H}^1(\Omega)$ is not continuous (even worse, it is generally defined only almost everywhere in Ω), so the values of f on $\partial\Omega$ have no meaning. However, these boundary values can be defined in a weaker sense, based on the following result, which is proved, for instance, in Neças [176, Sections 5.4–5.5].

Theorem 13.6.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. Then the mapping $\gamma_0 : C^1(\mathrm{clos}\ \Omega) \to C^0(\partial\Omega)$, defined by*

$$\gamma_0 f = f|_{\partial\Omega} \qquad \forall\, f \in C^1(\mathrm{clos}\ \Omega),$$

has a unique extension to a bounded linear operator from $\mathcal{H}^1(\Omega)$ onto $\mathcal{H}^{\frac{1}{2}}(\partial\Omega)$.

If $f \in \mathcal{H}^1(\Omega)$, then we call $\gamma_0 f$ the *Dirichlet trace* of f on $\partial\Omega$. For the sake of simplicity, we sometimes write $f(x)$ instead of $(\gamma_0 f)(x)$ (where $x \in \partial\Omega$). The space $\mathcal{H}^1_0(\Omega)$ introduced in Definition 13.4.6 can be characterized as follows.

Proposition 13.6.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Then*

$$\mathcal{H}^1_0(\Omega) = \{f \in \mathcal{H}^1(\Omega) \mid \gamma_0 f = 0\}.$$

For a proof of the above proposition we refer the reader to [176, p. 87].

Definition 13.6.3. If Ω is a bounded open set with a Lipschitz boundary, then the *unit outward normal vector field* is defined for almost all $x \in \partial\Omega$, using local coordinates as in Definition 13.5.1 (such that x has the coordinates $(y', \varphi(y'))$), as follows:

$$\nu(x) = \frac{1}{\sqrt{1 + \left[\frac{\partial\varphi}{\partial y_1}(y')\right]^2 + \cdots + \left[\frac{\partial\varphi}{\partial y_{n-1}}(y')\right]^2}} \begin{bmatrix} -\frac{\partial\varphi}{\partial y_1}(y') \\ \vdots \\ -\frac{\partial\varphi}{\partial y_{n-1}}(y') \\ 1 \end{bmatrix}. \tag{13.6.1}$$

This vector field can be extended to almost every point in the rectangular open set V by defining it to be independent of y_n (the last local coordinate). Now let $(\partial\Omega \cap V_j, \Phi_j^{-1})_{j=1}^J$ be an atlas of $\partial\Omega$, where V_j is rectangular and Φ_j corresponds to $\varphi_{j,n}$ as in (13.5.1). By a partition of unity subordinated to the compact set $\partial\Omega$ and its covering $(V_j)_{j=1}^J$ (see Proposition 13.1.6), we can define a vector field ν in a neighborhood of clos Ω coinciding with the outward unit normal almost everywhere on $\partial\Omega$. If Ω is only Lipschitz, then all what we can say about the vector field ν is that it is almost everywhere defined on $\partial\Omega$ and measurable (and obviously bounded). If Ω is of class C^m (or $C^{m,1}$), with $m \in \mathbb{N}$, then ν is of class C^{m-1} (or $C^{m-1,1}$).

Definition 13.6.4. If $f \in C^1(\text{clos } \Omega)$, then the scalar field on $\partial\Omega$, defined by

$$\frac{\partial f}{\partial \nu}(x) = \nabla f(x) \cdot \nu(x) \quad \text{for almost all} \quad x \in \partial\Omega, \tag{13.6.2}$$

is called the *normal derivative of f* on $\partial\Omega$.

Remark 13.6.5. Theorem 13.6.1 allows us to extend the definition of the normal derivative for any function $f \in \mathcal{H}^2(\Omega)$, and we obtain that $\frac{\partial f}{\partial \nu} \in L^2(\partial\Omega)$ (this is still for Ω bounded, open and with a Lipschitz boundary, which implies that $\nu \in L^\infty(\partial\Omega)$). Thus, for any bounded domain with Lipschitz boundary,

$$\gamma_1 \in \mathcal{L}(\mathcal{H}^2(\Omega), L^2(\partial\Omega)).$$

With more smoothness imposed on the boundary, we get the following stronger result; see Grisvard [77, Theorem 1.5.1.2] and Lions and Magenes [157, Chapter 1, Theorem 8.3] (the latter actually assumes C^∞ boundary).

Theorem 13.6.6. *Let Ω be a bounded open set in $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^2. Let $\gamma_1 : C^2(\text{clos } \Omega) \to L^2(\partial\Omega)$ be the mapping*

$$(\gamma_1 f)(x) = \frac{\partial f}{\partial \nu}(x) \quad \textit{almost everywhere in} \quad \partial\Omega,$$

where $\frac{\partial f}{\partial \nu}(x)$ has been defined in (13.6.2). Then γ_1 has a unique extension as a bounded operator from $\mathcal{H}^2(\Omega)$ onto $\mathcal{H}^{\frac{1}{2}}(\partial\Omega)$.

If we restrict the trace operator γ_1 to $\mathcal{H}^2(\Omega) \cap \mathcal{H}^1_0(\Omega)$, then it is still onto $\mathcal{H}^{\frac{1}{2}}(\partial\Omega)$.

If $f \in \mathcal{H}^2(\Omega)$, we call $\gamma_1 f$ the *Neumann trace* of f on $\partial\Omega$ and, for the sake of simplicity, we often denote $(\gamma_1 f)(x) = \frac{\partial f}{\partial \nu}(x)$.

The space $\mathcal{H}^2_0(\Omega)$ from Definition 13.4.6 can be characterized as follows.

Proposition 13.6.7. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a C^2 boundary. Then*

$$\mathcal{H}^2_0(\Omega) = \{f \in \mathcal{H}^2(\Omega) \mid \gamma_0 f = 0, \quad \gamma_1 f = 0\}.$$

For a proof of the above result we refer the reader to [176, p. 90].

By combining Proposition 13.5.8 and Theorem 13.6.6, we obtain the following:

Corollary 13.6.8. *Let Ω be a bounded open set of $\mathbb{R}^n$ with a boundary $\partial\Omega$ of class C^2 and let γ_1 be the Neumann trace operator on $\partial\Omega$.*

Then γ_1 is a compact operator from $\mathcal{H}^2(\Omega)$ into $L^2(\partial\Omega)$.

Let Ω be a bounded open and connected set in $\mathbb{R}^n$ with Lipschitz boundary and let Γ_0, Γ_1 be open subsets of $\partial\Omega$ such that

$$\operatorname{clos}\Gamma_0 \cup \operatorname{clos}\Gamma_1 = \partial\Omega\,, \qquad \Gamma_0 \cap \Gamma_1 = \emptyset\,. \tag{13.6.3}$$

We define

$$\mathcal{H}^1_{\Gamma_0}(\Omega) = \left\{ f \in \mathcal{H}^1(\Omega) \mid \gamma_0 f_{|\Gamma_0} = 0 \right\},$$

which we regard as a closed subspace of $\mathcal{H}^1(\Omega)$. (The formula $\gamma_0 f_{|\Gamma_0} = 0$ has to be understood as an equality in $L^2(\partial\Omega)$, i.e., with equality almost everywhere.) According to Proposition 13.6.2 we have $\mathcal{H}^1_0(\Omega) \subset \mathcal{H}^1_{\Gamma_0}(\Omega)$. We show that the Poincaré inequality proved in Proposition 13.4.10 for functions in $\mathcal{H}^1_0(\Omega)$ still holds in this larger space. Here (unlike in Proposition 13.4.10) Ω has to be bounded and we do not obtain an explicit expression for the constant in the inequality.

Theorem 13.6.9. *With Ω, Γ_0 and Γ_1 as above, assume that $\Gamma_0 \neq \emptyset$. Then there exists a constant $c > 0$, depending only on Ω and Γ_0, such that*

$$\int_\Omega |f(x)|^2 \mathrm{d}x \leqslant c^2 \int_\Omega \|\nabla f(x)\|^2 \mathrm{d}x \qquad \forall\, f \in \mathcal{H}^1_{\Gamma_0}(\Omega)\,.$$

Proof. We use a contradiction argument. Assume that the conclusion of the theorem is false. This implies the existence of a sequence (f_n) in $\mathcal{H}^1_{\Gamma_0}(\Omega)$ such that

$$\|f_n\|_{L^2(\Omega)} = 1 \qquad \forall\, n \in \mathbb{N}, \tag{13.6.4}$$

$$\|\nabla f_n\|_{L^2(\Omega)} \to 0\,. \tag{13.6.5}$$

Clearly (f_n) is bounded in $\mathcal{H}^1_{\Gamma_0}(\Omega)$. According to to Alaoglu's theorem (see Lemma 12.2.4), there exist $f \in \mathcal{H}^1_{\Gamma_0}(\Omega)$ and a subsequence (f_{n_k}) such that

$$\lim_{k\to\infty} \langle f_{n_k}, \varphi\rangle_{\mathcal{H}^1} = \langle f, \varphi\rangle_{\mathcal{H}^1} \qquad \forall\, \varphi \in \mathcal{H}^1_{\Gamma_0}(\Omega)\,. \tag{13.6.6}$$

Since $\nabla \in \mathcal{L}(\mathcal{H}^1_{\Gamma_0}(\Omega), L^2(\Omega))$, it follows that

$$\lim_{k\to\infty} \langle \nabla f_{n_k}, \psi\rangle = \langle \nabla f, \psi\rangle \qquad \forall\, \psi \in L^2(\Omega)\,,$$

where the inner products are taken in $L^2(\Omega)$. The above formula with (13.6.5) imply that $\nabla f = 0$ in Ω. By Theorem 13.3.2, it follows that f is a constant function in Ω. Since $f \in \mathcal{H}^1_{\Gamma_0}(\Omega)$, the trace of this constant on Γ_0 must be zero. Since the $(n-1)$-dimensional measure of Γ_0 is not zero, we obtain that $f = 0$.

On the other hand, (13.6.6) implies, because of the compact embedding of $\mathcal{H}^1_{\Gamma_0}(\Omega)$ in $L^2(\Omega)$ (see Theorem 13.5.3) and because of Proposition 12.2.5, that $f_{n_k} \to f$ in $L^2(\Omega)$. This fact, combined with (13.6.4), yields that $\|f\|_{L^2(\Omega)} = 1$, which clearly contradicts the previously established fact that $f = 0$. $\square$

With Ω, Γ_0 and Γ_1 as in (13.6.3), we regard $L^2(\Gamma_1)$ as a closed subspace of $L^2(\partial\Omega)$, consisting of those $f \in L^2(\partial\Omega)$ for which $f(x) = 0$ for almost every $x \in \partial\Omega \setminus \Gamma_1$. (This condition is in general stronger than $f(x) = 0$ for almost every $x \in \Gamma_0$.)

Theorem 13.6.10. *With Ω, Γ_0 and Γ_1 as in* (13.6.3)*, the space*

$$\mathcal{V}(\Gamma_1) = \left\{ f \in \gamma_0 \mathcal{H}^1(\Omega) \mid \operatorname{supp} f \subset \Gamma_1 \right\}$$

is dense in $L^2(\Gamma_1)$.

Proof. Let $f \in L^2(\Gamma_1)$ and $\varepsilon > 0$. *The first step* is to construct $f_\varepsilon \in C(\Gamma_1)$, with compact support contained in Γ_1, which is a good approximation of f.

Let $(\partial\Omega \cap V_j, \Phi_j^{-1})_{j=1}^J$ be an atlas of $\partial\Omega$ as in (13.5.2). Clearly, $V_1, \dots, V_J$ is an open covering of the compact set $\operatorname{clos} \Gamma_1$. Let $\psi_1, \dots, \psi_J$ be a partition of unity subordinated to $\operatorname{clos} \Gamma_1$ and its covering $V_1, \dots, V_J$ (see Proposition 13.1.6), so that

$$f = f\psi_1 + \cdots + f\psi_J .$$

Then $f\psi_j \in L^2(\Gamma_1 \cap V_j)$, or equivalently,

$$(f\psi_j) \circ \Phi_j \in L^2(\Phi_j^{-1}(\Gamma_1 \cap V_j)) \qquad (1 \leqslant j \leqslant J).$$

Note that $\Phi_j^{-1}(\Gamma_1 \cap V_j)) \subset V_j'$ is open in $\mathbb{R}^{n-1}$ (V_j' is the $(n-1)$-dimensional rectangle at the basis of V_j, as in Definition 13.5.1). Since $\mathcal{D}(\Phi_j^{-1}(\Gamma_1 \cap V_j))$ is dense in $L^2(\Phi_j^{-1}(\Gamma_1 \cap V_j))$ (see Proposition 13.1.9), we can find $\tilde{f}_{j,\varepsilon} \in \mathcal{D}(\Phi_j^{-1}(\Gamma_1 \cap V_j))$ such that

$$\|(f\psi_j) \circ \Phi_j - \tilde{f}_{j,\varepsilon}\|_{L^2(V_j')} \leqslant \varepsilon .$$

For all $j \in \{1, \dots, J\}$ we define $f_{j,\varepsilon} \in C(\Gamma_1 \cap V_j)$ by

$$f_{j,\varepsilon} = \tilde{f}_{j,\varepsilon} \circ \Phi_j^{-1} ,$$

and we extend $f_{j,\varepsilon}$ to a function in $C(\partial\Omega)$ by making it equal to zero in all the other points of $\partial\Omega$. Note that $\operatorname{supp} f_{j,\varepsilon} \subset \Gamma_1 \cap V_j$. Let us denote by φ_j the last component of Φ_j (this is the scalar Lipschitz function as in Definition 13.5.1, whose graph is $\partial\Omega \cap V_j$). Let L_j be the Lipschitz constant of φ_j. Then

$$\begin{aligned}\|f\psi_j - f_{j,\varepsilon}\|_{L^2(\partial\Omega)} &= \|f\psi_j - f_{j,\varepsilon}\|_{L^2(\Gamma_1 \cap V_j)} \\ &\leqslant (1 + L_j^2)^{\frac{1}{2}} \|(f\psi_j) \circ \Phi_j - \tilde{f}_{j,\varepsilon}\|_{L^2(V_j')} \leqslant (1 + L_j^2)^{\frac{1}{2}} \varepsilon .\end{aligned}$$

It follows that the function $f_\varepsilon \in C(\partial\Omega)$, defined by

$$f_\varepsilon = \sum_{j=1}^J f_{j,\varepsilon} ,$$

satisfies

$$\|f - f_\varepsilon\|_{L^2(\partial\Omega)} \leqslant \varepsilon \sum_{j=1}^{J} (1 + L_j^2)^{\frac{1}{2}} .$$

This shows that (by choosing ε) we can choose f_ε as close as we wish to f.

The *second step* is to show that for each $j \in \{1, \dots, J\}$ we have $f_{j,\varepsilon} \in \mathcal{H}^1(\partial\Omega)$. This is equivalent to the statement that for each $j \in \{1, \dots, J\}$,

$$f_{j,\varepsilon} \circ \Phi_k \in \mathcal{H}^1(V_k') \qquad \forall\, k \in \{1, \dots, J\} . \tag{13.6.7}$$

We denote

$$V_{k,j}' = \{ y \in V_k' \mid \Phi_k(y) \in \Gamma_1 \cap V_j \} \qquad \forall\, j, k \in \{1, \dots, J\} .$$

Then (using that $\operatorname{supp} f_{j,\varepsilon} \subset \Gamma_1 \cap V_j$) (13.6.7) is equivalent to

$$\tilde{f}_{j,\varepsilon} \circ \Phi_j^{-1} \circ \Phi_k \in \mathcal{H}^1(V_{k,j}') \qquad \forall\, k \in \{1, \dots, J\} .$$

Since both $\tilde{f}_{j,\varepsilon}$ and $\Phi_j^{-1} \circ \Phi_k$ are Lipschitz, the above statement is true.

The third step is to show that $f_\varepsilon \in \mathcal{V}(\Gamma_1)$. For this, clearly it will be enough to show that each term $f_{j,\varepsilon}$ is in this space (where $j \in \{1, \dots, J\}$). We already know from the second step and from Proposition 13.5.8 that

$$f_{j,\varepsilon} \in \mathcal{H}^1(\partial\Omega) \subset \mathcal{H}^{\frac{1}{2}}(\partial\Omega) .$$

According to Theorem 13.6.1, there exist functions $g_{j,\varepsilon} \in \mathcal{H}^1(\Omega)$ such that

$$\gamma_0 \, g_{j,\varepsilon} = f_{j,\varepsilon} .$$

From $\operatorname{supp} f_{j,\varepsilon} \subset \Gamma_1 \cap V_j$ we see that indeed $f_{j,\varepsilon} \in \mathcal{V}(\Gamma_1)$. □

Corollary 13.6.11. *With Ω, Γ_0 and Γ_1 as in* (13.6.3), *$\mathcal{H}^{\frac{1}{2}}(\Gamma_1)$ is dense in $L^2(\Gamma_1)$.*

Indeed, this follows from the last theorem since, by Theorem 13.6.1,

$$\mathcal{V}(\Gamma_1) \subset \mathcal{H}^{\frac{1}{2}}(\Gamma_1) .$$

In the following four remarks we continue using the notation from (13.6.3).

Remark 13.6.12. In general, $\mathcal{V}(\Gamma_1) \subset \gamma_0 \mathcal{H}^1_{\Gamma_0}(\Omega)$, since

$$\gamma_0 \mathcal{H}^1_{\Gamma_0}(\Omega) = \left\{ f \in \gamma_0 \mathcal{H}^1(\Omega) \mid \operatorname{supp} f \subset (\partial\Omega \setminus \Gamma_0) \right\} .$$

The inclusion may be strict, because the inclusion $\Gamma_1 \subset \partial\Omega \setminus \Gamma_0$ may be strict.

Remark 13.6.13. We denote by $\partial\Gamma_0$ and $\partial\Gamma_1$ the boundaries of Γ_0 and Γ_1 in $\partial\Omega$. In general, it seems that $\gamma_0\mathcal{H}^1_{\Gamma_0}(\Omega)$ is not a subspace of $L^2(\Gamma_1)$. However, if $\partial\Gamma_0$ and $\partial\Gamma_1$ have surface measure zero, then Γ_1 and $\partial\Omega\setminus\Gamma_0$ differ by a set of measure zero, so that $\gamma_0\mathcal{H}^1_{\Gamma_0}(\Omega) \subset L^2(\Gamma_1)$. This is the case, for example, if $\partial\Gamma_0 = \partial\Gamma_1 = \emptyset$ or if $\partial\Gamma_0$ and $\partial\Gamma_1$ are Lipschitz in $\partial\Omega$, as in Definition 13.5.9 (and then $\partial\Gamma_0 = \partial\Gamma_1$). If $\gamma_0\mathcal{H}^1_{\Gamma_0}(\Omega)$ is a subspace of $L^2(\Gamma_1)$, then clearly it is a subspace of $\mathcal{H}^{\frac{1}{2}}(\Gamma_1)$ (because, according to Theorem 13.6.1, it is a subspace of $\mathcal{H}^{\frac{1}{2}}(\partial\Omega)$).

Remark 13.6.14. By combining Theorem 13.6.10 and Remarks 13.6.12 and 13.6.13, we obtain the following statement: If $\partial\Gamma_0$ and $\partial\Gamma_1$ have surface measure zero, then $\gamma_0\mathcal{H}^1_{\Gamma_0}(\Omega)$ is a dense subspace of $L^2(\Gamma_1)$.

Remark 13.6.15. Suppose that $\partial\Gamma_0 = \partial\Gamma_1 = \emptyset$ (equivalently, $\Gamma_0 = \operatorname{clos}\Gamma_0$ and $\Gamma_1 = \operatorname{clos}\Gamma_1$, or still equivalently, $\Gamma_1 = \partial\Omega \setminus \Gamma_0$). Intuitively, this means that Γ_0 and Γ_1 do not touch, like in Section 7.6. Then

$$\mathcal{V}(\Gamma_1) = \gamma_0\mathcal{H}^1_{\Gamma_0}(\Omega) = \mathcal{H}^{\frac{1}{2}}(\Gamma_1).$$

Indeed, the inclusions $\mathcal{V}(\Gamma_1) \subset \gamma_0\mathcal{H}^1_{\Gamma_0}(\Omega) \subset \mathcal{H}^{\frac{1}{2}}(\Gamma_1)$ follow from Remarks 13.6.12 and 13.6.13. If $f \in \mathcal{H}^{\frac{1}{2}}(\Gamma_1)$, then we extend it to be zero on Γ_0 and we obtain a function in $\mathcal{H}^{\frac{1}{2}}(\partial\Omega)$, which has support in Γ_1. Because of the "onto" statement in Theorem 13.6.1, $f \in \mathcal{V}(\Gamma_1)$. Thus, $\mathcal{H}^{\frac{1}{2}}(\Gamma_1) \subset \mathcal{V}(\Gamma_1)$, which concludes the proof.

The space $\mathcal{H}^1_{\Gamma_0}(\Omega)$ provides a natural framework to study the Laplace operator with mixed boundary conditions. In particular, the regularity result in Theorem 13.5.5 can be extended, with appropriate assumptions on Ω, Γ_0 and Γ_1, to the case in which the Dirichlet boundary conditions hold only on Γ_0 and with Neumann boundary conditions on Γ_1. More precisely, using a result which is difficult, but well known in the literature on elliptic PDEs (see, for instance, Grisvard [77, Theorem 2.4.1.3]), it is not difficult to establish the following proposition.

Proposition 13.6.16. *With the assumptions and the notation of Remark* 13.6.15*, suppose that $\partial\Omega$ is of class C^2 and that $\Gamma_0 \neq \emptyset$. Then the operator*

$$T\phi = \begin{bmatrix} \Delta\phi \\ \gamma_1\phi|_{\Gamma_1} \end{bmatrix}$$

is an isomorphism from $\mathcal{H}^2(\Omega) \cap \mathcal{H}^1_{\Gamma_0}(\Omega)$ onto $L^2(\Omega) \times \mathcal{H}^{\frac{1}{2}}(\Gamma_1)$.

13.7 The Green formulas and extensions of trace operators

Using trace operators, we derive in this section two identities called the Green formulas. The use of the Green formulas in computations is also called integration by parts. Using the Green formulas, we introduce some extensions of trace operators.

The results in Section 13.6 allow us to define the Dirichlet or the Neumann trace of a function $f \in \mathcal{H}^s(\Omega)$ for certain values of s. It has been shown in [157] that if a function f satisfies an elliptic PDE, then f and its derivatives have traces on the boundary, provided that $f \in \mathcal{H}^s(\Omega)$, without any restriction on $s \in \mathbb{R}$. We shall present here some particular cases of such extended trace operators, which are relevant for the other chapters.

We need the following Green-type formula, given in Neças [176, Theorem 1.1, Chapter 3] (see also Lions and Magenes [157, Chapter 2, Theorem 5.4]).

Theorem 13.7.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$, let $f, g \in \mathcal{H}^1(\Omega)$ and let $l \in \{1, \ldots, n\}$. Then we have*

$$\int_\Omega \frac{\partial f}{\partial x_l} g \,\mathrm{d}x + \int_\Omega f \frac{\partial g}{\partial x_l} \,\mathrm{d}x = \int_{\partial\Omega} (\gamma_0 f)(\gamma_0 g)\nu_l \,\mathrm{d}\sigma \tag{13.7.1}$$

("integration by parts"), where ν_l denotes the lth component of the unit outward normal vector field from Definition 13.6.3.

Remark 13.7.2. Suppose that $v \in \mathcal{H}^1(\Omega; \mathbb{C}^n)$ and $g \in \mathcal{H}^1(\Omega)$. If we take $f = v_l$ in (13.7.1) and do a summation over all $l = 1, 2, \ldots, n$, we obtain:

$$\int_\Omega (\mathrm{div}\; v) g \,\mathrm{d}x + \int_\Omega v \cdot \nabla g \,\mathrm{d}x = \int_{\partial\Omega} (v \cdot \nu) g \,\mathrm{d}\sigma. \tag{13.7.2}$$

In particular, for $g(x) = 1$, we obtain the *Gauss formula*

$$\int_\Omega \mathrm{div}\; v \,\mathrm{d}x = \int_{\partial\Omega} v \cdot \nu \,\mathrm{d}\sigma. \tag{13.7.3}$$

Remark 13.7.3. Formula (13.7.2) is often encountered in the following particular form: Suppose that Ω is as in Theorem 13.7.1, $h \in \mathcal{H}^2(\Omega)$ and $g \in \mathcal{H}^1(\Omega)$. If we denote $v = \mathrm{grad}\, h$ and apply (13.7.2), we obtain

$$\int_\Omega (\Delta h) g \,\mathrm{d}x + \int_\Omega \nabla h \cdot \nabla g \,\mathrm{d}x = \int_{\partial\Omega} (\gamma_1 h)(\gamma_0 g) \,\mathrm{d}\sigma.$$

(Here $\gamma_1 h$ is defined as in Remark 13.6.5.) This is sometimes called the *first Green formula*. If we interchange the roles of h and g and subtract the equations, we obtain

$$\int_\Omega (\Delta h) g \,\mathrm{d}x - \int_\Omega h(\Delta g) \,\mathrm{d}x = \int_{\partial\Omega} (\gamma_1 h)(\gamma_0 g) \,\mathrm{d}\sigma - \int_{\partial\Omega} (\gamma_0 h)(\gamma_1 g) \,\mathrm{d}\sigma,$$

which holds if $h, g \in \mathcal{H}^2(\Omega)$. This is called the *second Green formula*.

Remark 13.7.4. The Gauss formula (13.7.3) does not have to hold on unbounded domains. For example, let Ω be the exterior of the unit ball: $\Omega = \{x \in \mathbb{R}^n \mid |x| > 1\}$, with $n \geqslant 3$ and define the regular $\mathbb{C}^n$-valued distribution v on Ω by

$$v(x) = \frac{1}{|x|^n}\, x\,.$$

It is easy to verify that $v \in \mathcal{H}^1(\Omega;\mathbb{C}^n)$. It follows from (13.3.6) that $\operatorname{div} v = 0$. The left-hand side of (13.7.3) is clearly zero, while the right-hand side is $-A_n$, where $A_n = nV_n$ is the area of the unit sphere in $\mathbb{R}^n$ (V_n is the volume of the unit ball). However, this is not really surprising, because (13.7.3) has been derived from (13.7.2) using $g(x) = 1$, and this g is not in $\mathcal{H}^1(\Omega)$.

Example 13.7.5. Consider the following function (regular distribution) on $\mathbb{R}^n$:

$$f(x) = \frac{1}{|x|^{n-2}}\,.$$

We have seen in Remark 13.3.1 that

$$\nabla f = \frac{2-n}{|x|^n}\, x\,, \tag{13.7.4}$$

which is still a (vector-valued) regular distribution on $\mathbb{R}^n$. Our goal in this example is to compute $\Delta f = \operatorname{div}(\nabla f)$. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and let $R > 0$ sufficiently large so that $\operatorname{supp}\varphi \subset B(0,R)$ (here $B(0,R)$ denotes, as usual, the open ball of radius R centered at zero). According to (13.3.3), we have

$$\begin{aligned}
\langle \Delta f, \varphi\rangle &= -\langle \nabla f, \nabla\varphi\rangle \\
&= -\int_{B(0,R)} (\nabla f)(x)\cdot(\nabla\varphi)(x)\,\mathrm{d}x \\
&= -\lim_{\varepsilon\to 0}\int_{B(0,R)\setminus B(0,\varepsilon)} (\nabla f)(x)\cdot(\nabla\varphi)(x)\,\mathrm{d}x\,.
\end{aligned} \tag{13.7.5}$$

We shall now use the first Green formula (from Remark 13.7.3) on the domain $\Omega_\varepsilon = B(0,R)\setminus B(0,\varepsilon)$, where $\varepsilon > 0$. We take $h = f$, which is in $\mathcal{H}^2(\Omega_\varepsilon)$, and we take $g = \varphi$. Since $\Delta f = 0$ on Ω_ε (see (13.3.6)), we obtain

$$\int_{\Omega_\varepsilon} (\nabla f)(x)\cdot(\nabla\varphi)(x)\,\mathrm{d}x = \int_{\partial\Omega_\varepsilon} (\gamma_1 f)(\gamma_0\varphi)\,\mathrm{d}\sigma\,.$$

From (13.7.4) we see that $\gamma_1 f = \frac{n-2}{|x|^{n-1}}$, so that (13.7.5) becomes

$$\langle \Delta f, \varphi\rangle = -(n-2)\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \frac{\varphi(x)}{|x|^{n-1}}\,\mathrm{d}\sigma = -(n-2)A_n\varphi(0)\,,$$

where A_n is again the area of the unit sphere in $\mathbb{R}^n$. Thus,

$$\Delta \frac{1}{|x|^{n-2}} = -(n-2)A_n\delta_0,$$

where δ_0 is the Dirac mass at 0 (defined in Example 13.2.3).

With Ω as in Theorem 13.7.1, we denote by $\mathcal{H}^{-\frac{1}{2}}(\partial\Omega)$ the dual of $\mathcal{H}^{\frac{1}{2}}(\partial\Omega)$ with respect to the pivot space $L^2(\partial\Omega)$. We also introduce the space

$$\mathcal{D}(\Delta) = \{f \in \mathcal{H}^1(\Omega) \mid \Delta f \in L^2(\Omega)\},$$

where Δ is the Laplacian in the sense of distributions. Endowed with the norm

$$\|f\|_{\mathcal{D}(\Delta)} = \sqrt{\|f\|^2_{\mathcal{H}^1(\Omega)} + \|\Delta f\|^2_{L^2(\Omega)}} \qquad \forall\, f \in \mathcal{D}(\Delta),$$

$\mathcal{D}(\Delta)$ is clearly a Hilbert space.

Theorem 13.7.6. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$. Then the Neumann trace operator γ_1 (which until now was defined on $\mathcal{H}^2(\Omega)$) has an extension that is a bounded operator from $\mathcal{D}(\Delta)$ into $\mathcal{H}^{-\frac{1}{2}}(\partial\Omega)$.*

Proof. According to Theorem 13.6.1, we have that $\gamma_0 \in \mathcal{L}(\mathcal{H}^1(\Omega), \mathcal{H}^{\frac{1}{2}}(\partial\Omega))$ and this operator is onto. From Proposition 12.1.3 we conclude that $\gamma_0\gamma_0^*$ is a strictly positive (hence, invertible) operator on $\mathcal{H}^{\frac{1}{2}}(\partial\Omega)$.

Suppose that $f \in \mathcal{H}^2(\Omega)$ and consider an arbitrary $\varphi \in \mathcal{H}^{\frac{1}{2}}(\partial\Omega)$. Define $\widetilde{\varphi} \in \mathcal{H}^1(\Omega)$ by $\widetilde{\varphi} = \gamma_0^*(\gamma_0\gamma_0^*)^{-1}\varphi$. Denoting $c = \|\gamma_0^*(\gamma_0\gamma_0^*)^{-1}\|$, we have

$$\gamma_0\widetilde{\varphi} = \varphi, \quad \|\widetilde{\varphi}\|_{\mathcal{H}^1(\Omega)} \leqslant c\|\varphi\|_{\mathcal{H}^{\frac{1}{2}}(\partial\Omega)}.$$

From the first Green formula (Remark 13.7.3) we have

$$\int_{\partial\Omega} (\gamma_1 f)\,\varphi \,\mathrm{d}\sigma = \int_\Omega \Delta f\,\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega \nabla f \cdot \nabla\widetilde{\varphi}\,\mathrm{d}x. \tag{13.7.6}$$

This implies that

$$\left|\int_{\partial\Omega} (\gamma_1 f)\,\varphi\,\mathrm{d}\sigma\right| \leqslant c\|f\|_{\mathcal{D}(\Delta)}\|\varphi\|_{\mathcal{H}^{\frac{1}{2}}(\partial\Omega)} \qquad \forall\, \varphi \in \mathcal{H}^{\frac{1}{2}}(\partial\Omega),$$

which implies that $\|\gamma_1 f\|_{\mathcal{H}^{-\frac{1}{2}}} \leqslant c\|f\|_{\mathcal{D}(\Delta)}$. Hence, γ_1 can be extended as claimed. $\square$

Remark 13.7.7. In the last theorem, we did not claim the unicity of the extension of γ_1. If we would know that $\mathcal{H}^2(\Omega)$ is dense in $\mathcal{D}(\Delta)$, then of course the extension would be unique. However, we do not know if this is the case. The easiest way to define an extension of γ_1 is via (13.7.6) with $\widetilde{\varphi} = \gamma_0^*(\gamma_0\gamma_0^*)^{-1}\varphi$. Possibly different extensions can be obtained using (13.7.6) and a different definition of $\widetilde{\varphi}$.

Now we show that the Dirichlet trace operator γ_0 can also be extended. We introduce the space

$$\mathcal{W}(\Delta) = \{g \in L^2(\Omega) \mid \Delta g \in \mathcal{H}^{-1}(\Omega)\},$$

which is a Hilbert space with the norm

$$\|g\|_{\mathcal{W}(\Delta)} = \sqrt{\|g\|^2_{L^2(\Omega)} + \|\Delta g\|^2_{\mathcal{H}^{-1}(\Omega)}} \qquad \forall\, g \in \mathcal{W}(\Delta).$$

Proposition 13.7.8. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^2. Then the Dirichlet trace operator γ_0 (which until now was defined on $\mathcal{H}^1(\Omega)$) has an extension that is a bounded operator from $\mathcal{W}(\Delta)$ into $\mathcal{H}^{-\frac{1}{2}}(\partial\Omega)$.*

Proof. According to the last part of Theorem 13.6.6, we have that

$$\gamma_1 \in \mathcal{L}(\mathcal{H}^2(\Omega) \cap \mathcal{H}^1_0(\Omega), \mathcal{H}^{\frac{1}{2}}(\partial\Omega))$$

and this operator is onto. From Proposition 12.1.3 we conclude that $\gamma_1\gamma_1^*$ is a strictly positive (hence, invertible) operator on $\mathcal{H}^{\frac{1}{2}}(\partial\Omega)$.

Take $g \in \mathcal{H}^2(\Omega)$ and consider an arbitrary $\varphi \in \mathcal{H}^{\frac{1}{2}}(\partial\Omega)$. Define the function $\widetilde{\varphi} \in \mathcal{H}^2(\Omega) \cap \mathcal{H}^1_0(\Omega)$ by $\widetilde{\varphi} = \gamma_1^*(\gamma_1\gamma_1^*)^{-1}\varphi$. Denoting $\kappa = \|\gamma_1^*(\gamma_1\gamma_1^*)^{-1}\|$, we have

$$\gamma_0\widetilde{\varphi} = 0, \quad \gamma_1\widetilde{\varphi} = \varphi, \quad \|\widetilde{\varphi}\|_{\mathcal{H}^2(\Omega)} \leqslant \kappa\|\varphi\|_{\mathcal{H}^{\frac{1}{2}}(\partial\Omega)}.$$

From the second Green formula (Remark 13.7.3, with $h = \widetilde{\varphi}$) we have

$$\int_{\partial\Omega} (\gamma_0 g)\varphi \mathrm{d}\sigma = \int_\Omega g\, \Delta\widetilde{\varphi} \mathrm{d}x - \int_\Omega (\Delta g)\widetilde{\varphi} \mathrm{d}x. \tag{13.7.7}$$

This implies that for every $\varphi \in \mathcal{H}^{\frac{1}{2}}(\partial\Omega)$,

$$\left| \int_{\partial\Omega} (\gamma_0 g)\varphi \mathrm{d}\sigma \right| \leqslant \|g\|_{\mathcal{W}(\Delta)} \|\widetilde{\varphi}\|_{\mathcal{H}^2(\Omega)} \leqslant \kappa\|g\|_{\mathcal{W}(\Delta)}\|\varphi\|_{\mathcal{H}^{\frac{1}{2}}(\partial\Omega)},$$

which in turn implies that $\|\gamma_0 g\|_{\mathcal{H}^{-\frac{1}{2}}} \leqslant \kappa\|g\|_{\mathcal{W}(\Delta)}$. Hence, γ_0 can be extended from the domain $\mathcal{H}^2(\Omega)$ to the domain $\mathcal{W}(\Delta)$, as stated. On $\mathcal{H}^1(\Omega)$ this extension coincides with the one introduced in Theorem 13.6.1, because $\mathcal{H}^2(\Omega)$ is dense in $\mathcal{H}^1(\Omega)$ (this follows from the first part of Theorem 13.5.4). □

Remark 13.7.9. The extension of γ_0 (whose existence is stated in the last proposition) is not unique. The story is similar to Remark 13.7.7: the easiest way to specify an extension of γ_0 is to require that (13.7.7) should hold for all $g \in \mathcal{W}(\Delta)$ and for all $\varphi \in \mathcal{H}^{\frac{1}{2}}(\partial\Omega)$. Now, the integral on the left of (13.7.7) and one of the integrals on the right should be replaced by duality pairings:

$$\langle \gamma_0 g, \overline{\varphi}\rangle_{\mathcal{H}^{-\frac{1}{2}}(\partial\Omega), \mathcal{H}^{\frac{1}{2}}(\partial\Omega)} = \int_\Omega g\, \Delta\widetilde{\varphi} \mathrm{d}x - \langle \Delta g, \overline{\widetilde{\varphi}}\rangle_{\mathcal{H}^{-1}(\Omega), \mathcal{H}^1_0(\Omega)}.$$

Thus, γ_0 has a unique extension to $\mathcal{W}(\Delta)$ that satisfies the above formula.

Chapter 14

Appendix III: Some Background on Differential Calculus

The aim of this chapter is to provide an elementary proof of Theorem 9.4.3, after introducing the necessary tools from differential calculus. First we recall some basic concepts and prove a classical result of Sard. Then we give the detailed construction of η_0 from Theorem 9.4.3. Our method requires only a particular case of Sard's theorem (which is proved below). We refer the reader to Coron [36, Lemma 2.68] and Fursikov and Imanuvilov [69, Lemma 1.1] for related proofs.

Notation. In this chapter $n, p \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ is open, bounded and connected, with boundary of class C^m, with $m \geqslant 2$, and $\mathcal{O}$ is an open subset of Ω. For $a \in \mathbb{R}^n$ and $r > 0$ we denote by $B(a, r)$ the open ball in $\mathbb{R}^n$ of center a and radius r.

14.1 Critical points and Sard's theorem

Definition 14.1.1. Let $V \subset \mathbb{R}^n$ be open and $a \in V$. A mapping $f : V \to \mathbb{R}^p$ is called *differentiable at* a if there exists $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ such that

$$\lim_{\|h\| \to 0} \frac{1}{\|h\|}(f(a+h) - f(a) - Lh) = 0.$$

It is well known (see, for instance, Spivak [208, p. 16]) that there exists at most one linear map satisfying the above definition. This linear map will be denoted by $\mathrm{D}f(a)$ and it is called *the differential of* f *at* a (also called *the Jacobian of* f *at* a). The function f is in $C^1(V, \mathbb{R}^p)$ (or simply $C^1(V)$ for $p = 1$) if f is differentiable at each $a \in V$ and the map $a \mapsto \mathrm{D}f(a)$ is continuous from V to $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$. We recall a classical result in differential calculus, called *the inverse function theorem.*

Theorem 14.1.2. *Let* $f : V \to \mathbb{R}^n$ *be a* C^1 *function and let* $a \in V$ *be such that* $\mathrm{D}f(a)$ *is an invertible linear operator. Then there exist an open set* $U \subset V$ *containing* a *and an open set* $W \subset \mathbb{R}^n$ *containing* $f(a)$ *such that* f *is an invertible mapping from* U *onto* W *and the inverse map* $f^{-1} : W \to U$ *is* C^1.

For a proof of this result we refer the reader to [208, p. 35].

In the case $p = 1$, it is easy to check that if f is differentiable at $a \in V$, then, with the notation in Section 13.3,

$$\mathrm{D}f(a)h = \langle \nabla f(a), h\rangle \qquad \forall\, h \in \mathbb{R}^n .$$

Definition 14.1.3. Let $p \in \mathbb{N}$ and let $f : V \to \mathbb{R}^p$ be a C^1 function. We say that $a \in V$ is a *critical point* of f if $\operatorname{Ran} \mathrm{D}f(a) \neq \mathbb{R}^p$.

We also recall a well-known property which is a consequence of a result called the *chain rule* (see, for instance, [208, p. 19]).

Proposition 14.1.4. *Let $p \in \mathbb{N}$, let $W \subset \mathbb{R}^p$ be an open set and let $f : V \to W$ and $\gamma : W \to \mathbb{R}^q$, with $q \in \mathbb{N}$, be two functions which are differentiable at any point $a \in V$, respectively any $b \in W$. Then the function $g : W \to \mathbb{R}^q$ defined by $g = f \circ \gamma$ is differentiable at any point $b \in W$ and*

$$\mathrm{D}g(b)h = \mathrm{D}f(\gamma(b))[\mathrm{D}\gamma(b)h] \qquad \forall\, h \in \mathbb{R}^p .$$

Remark 14.1.5. If f is C^1 on the open convex set V and K is a compact convex subset of V, then there exist $\alpha > 0$ and an increasing function $\lambda : [0, \alpha] \to [0, \infty)$ such that $\lim_{t\to 0} \lambda(t) = 0$ and

$$\|f(y) - f(x) - \mathrm{D}f(x)(x - y)\| \leqslant \lambda(\|x - y\|)\|x - y\| \tag{14.1.1}$$

for every $x,\ y \in K$ with $\|x - y\| < \alpha$. Indeed, by applying Proposition 14.1.4 it follows that

$$f(y) - f(x) = \int_0^1 \mathrm{D}f(x + t(y - x))(y - x)\,\mathrm{d}t,$$

so that

$$\begin{aligned}&\|f(y) - f(x) - \mathrm{D}f(x)(x - y)\|\\ &\quad\leqslant \int_0^1 \|\mathrm{D}f(x + t(y - x))(y - x) - \mathrm{D}f(x)(x - y)\|\,\mathrm{d}t,\end{aligned}$$

and (14.1.1) follows by using the uniform continuity of $\mathrm{D}f$ on K. Note that the resulting λ is increasing.

The following result is a particular case of Sard's theorem. We refer the reader to Sternberg [211, p. 47] for stronger versions of this result.

Theorem 14.1.6. *Let $f : V \to \mathbb{R}^n$ be a C^1 function and let B be the set of all the critical points of f. Then the Lebesgue measure of $f(B)$ in $\mathbb{R}^n$ is zero.*

Proof. Let $x \in B$. Since the linear operator $\mathrm{D}f(x)$ is not invertible, $\operatorname{Ran} \mathrm{D}f(x)$ is contained in a subspace P of $\mathbb{R}^n$ with dimension at most $n - 1$. Denote

$$\widetilde{P} = \{w + f(x) \ \mid\ w \in P\},$$

which is the affine hyperplane parallel to P and passing by $f(x)$.

For $r > 0$ we denote (as usual) by $B(x,r)$ the open ball of center x and of radius r in $\mathbb{R}^n$. Let $r > 0$ be small enough such that $B(x,r) \subset V$ and let $y \in B(x,r)$. Since $f(x) + \mathrm{D}f(x)(x-y)$ belongs to $\widetilde{P}$, the distance of $f(y)$ to $\widetilde{P}$ is smaller than $\|f(y) - f(x) - \mathrm{D}f(x)(x-y)\|$. This fact and (14.1.1) imply that the distance of $f(y)$ to $\widetilde{P}$ is smaller than $\lambda(r)r$. Let

$$K = \sup_{z \in B(x,r)} \|\mathrm{D}f(z)\|.$$

Then

$$\|f(y) - f(x)\| = \left\| \int_0^1 \mathrm{D}f(x + t(y-x))(y-x)\,\mathrm{d}t \right\| \leqslant Kr \quad \forall\, y \in B(x,r).$$

The above facts show that f maps $B(x,r)$ into a cylinder $C(x,r)$ whose base is the $(n-1)$-dimensional ball $\widetilde{P} \cap B(f(x), Kr)$ and whose height is $2\lambda(r)r$. Let V_{n-1} be the volume of the $(n-1)$-dimensional unit ball. Then the volume (or the n-dimensional Lebesgue measure) of $C(x,r)$ is

$$\mathrm{Vol}(C(x,r)) = 2\lambda(r)r(V_{n-1}(Kr)^{n-1}) = 2V_{n-1}K^{n-1}\lambda(r)r^n\,.$$

It follows that

$$\mathrm{Vol}(f(B(x,r))) \leqslant \mathrm{Vol}(C(x,r)) = 2V_{n-1}K^{n-1}\lambda(r)r^n\,. \tag{14.1.2}$$

Let $k \in \mathbb{N}$ be such that the cube whose side length is $\frac{1}{k}$ is contained in V and let $m \in \mathbb{N}$. The cube A can be divided in at most m^n cubes whose side length is $\frac{1}{mk}$. It is easy to see that if one of these cubes contains some $x \in B$, then it is contained in $B(x, 2\frac{\sqrt{n}}{mk})$. Hence, $A \cap B$ is contained in at most m^n balls whose center is the image of a point of B through f and whose radius is $2\frac{\sqrt{n}}{mk}$. From (14.1.2) it follows that

$$\mathrm{Vol}(f(A \cap B)) \leqslant m^n 2V_{n-1}K^{n-1}\lambda\left(2\frac{\sqrt{n}}{mk}\right)\left(2\frac{\sqrt{n}}{mk}\right)^n = C(n,k,K)\lambda\left(2\frac{\sqrt{n}}{mk}\right),$$

where $C(n,k,K)$ is a positive constant depending on n, k and K but *not* on m. So, letting m go to $+\infty$, we obtain that $\mathrm{Vol}(f(A \cap B)) = 0$. Covering V by a countable number of such cubes, we get that $\mathrm{Vol}(f(B)) = 0$. □

14.2 Existence of Morse functions on Ω

Let $f \in C^2(\mathrm{clos}\,\Omega; \mathbb{R})$ and let $a \in \Omega$ be a critical point of f. We say that a is *non-degenerate* if the Hessian matrix $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right]$ is invertible.

Definition 14.2.1. A *Morse function* on Ω is an $f \in C^2(\mathrm{clos}\,\Omega, \mathbb{R})$ such that

$$f(x) = 0, \qquad \nabla f(x) \neq 0 \qquad \forall\, x \in \partial\Omega$$

and all the critical points of f in Ω are non-degenerate.

Proposition 14.2.2. *Let f be a Morse function on Ω. Then f has a finite number of critical points.*

Proof. Let $g : \operatorname{clos}\Omega \to \mathbb{R}^n$ be the C^1 function defined by

$$g(x) = \nabla f(x) \qquad \forall\, x \in \operatorname{clos}\Omega .$$

The fact that f is a Morse function implies that $\mathrm{D}g(a)$ is invertible at every point a such that $g(a) = 0$. According to Theorem 14.1.2, it follows that for any critical point a of f, there exists an open set $V_a \subset \Omega$ such that $\nabla f(x) \neq 0$ for every $x \in V_a$ that is different from a. Thus, a is isolated (within the set of critical points of f). The critical points cannot have a limit point on the boundary, because of the second condition in the definition of a Morse function. Therefore, the set of critical points is closed. Since it consists of isolated points, this set is finite. $\square$

The main result of this section is the following.

Theorem 14.2.3. *There exists a Morse function f on Ω such that $f \in C^m(\operatorname{clos}\Omega)$ and $f(x) > 0$ for every $x \in \Omega$.*

One of the main ingredients of the proof of Theorem 14.2.3 is the following result.

Lemma 14.2.4. *Let V be an open bounded subset of $\mathbb{R}^n$ and let $g \in C^2(\operatorname{clos} V; \mathbb{R})$. Then there exists a sequence (l_k) in $\mathbb{R}^n$ such that $\lim l_k = 0$ and for every $k \in \mathbb{N}$ the map*

$$x \mapsto f(x) + \langle l_k, x\rangle$$

has only non-degenerate critical points.

Proof. Let $G : V \to \mathbb{R}^n$ be defined by

$$G(x) = -\nabla f(x) \qquad \forall\, x \in V .$$

Let $B \subset V$ be the set of critical values of G and let $l \in \mathbb{R}^n$. For $l \notin G(B)$ we consider the map

$$x \mapsto f(x) + \langle l, x\rangle . \tag{14.2.1}$$

If $a \in V$ is a critical point of the above map, then $l = -\nabla f(a) = G(a)$. Since $l \notin G(B)$, it follows that $a \notin B$, so that $DG(a)$ is invertible. It is easy to see that the Hessian at a of the map defined in (14.2.1) is $-DG(a)$ so that the critical point a is non-degenerate. We have thus shown that if $l \notin G(B)$, then the map defined in (14.2.1) has only non-degenerate critical points.

On the other hand, by applying Sard's theorem (Theorem 14.1.6) to the function G, it follows that for every $k \in \mathbb{N}$ there exists $l_k \in B(0, \frac{1}{k}) \setminus G(B)$. The sequence (l_k) clearly has the required properties. $\square$

We are now in a position to prove the main result of this section.

Proof of Theorem 14.2.3. proof is divided in two steps.

Step 1. We show that there exists a function $v : \operatorname{clos}\,\Omega \to \mathbb{R}$ of class C^m satisfying

(P1) $v > 0$ in Ω, $v = 0$ on $\partial\Omega$,

(P2) v has no critical point in $V = \operatorname{clos}\,\mathcal{N} \cap \operatorname{clos}\,\Omega$, where $\mathcal{N}$ is an open neighborhood of $\partial\Omega$ in $\mathbb{R}^n$.

Since the open set Ω is of class C^m, by using Definition 13.5.2, it is not difficult to prove that there exists an open covering $(V^k)_{k=0,p}$ of $\operatorname{clos}\,\Omega$ such that $V^0 \cap \partial\Omega = \emptyset$, $(V^k)_{k=1,p}$ is a covering of $\partial\Omega$ and such that for every $k \in \{1, \ldots, p\}$ there exists a system of orthonormal coordinates $(y_1^k, \ldots, y_n^k)$ such that

1. V^k is a hypercube in the new coordinates;
2. for every $k \in \{1, \ldots, p\}$ there exists a C^m function φ^k of the arguments $(y_1^k, \ldots, y_{n-1}^k)$ that vary in the basis of the hypercube V^k, such that

$$\Omega \cap V^k = \{y = (y \in V^k \mid y_n^k < \varphi^k(y_1^k, \ldots, y_{n-1}^k)\},$$
$$\partial\Omega \cap V^k = \{y \in V^k \mid y_n^k = \varphi(y_1^k, \ldots, y_{n-1}^k)\}.$$

Let $v^0 : V^0 \to \mathbb{R}$ be defined by

$$v^0(x) = 1 \qquad \forall\, x \in V^0. \tag{14.2.2}$$

Moreover, for $1 \leqslant k \leqslant p$ we define $v^k : V^k \to \mathbb{R}$ by

$$v^k(y_1^k, \ldots, y_n^k) = y_n^k - \varphi^k(y_1^k, \ldots, y_{n-1}^k) \qquad \forall\, (y_1^k, \ldots, y_n^k) \in V_k\,.$$

We clearly have

$$v^k = 0 \text{ on } \partial\Omega \cap V^k. \tag{14.2.3}$$

Moreover, if ν is the unit outward normal vector field, defined by (13.6.1), then for every $y \in V^k \cap \partial\Omega$ we have

$$\nabla v^k(y) \cdot \nu(y) = \sqrt{1 + \left[\frac{\partial \varphi^k}{\partial y_1}(y')\right]^2 + \cdots + \left[\frac{\partial \varphi^k}{\partial y_{n-1}}(y')\right]^2} > 0,$$

where $y = (y_1^k, \ldots, y_n^k)$ and $y' = (y_1^k, \ldots, y_{n-1}^k)$. By using the compactness of $\partial\Omega$ it follows that there exists $m_0 > 0$ such that, for every $k \in \{1, \ldots, p\}$ we have

$$\nabla v^k(y) \cdot \nu(y) \geqslant m_0 \qquad \forall\, y \in \partial\Omega \cap V^k. \tag{14.2.4}$$

Let $\psi_1, \ldots, \psi_p$ be a partition of unity subordinated to the compact $\operatorname{clos}\,\Omega$ and to its covering $V^1, \ldots V^p$, as in Proposition 13.1.6. We next define $v \in \mathcal{D}(\mathbb{R}^n)$ by

$$v(x) = \sum_{k=1}^{p} \psi_k(x) v^k(x) \qquad \forall\, x \in \mathbb{R}^n. \tag{14.2.5}$$

We clearly have $v \in C^m(\text{clos}\,\Omega)$. Moreover, from (14.2.2), (14.2.3) and the properties of (ψ_k) it is easy to see that v satisfies property (P1) above.

On the other hand, by combining (14.2.3) and (14.2.5) it follows that

$$\nabla v(x) = \sum_{k=1}^{p} \psi_k(x) \nabla v_k(x) \qquad \forall\, x \in \partial\Omega.$$

From the above formula and (14.2.4) it follows that

$$\nabla v(x) \cdot \nu(x) \geqslant m_0 > 0 \qquad \forall\, x \in \partial\Omega.$$

Since $v = 0$ on $\partial\Omega$ it follows that

$$\|\nabla v(x)\| \geqslant m_0 \qquad \forall\, x \in \partial\Omega.$$

Using the continuity of the map $x \mapsto \nabla v(x)$ yields the fact that v also satisfies property (P2). This ends the first step of the proof.

Step 2. Let $v \in C^m(\text{clos}\,\Omega)$ be the function constructed in Step 1, let $V = \text{clos}\,\mathcal{N} \cap \text{clos}\,\Omega$ be the set constructed in Step 1, such that v has no critical point in V, and consider the open set $W = \Omega \setminus V$. By Proposition 13.1.5, there exists a smooth function $\eta \in \mathcal{D}(\Omega)$ with $0 \leqslant \eta \leqslant 1$ and $\eta = 1$ on $\text{clos}\, W$. We set

$$\varepsilon = \inf_{x \in \text{supp}\,(\eta) \cap V} \|\nabla v(x)\|, \tag{14.2.6}$$

so that $\varepsilon > 0$. By Lemma 14.2.4 there exists $l \in \mathbb{R}^n$ such that the map $x \mapsto v(x) + \langle l, x\rangle$ has the following properties:

(H1) it has only non-degenerate critical points on W;

(H2) the gradient of the map

$$x \mapsto \eta(x)\langle l, x\rangle \tag{14.2.7}$$

is smaller than $\frac{\varepsilon}{2}$ for $x \in \text{supp}\,(\eta) \cap V$;

(H3) the map defined in (14.2.7) is positive on $\text{supp}\,(\eta)$.

Let

$$f(x) = v(x) + \eta(x)\langle l, x\rangle \qquad \forall\, x \in \text{clos}\,\Omega.$$

By using the properties of v, of η and of l we easily see that $f \in C^m(\text{clos}\,\Omega)$, $f = 0$ on $\partial\Omega$ and $f > 0$ in Ω. We still have to show that f has only non-degenerate critical points. This will be done by noting first that

$$\overline{\Omega} = W \cup (\text{supp}(\eta) \cap V) \cup (\overline{\Omega} \setminus \text{supp}\,(\eta)).$$

Since

$$f(x) = v(x) + \langle l, x\rangle \qquad \forall\, x \in \text{clos}\, W,$$

it follows that all the critical points of f in $\text{clos}\, W$ are non-degenerate. On the other hand, for $x \in \text{supp}\,(\eta) \cap V$ we can combine (14.2.6) and condition (H2) above to obtain that f has no critical points in $\mathrm{supp}(\eta)\cap V$. Finally, on $\overline{\Omega}\setminus \mathrm{supp}\,(\eta) \subset V$ we have $f = v$, so that f has no critical points in $\overline{\Omega} \setminus \mathrm{supp}\,(\eta)$. Consequently, all the critical points of f in $\mathrm{clos}\,\Omega$ are non-degenerate, so that f satisfies all the conditions required in Theorem 14.2.3. $\square$

14.3 Proof of Theorem 9.4.3

The main ingredients of the proof of Theorem 9.4.3 are Theorem 14.2.3 and the following result. Recall the standing assumptions on Ω and $\mathcal{O}$, from the beginning of the chapter.

Proposition 14.3.1. *Let $l \in \mathbb{N}$ and let $\{a_1, \ldots, a_l\} \in \Omega$. Then there exists a C^∞ diffeomorphism $\Phi : \mathrm{clos}\,\Omega \to \mathrm{clos}\,\Omega$ such that $\Phi(x) = x$ for $x \in \partial\Omega$ and*

$$\Phi(a_k) \in \mathcal{O} \qquad \forall\, k \in \{1, \ldots, l\}.$$

For the proof we begin with the following lemma.

Lemma 14.3.2. *Let $W \subset \mathbb{R}^n$ be an open connected set. Then for any $x, y \in W$ there exists a C^∞ simple regular curve contained in W going from x to y. In other words, for every $x, y \in W$ there exists a C^∞ function $\gamma : [0, 1] \to W$ such that*

- *$\gamma(0) = x$, $\gamma(1) = y$;*
- *for every $t_1, t_2 \in [0, 1]$ with $t_1 \neq t_2$ we have $\gamma(t_1) \neq \gamma(t_2)$;*
- *$\dot\gamma(t) \neq 0$ on $[0, 1]$.*

Proof. For $x \in W$ we define W_x to be the set of the points $y \in W$ for which there exists a continuous piecewise linear map $\beta : [0, 1] \to$W such that

(PL1) $\beta(0) = x$, $\beta(1) = y$;
(PL2) for every $t_1, t_2 \in [0, 1]$ with $t_1 \neq t_2$ we have $\beta(t_1) \neq \beta(t_2)$;
(PL3) β is piecewise linear.

It is easy to check that W_x is open and non-empty. In order to show that $\Omega \setminus W_x$ is empty, we use a contradiction argument. If $\Omega \setminus W_x$ is not empty, take $a \in W_x$ such that x is closer to $\Omega \setminus W_x$ than to $\partial\Omega$. Let β_0 be a piecewise linear curve lying in W and leading from x to a. Let r_0 be the distance from a to $\Omega\setminus W_x$. Let $b \in \Omega\setminus W_x$ such that $|b-a| = r_0$. It is possible to find a piecewise linear curve lying in $B(a, r_0) \subset W_x$ leading from a to b which does not intersect β_0 (in most cases this will be just a straight line). Joining the two curves in a suitable way, we obtain a curve joining x and b, so that $b \in W_x$, which is a contradiction. We have thus shown that $W_x = W$, i.e., for every $x, y \in W$ there exists a path β satisfying (PL1)–(PL3). Let $0 = t_0 < t_1 < \cdots < t_{r-1} < t_r = 1$, with $r \in \mathbb{N}$, be such that β is an affine function on each interval $[t_k, t_{k+1}]$, with $k \in \{0, \ldots, r-1\}$. We extend β to a function defined on $\mathbb{R}$ (still denoted by β) which is affine on $(-\infty, t_1]$ and

$[t_{r-1},\infty)$. If $\varphi \in \mathcal{D}(\mathbb{R})$ is such that $\int_{\mathbb{R}} \varphi(t)\mathrm{d}t = 1$, $\int_{\mathbb{R}} t\varphi(t)\mathrm{d}t = 0$ and $\operatorname{supp}\varphi$ is a sufficiently small interval centered at 0, then the function

$$\gamma(t) = \int_{\mathbb{R}} \varphi(t-s)\beta(s)\,\mathrm{d}s \qquad \forall\, t \in [0,1]$$

satisfies the three conditions required in the lemma (in other words, the convolution with φ "rounds the corners" of β). $\square$

We also need the following result, which looks obvious but for which we did not find a proof in the literature. We give below a simple proof.

Lemma 14.3.3. *Let $W \subset \mathbb{R}^n$, with $n \geqslant 2$, be an open connected set, let $x, y \in W$ and let $\gamma : [0,1] \to W$ be a C^∞ curve satisfying the three conditions in Lemma 14.3.2. Then $\Omega \setminus \gamma([0,1])$ is an open connected set.*

Proof. Denote

$$I_c = \{t \in [0,1] \ \mid\ \Omega \setminus \gamma([0,t]) \text{ is connected }\}, \tag{14.3.1}$$

$$D = \{x \in \mathbb{R}^n \ \mid\ x_1 \in [0,1], \ \ x_2 = \cdots = x_n = 0\}. \tag{14.3.2}$$

Clearly $0 \in I_c$ so that $I_c \neq \emptyset$. We show that I_c is open in $[0,1]$. Let $t \in I_c$. Let $\varepsilon > 0$ be small enough such that $B(\gamma(t),\varepsilon)\setminus\gamma([0,t])$ is diffeomorphic to $B(0,1)\setminus D$. Since $B(0,1)\setminus D$ is connected, the same property holds for $B(\gamma(t),\varepsilon)\setminus\gamma([0,t])$. It is easy to see that for $\delta > 0$ small enough, $B(\gamma(t),\varepsilon)\setminus\gamma([0,t+\delta])$ remains connected. Let $p, q \in \Omega\setminus\gamma([0,t+\delta])$. Since $t \in I_c$, we can find a continuous path $g : [0,1] \to \Omega\setminus\gamma([0,t])$ with $g(0) = p$ and $g(1) = q$. If $g([0,1]) \cap B(\gamma(t),\varepsilon) = \emptyset$, then the path g goes from p to q and it is contained in $\Omega\setminus\gamma([0,t+\delta])$. Assume that $g([0,1]) \cap B(\gamma(t),\varepsilon) \neq \emptyset$ and denote

$$t_0 = \inf\{t > 0 \mid g(t) \in B(\gamma(t),\varepsilon)\}, \quad t_1 = \sup\{t > 0 \mid g(t) \in B(\gamma(t),\varepsilon)\}.$$

Using the fact that $B(\gamma(t),\varepsilon)\setminus\gamma([0,t+\delta])$ is connected, it follows that there exists a continuous function $f : [t_0,t_1] \to B(\gamma(t),\varepsilon)\setminus\gamma([0,t+\delta])$ such that $f(t_0) = g(t_0)$ and $f(t_1) = g(t_1)$. Define $\widetilde{g} : [0,1] \to \Omega$ by

$$\widetilde{g}(t) = \begin{cases} f(t) & \text{for } \ t \in [t_0,t_1], \\ g(t) & \text{for } \ t \in [0,1]\setminus[t_0,t_1]. \end{cases}$$

The function $\widetilde{g}$ is clearly continuous with $\widetilde{g}(0) = p$, $\widetilde{g}(1) = q$ and $\widetilde{g}(t) \in \Omega\setminus\gamma([0,t+\delta])$ for every $t \in [0,1]$. We have thus shown that $\Omega\setminus\gamma([0,t+\delta])$ is connected, thus, for $\delta > 0$ small enough, we have that $t+\delta \in I_c$, where I_c has been defined in (14.3.1). It follows that I_c is an open subset of $[0,1]$.

The set I_c is also closed in $[0,1]$. Indeed, let (t_k) be an increasing sequence of points of I_c converging to $t_\infty \in [0,1]$. Let ε be small enough in order to have that $B(\gamma(t_\infty),\varepsilon)\setminus\gamma([0,t_\infty])$ is diffeomorphic to $B(0,1)\setminus D$, where D has been defined in (14.3.2). By following the method used in order to show that I_c is open, we can construct a continuous path linking any two points of $\Omega \setminus \gamma([0,t_\infty])$ which does not intersect $\gamma([0,t_\infty])$, so that $t_\infty \in I_c$.

We have thus shown that the non-empty set I_c is both open and closed in $[0,1]$, so that $I_c = [0,1]$. We conclude that $\Omega \setminus \gamma([0,1])$ is connected. $\square$

We are now in a position to prove Proposition 14.3.1.

Proof of Proposition 14.3.1. Let $\{b_1,\dots,b_l\} \subset \mathcal{O}$. By applying recursively Lemmas 14.3.2 and 14.3.3, it follows that there exist the C^∞ functions $\gamma_1,\dots,\gamma_l : [0,1] \to \Omega$ such that

(SC1) for every $k \in \{1,\dots,l\}$ we have $\gamma_k(0) = a_k$, $\gamma_k(1) = b_k$;
(SC2) for every $k \in \{1,\dots,l\}$ and $t \in [0,1]$ we have $\dot\gamma_k(t) \neq 0$;
(SC3) for every $k, j \in \{1,\dots,l\}$ with $k \neq j$ we have $\gamma_k([0,1]) \cap \gamma_j([0,1]) = \emptyset$;
(SC4) for every $k \in \{1,\dots,l\}$ and $s, t \in [0,1]$ with $s \neq t$ we have $\gamma_k(t) \neq \gamma_k(s)$.

The next step is to construct a C^∞ vector field $X \in \mathcal{D}(\Omega,\mathbb{R}^n)$ such that

$$X(\gamma_k(t)) = \dot\gamma_k(t) \qquad \forall\, t \in [0,1], \quad k \in \{1,\dots,l\}.$$

This can be done first locally, around each curve by using property (SC3) and then by multiplying by an appropriate cutoff function. Let $\Phi : \Omega \times [0,\infty) \to \Omega$ be the flow associated with the vector field X. This means that for every $x \in \Omega$ the function $t \mapsto \Psi(x,t)$ is the solution of the initial value problem

$$\frac{\partial \Psi}{\partial t}(x,t) = X(\Psi(x,t),t) \qquad \forall\, t \geqslant 0, \qquad \Psi(x,0) = x.$$

Classical results on differential equations (see, for instance, Hartman [96, p. 100]) imply that Ψ is well defined and that $\Psi(\cdot,t)$ is a C^∞ diffeomorphism of Ω with $\Psi(\partial\Omega,t) = \partial\Omega$ for every $t \geqslant 0$. In particular, the map Φ, defined by

$$\Phi(x) = \Psi(1,x) \qquad \forall\, x \in \Omega,$$

is the desired diffeomorphism. Indeed, besides the properties inherited from Ψ, it is easily seen that $\Phi(a_k) = b_k$ for every $k \in \{1,\dots,l\}$. $\square$

Proof of Theorem 9.4.3. Let $f \in C^m(\operatorname{clos}\Omega,\mathbb{R})$ be a Morse function on Ω, with $f > 0$ on Ω as in Theorem 14.2.3. This means, in particular, that f has only a finite number of critical points $a_1,\dots,a_l$ in Ω, where $l \in \mathbb{N}$. Let Φ be the map constructed in Proposition 14.3.1 and let $\eta_0 = f \circ \Phi^{-1}$. We clearly have $\eta_0 \in C^m(\operatorname{clos}\Omega)$, $\eta_0 = 0$ on $\partial\Omega$ and $\eta_0 > 0$ in Ω.

For $q \in \operatorname{clos}\Omega$ we have, by the chain rule (see Proposition 14.1.4) $\mathrm{D}\eta_0(q) = \mathrm{D}f(\Phi^{-1}(q)) \circ \mathrm{D}\Phi^{-1}(q)$ and since $\mathrm{D}\Phi^{-1}(q) \in \mathcal{L}(\mathbb{R}^n)$ is an isomorphism, q is a critical point for η_0 iff $\Phi^{-1}(q)$ is a critical point of f. Since for every critical point p of f, we have $\Phi(p) \in \mathcal{O}$ it follows that all the critical points of η_0 belong to $\mathcal{O}$. This concludes the proof of Theorem 9.4.3. $\square$

Chapter 15

Appendix IV: Unique Continuation for Elliptic Operators

15.1 A Carleman estimate for elliptic operators

In this section we provide an elementary proof of a Carleman estimate for second-order elliptic operators. As it has already been remarked by Carleman in [29], this kind of estimates provides a powerful tool for proving unique continuation results for linear elliptic PDEs. Our approach is essentially based on Burq and Gérard [26]. More sophisticated versions of Carleman estimates are currently applied to quite general linear partial differential operators (see, for instance, Hörmander [103], Fursikov and Imanuvilov [69], Tataru [214, 216], Imanuvilov and Puel [106] and Lebeau and Robbiano [151, 152]).

Throughout this section, $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ is an open bounded set and the family of $C^2(\Omega)$ real-valued functions a_{kl}, with $k, l \in \{1, \dots, n\}$, is such that $a_{kl} = a_{lk}$ for every $k, l \in \{1, \dots, n\}$ and, for some constant $\delta > 0$,

$$\sum_{l,k=1}^{n} a_{kl}(x)\xi_k\xi_l \geqslant \delta \sum_{k=1}^{n} |\xi_k|^2 \qquad \forall\, x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^n. \tag{15.1.1}$$

We define the differential operator $P : \mathcal{H}^2(\Omega) \to L^2(\Omega)$ by

$$P\varphi = \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k}\left(a_{kl}\frac{\partial \varphi}{\partial x_l}\right).$$

With the above assumptions the operator P is *uniformly elliptic* on Ω. For $f \in L^2(\Omega)$ we denote by $\|f\|$ the norm of f in $L^2(\Omega)$, whereas for $x \in \mathbb{R}^n$ we denote by $|x|$ the Euclidian norm of x. The standard inner product in $L^2(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle$.

For $\lambda,\ s > 0$ we define the functions

$$\alpha(x) = e^{\lambda x_n} \qquad \forall\, x \in \mathbb{R}^n, \tag{15.1.2}$$

and the operator $P_{s,\lambda}$ is defined by

$$P_{s,\lambda}\varphi = e^{s\alpha}P(e^{-s\alpha}\varphi) \qquad \forall\, \varphi \in H^2(\Omega).$$

The main result of this section is the following.

Theorem 15.1.1. *Let K be a compact subset of Ω. Then there exist $C>0$, $\lambda_0>0$ and $s_0>0$ such that for every $s \geqslant s_0$ and every $\varphi \in \mathcal{H}^2(\Omega)$ supported in K,*

$$s\|\nabla\varphi\|^2 + s^3\|\varphi\|^2 \leqslant C\|P_{s,\lambda_0}\varphi\|^2. \tag{15.1.3}$$

Proof. We may assume, without loss of generality, that φ is real-valued. From (15.1.2) it follows that

$$\frac{\partial}{\partial x_l}(e^{-s\alpha}) = -\lambda s\delta_{ln}\,\alpha e^{-s\alpha}, \tag{15.1.4}$$

where δ_{ln} is the Kronecker symbol. It follows that for every $k\in\{1,\dots,n\}$ we have

$$\sum_{l=1}^n a_{kl}\frac{\partial}{\partial x_l}(e^{-s\alpha}\varphi) = e^{-s\alpha}\left(\sum_{l=1}^n a_{kl}\frac{\partial\varphi}{\partial x_l} - \lambda s\alpha a_{kn}\varphi\right).$$

From the above formula it follows that, for every $s,\ \lambda>0$ we have

$$\sum_{k,l=1}^n \frac{\partial}{\partial x_k}\left[a_{kl}\frac{\partial}{\partial x_l}(e^{-s\alpha}\varphi)\right] = \sum_{k=1}^n \frac{\partial}{\partial x_k}\left[e^{-s\alpha}\left(\sum_{l=1}^n a_{kl}\frac{\partial\varphi}{\partial x_l} - \lambda s\alpha a_{kn}\varphi\right)\right],$$

which, combined with (15.1.4), implies that

$$\begin{aligned}\sum_{k,l=1}^n \frac{\partial}{\partial x_k}\left[a_{kl}\frac{\partial}{\partial x_l}(e^{-s\alpha}\varphi)\right] &= -\lambda s\alpha e^{-s\alpha}\left(\sum_{l=1}^n a_{nl}\frac{\partial\varphi}{\partial x_l} - \lambda s\alpha a_{nn}\varphi\right)\\ &\quad + e^{-s\alpha}\sum_{k,l=1}^n \frac{\partial}{\partial x_k}\left(a_{kl}\frac{\partial\varphi}{\partial x_l}\right) - e^{-s\alpha}\sum_{k=1}^n \lambda s\frac{\partial}{\partial x_k}(\alpha a_{kn}\varphi).\end{aligned}$$

From the above formula it follows that

$$P_{s,\lambda}\varphi = L_1\varphi - L_2\varphi, \tag{15.1.5}$$

where

$$L_1\varphi = P\varphi + \lambda^2 s^2\alpha^2 a_{nn}\varphi, \tag{15.1.6}$$

$$L_2\varphi = \lambda s\left(\sum_{k=1}^n \alpha a_{nk}\frac{\partial\varphi}{\partial x_k} + \sum_{k=1}^n \frac{\partial}{\partial x_k}(\alpha a_{kn}\varphi)\right). \tag{15.1.7}$$

It is not difficult to check that L_1 is "symmetric" and L_2 is "skew-symmetric", in the sense that

$$\langle L_1\varphi,\psi\rangle = \langle L_1\psi,\varphi\rangle, \quad \langle L_2\varphi,\psi\rangle = -\langle L_2\psi,\varphi\rangle$$

for every $\varphi,\ \psi \in H^2(\Omega)$ supported in K. This property, combined with (15.1.5), implies that for every $s,\ \lambda > 0$ and every $\varphi,\ \psi \in H^2(\Omega)$ supported in K, we have

$$\|P_{s,\lambda}\varphi\|^2 = \|L_1\varphi\|^2 + \|L_2\varphi\|^2 + \langle (L_2L_1 - L_1L_2)\varphi, \varphi\rangle . \tag{15.1.8}$$

Let us estimate the last term on the right-hand side of the above formula. We have

$$\begin{aligned}
L_1L_2\varphi = \lambda s &\sum_{p,q,k=1}^n \frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial}{\partial x_q}\left(\alpha a_{nk}\frac{\partial\varphi}{\partial x_k} + \frac{\partial}{\partial x_k}(\alpha a_{kn}\varphi)\right)\right) \\
&+ \lambda^3 s^3\alpha^3 a_{nn}\sum_{k=1}^n a_{nk}\frac{\partial\varphi}{\partial x_k} + \lambda^3 s^3\alpha^2 a_{nn}\sum_{k=1}^n \frac{\partial}{\partial x_k}(\alpha a_{kn}\varphi),
\end{aligned}$$

$$\begin{aligned}
L_2L_1\varphi = \lambda s &\sum_{k,p,q=1}^n \alpha a_{nk}\frac{\partial^2}{\partial x_k\partial x_p}\left(a_{pq}\frac{\partial\varphi}{\partial x_q}\right) + \lambda^3 s^3\alpha\sum_{k=1}^n a_{nk}\frac{\partial}{\partial x_k}(\alpha^2 a_{nn}\varphi) \\
&+ \lambda s\sum_{k,p,q=1}^n \frac{\partial}{\partial x_k}\left(\alpha a_{kn}\frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial\varphi}{\partial x_q}\right)\right) + \lambda^3 s^3\sum_{k=1}^n \frac{\partial}{\partial x_k}\left(a_{kn}\alpha^3 a_{nn}\varphi\right),
\end{aligned}$$

so that

$$\begin{aligned}
L_2L_1\varphi = \lambda s &\sum_{k,p,q=1}^n \alpha a_{nk}\frac{\partial^2}{\partial x_k\partial x_p}\left(a_{pq}\frac{\partial\varphi}{\partial x_q}\right) + \lambda s\sum_{k,p,q=1}^n \frac{\partial}{\partial x_k}\left(\alpha a_{kn}\frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial\varphi}{\partial x_q}\right)\right) \\
&+ 2\lambda^3 s^3\alpha\sum_{k=1}^n a_{nk}\frac{\partial}{\partial x_k}(\alpha^2 a_{nn})\varphi + \lambda^3 s^3\alpha^3 a_{nn}\sum_{k=1}^n a_{nk}\frac{\partial\varphi}{\partial x_k} \\
&+ \lambda^3 s^3\alpha^2 a_{nn}\sum_{k=1}^n \frac{\partial}{\partial x_k}\left(\alpha a_{kn}\varphi\right).
\end{aligned}$$

The above formulas imply that

$$\begin{aligned}
&(L_2L_1 - L_1L_2)\varphi \\
&= 2\lambda^3 s^3\alpha\sum_{k=1}^n a_{nk}\frac{\partial}{\partial x_k}(\alpha^2 a_{nn})\varphi \\
&\quad + \lambda s\sum_{k,p,q=1}^n \alpha a_{nk}\frac{\partial^2}{\partial x_k\partial x_p}\left(a_{pq}\frac{\partial\varphi}{\partial x_q}\right) + \lambda s\sum_{k,p,q=1}^n \frac{\partial}{\partial x_k}\left(\alpha a_{kn}\frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial\varphi}{\partial x_q}\right)\right) \\
&\quad - \lambda s\sum_{p,q,k=1}^n \frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial}{\partial x_q}\left(\alpha a_{nk}\frac{\partial\varphi}{\partial x_k} + \frac{\partial}{\partial x_k}(\alpha a_{kn}\varphi)\right)\right) \\
&= 4\lambda^4 s^3\alpha^3 a_{nn}^2\varphi + 2\lambda^3 s^3\alpha^3\sum_{k=1}^n a_{nk}\frac{\partial a_{nn}}{\partial x_k}\varphi + \lambda s(L_3\varphi + L_4\varphi - L_5\varphi),
\end{aligned} \tag{15.1.9}$$

where

$$L_3\varphi = 2\sum_{k,p,q=1}^{n} \alpha a_{nk}\frac{\partial^2}{\partial x_k \partial x_p}\Big(a_{pq}\frac{\partial \varphi}{\partial x_q}\Big), \tag{15.1.10}$$

$$L_4\varphi = \sum_{k,p,q=1}^{n} \frac{\partial(\alpha a_{kn})}{\partial x_k}\frac{\partial}{\partial x_p}\Big(a_{pq}\frac{\partial \varphi}{\partial x_q}\Big), \tag{15.1.11}$$

$$L_5\varphi = \sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial}{\partial x_q}\left(2\alpha a_{nk}\frac{\partial \varphi}{\partial x_k} + \frac{\partial(\alpha a_{kn})}{\partial x_k}\varphi\right)\right). \tag{15.1.12}$$

By simple calculations, (15.1.10) implies that

$$\begin{aligned}
L_3\varphi &= 2\alpha \sum_{k,p,q=1}^{n} a_{nk}\frac{\partial}{\partial x_k}\Big(\frac{\partial a_{pq}}{\partial x_p}\frac{\partial \varphi}{\partial x_q}\Big) + 2\alpha \sum_{k,p,q=1}^{n} a_{nk}\frac{\partial}{\partial x_k}\Big(a_{pq}\frac{\partial^2 \varphi}{\partial x_p \partial x_q}\Big) \\
&= 2\alpha \sum_{k,p,q=1}^{n} a_{nk}\frac{\partial^2 a_{pq}}{\partial x_k \partial x_p}\frac{\partial \varphi}{\partial x_q} + 2\alpha \sum_{k,p,q=1}^{n} a_{nk}\frac{\partial a_{pq}}{\partial x_p}\frac{\partial^2 \varphi}{\partial x_k \partial x_q} \\
&\quad + 2\alpha \sum_{k,p,q=1}^{n} a_{nk}\frac{\partial a_{pq}}{\partial x_k}\frac{\partial^2 \varphi}{\partial x_p \partial x_q} + 2\alpha \sum_{k,p,q=1}^{n} a_{nk}a_{pq}\frac{\partial^3 \varphi}{\partial x_k \partial x_p \partial x_q}.
\end{aligned} \tag{15.1.13}$$

We write the operator L_5 defined in (15.1.12) in a more convenient form:

$$\begin{aligned}
L_5\varphi &= 2\sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial}{\partial x_q}\left(\alpha a_{nk}\frac{\partial \varphi}{\partial x_k}\right)\right) + \sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial}{\partial x_q}\left(\frac{\partial(\alpha a_{kn})}{\partial x_k}\varphi\right)\right) \\
&= 2\sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial(\alpha a_{nk})}{\partial x_q}\frac{\partial \varphi}{\partial x_k}\right) + 2\sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p}\left(a_{pq}\alpha a_{nk}\frac{\partial^2 \varphi}{\partial x_q \partial x_k}\right) \\
&\quad + \sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial}{\partial x_q}\left(\frac{\partial(\alpha a_{kn})}{\partial x_k}\varphi\right)\right) \\
&= 2\sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial(\alpha a_{nk})}{\partial x_q}\frac{\partial \varphi}{\partial x_k}\right) + 2\sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p}\left(a_{pq}\alpha a_{nk}\right)\frac{\partial^2 \varphi}{\partial x_q \partial x_k} \\
&\quad + 2\alpha \sum_{p,q,k=1}^{n} a_{pq}a_{nk}\frac{\partial^3 \varphi}{\partial x_p \partial x_q \partial x_k} + \sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial}{\partial x_q}\left(\frac{\partial(\alpha a_{kn})}{\partial x_k}\varphi\right)\right).
\end{aligned} \tag{15.1.14}$$

From (15.1.11), (15.1.13) and (15.1.14) we see that the terms containing the third-order derivatives of φ in the last term on the right-hand side of (15.1.9) cancel, so that

$$\begin{aligned}
&L_3\varphi + L_4\varphi - L_5\varphi \\
&= 2\alpha \sum_{k,p,q=1}^{n} a_{nk} \frac{\partial^2 a_{pq}}{\partial x_k \partial x_p} \frac{\partial \varphi}{\partial x_q} + 2\alpha \sum_{k,p,q=1}^{n} a_{nk} \frac{\partial a_{pq}}{\partial x_p} \frac{\partial^2 \varphi}{\partial x_k \partial x_q} \\
&\quad + 2\alpha \sum_{k,p,q=1}^{n} a_{nk} \frac{\partial a_{pq}}{\partial x_k} \frac{\partial^2 \varphi}{\partial x_p \partial x_q} + \sum_{k,p,q=1}^{n} \frac{\partial (\alpha a_{kn})}{\partial x_k} \frac{\partial}{\partial x_p}\Big(a_{pq} \frac{\partial \varphi}{\partial x_q}\Big) \\
&\quad - 2 \sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p} \left(a_{pq} \frac{\partial (\alpha a_{nk})}{\partial x_q} \frac{\partial \varphi}{\partial x_k} \right) - 2 \sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p} \left(a_{pq} \alpha a_{nk} \right) \frac{\partial^2 \varphi}{\partial x_q \partial x_k} \\
&\quad - \sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p} \left(a_{pq} \frac{\partial^2 (\alpha a_{kn})}{\partial x_q \partial x_k} \varphi \right) - \sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p} \left(a_{pq} \frac{\partial (\alpha a_{kn})}{\partial x_k} \frac{\partial \varphi}{\partial x_q} \right). \qquad (15.1.15)
\end{aligned}$$

We remark that

$$\begin{aligned}
&2\alpha \sum_{k,p,q=1}^{n} a_{nk} \frac{\partial a_{pq}}{\partial x_p} \frac{\partial^2 \varphi}{\partial x_k \partial x_q} \\
&\qquad = 2 \sum_{k,p,q=1}^{n} \frac{\partial}{\partial x_q} \left(\alpha a_{nk} \frac{\partial a_{pq}}{\partial x_p} \frac{\partial \varphi}{\partial x_k} \right) - 2 \sum_{k,p,q=1}^{n} \frac{\partial}{\partial x_q} \left(\alpha a_{nk} \frac{\partial a_{pq}}{\partial x_p} \right) \frac{\partial \varphi}{\partial x_k},
\end{aligned}$$

$$\begin{aligned}
&2\alpha \sum_{k,p,q=1}^{n} a_{nk} \frac{\partial a_{pq}}{\partial x_k} \frac{\partial^2 \varphi}{\partial x_p \partial x_q} \\
&\qquad = 2 \sum_{k,p,q=1}^{n} \frac{\partial}{\partial x_p} \left(\alpha a_{nk} \frac{\partial a_{pq}}{\partial x_k} \frac{\partial \varphi}{\partial x_q} \right) - 2 \sum_{k,p,q=1}^{n} \frac{\partial}{\partial x_p} \left(\alpha a_{nk} \frac{\partial a_{pq}}{\partial x_k} \right) \frac{\partial \varphi}{\partial x_q},
\end{aligned}$$

$$\begin{aligned}
&\sum_{k,p,q=1}^{n} \frac{\partial (\alpha a_{kn})}{\partial x_k} \frac{\partial}{\partial x_p}\Big(a_{pq} \frac{\partial \varphi}{\partial x_q}\Big) \\
&\qquad = \sum_{k,p,q=1}^{n} \frac{\partial}{\partial x_p}\Big(\frac{\partial (\alpha a_{kn})}{\partial x_k} a_{pq} \frac{\partial \varphi}{\partial x_q}\Big) - \sum_{k,p,q=1}^{n} \frac{\partial}{\partial x_p}\Big(\frac{\partial (\alpha a_{kn})}{\partial x_k}\Big) a_{pq} \frac{\partial \varphi}{\partial x_q},
\end{aligned}$$

$$\begin{aligned}
&2 \sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p} \left(a_{pq} \alpha a_{nk} \right) \frac{\partial^2 \varphi}{\partial x_q \partial x_k} \\
&= 2 \sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_q} \left(\frac{\partial}{\partial x_p} \left(a_{pq} \alpha a_{nk} \right) \frac{\partial \varphi}{\partial x_k} \right) - 2 \sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_q} \left(\frac{\partial}{\partial x_p} \left(a_{pq} \alpha a_{nk} \right) \right) \frac{\partial \varphi}{\partial x_k}.
\end{aligned}$$

Substituting the last four formulas into (15.1.15), we obtain

$$\begin{aligned}
&L_3\varphi + L_4\varphi - L_5\varphi \\
&= -2\sum_{k,p,q=1}^{n} \frac{\partial(\alpha a_{nk})}{\partial x_p}\frac{\partial a_{pq}}{\partial x_k}\frac{\partial \varphi}{\partial x_q} - \sum_{k,p,q=1}^{n} \frac{\partial^2(\alpha a_{kn})}{\partial x_p \partial x_k} a_{pq}\frac{\partial \varphi}{\partial x_q} \\
&\quad + 2\sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_q}\left(a_{pq}\frac{\partial(\alpha a_{nk})}{\partial x_p}\right)\frac{\partial \varphi}{\partial x_k} - 4\sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial(\alpha a_{nk})}{\partial x_q}\frac{\partial \varphi}{\partial x_k}\right) \\
&\quad + 2\sum_{k,p,q=1}^{n} \frac{\partial}{\partial x_p}\left(\alpha a_{nk}\frac{\partial a_{pq}}{\partial x_k}\frac{\partial \varphi}{\partial x_q}\right) - \sum_{p,q,k=1}^{n} \frac{\partial}{\partial x_p}\left(a_{pq}\frac{\partial^2(\alpha a_{kn})}{\partial x_q \partial x_k}\varphi\right).
\end{aligned}$$

Taking the inner product of both sides with φ and integrating by parts, we notice that the contributions from the second and from the last term on the right-hand side of the above relation vanish, so that

$$\begin{aligned}
&\langle L_3\varphi + L_4\varphi - L_5\varphi, \varphi\rangle \\
&= -2\sum_{k,p,q=1}^{n} \left\langle \frac{\partial(\alpha a_{nk})}{\partial x_p}\frac{\partial a_{pq}}{\partial x_k}\frac{\partial \varphi}{\partial x_q}, \varphi\right\rangle + 2\sum_{p,q,k=1}^{n} \left\langle \frac{\partial}{\partial x_q}\left(a_{pq}\frac{\partial(\alpha a_{nk})}{\partial x_p}\right)\frac{\partial \varphi}{\partial x_k}, \varphi\right\rangle \\
&\quad + 4\sum_{p,q,k=1}^{n} \left\langle a_{pq}\frac{\partial(\alpha a_{nk})}{\partial x_q}\frac{\partial \varphi}{\partial x_k}, \frac{\partial \varphi}{\partial x_p}\right\rangle - 2\sum_{k,p,q=1}^{n} \left\langle \alpha a_{nk}\frac{\partial a_{pq}}{\partial x_k}\frac{\partial \varphi}{\partial x_q}, \frac{\partial \varphi}{\partial x_p}\right\rangle.
\end{aligned}$$

By developing $\frac{\partial(\alpha a_{nk})}{\partial x_q}$ in the third term on the right-hand side of the above relation, it follows that

$$\begin{aligned}
&\langle L_3\varphi + L_4\varphi - L_5\varphi, \varphi\rangle \\
&= -2\sum_{k,p,q=1}^{n} \left\langle \frac{\partial(\alpha a_{nk})}{\partial x_p}\frac{\partial a_{pq}}{\partial x_k}\frac{\partial \varphi}{\partial x_q}, \varphi\right\rangle + 2\sum_{p,q,k=1}^{n} \left\langle \frac{\partial}{\partial x_q}\left(a_{pq}\frac{\partial(\alpha a_{nk})}{\partial x_p}\right)\frac{\partial \varphi}{\partial x_k}, \varphi\right\rangle \\
&\quad + 4\lambda\sum_{p,k=1}^{n} \left\langle \alpha a_{pn}a_{nk}\frac{\partial \varphi}{\partial x_k}, \frac{\partial \varphi}{\partial x_p}\right\rangle + 4\alpha\sum_{p,q,k=1}^{n} \left\langle a_{pq}\frac{\partial a_{nk}}{\partial x_q}\frac{\partial \varphi}{\partial x_k}, \frac{\partial \varphi}{\partial x_p}\right\rangle \\
&\quad - 2\sum_{k,p,q=1}^{n} \left\langle \alpha a_{nk}\frac{\partial a_{pq}}{\partial x_k}\frac{\partial \varphi}{\partial x_q}, \frac{\partial \varphi}{\partial x_p}\right\rangle. \qquad (15.1.16)
\end{aligned}$$

The first term on the right-hand side of the above relation can be written as

$$\begin{aligned}
-2\sum_{k,p,q=1}^{n} \left\langle \frac{\partial(\alpha a_{nk})}{\partial x_p}\frac{\partial a_{pq}}{\partial x_k}\frac{\partial \varphi}{\partial x_q}, \varphi\right\rangle &= -\int_\Omega \frac{\partial(\alpha a_{nk})}{\partial x_p}\frac{\partial a_{pq}}{\partial x_k}\frac{\partial}{\partial x_q}|\varphi|^2\,\mathrm{d}x \\
&= \int_\Omega \frac{\partial}{\partial x_q}\left(\frac{\partial(\alpha a_{nk})}{\partial x_p}\frac{\partial a_{pq}}{\partial x_k}\right)|\varphi|^2\,\mathrm{d}x.
\end{aligned}$$

It follows that

$$\left\| -2 \sum_{k,p,q=1}^{n} \left\langle \frac{\partial(\alpha a_{nk})}{\partial x_p} \frac{\partial a_{pq}}{\partial x_k} \frac{\partial \varphi}{\partial x_q}, \varphi \right\rangle \right\| \leqslant C_1 C_2 \lambda^2 \|\varphi\|^2 \tag{15.1.17}$$

for every $\lambda \geqslant 1$, where

$$C_1 = \max_{x \in K} |\alpha(x)| \tag{15.1.18}$$

and $C_2 > 0$ depends only on (a_{kl}) and on K.

The second term on the right-hand side of (15.1.16) can be written as

$$\begin{aligned} 2 \sum_{p,q,k=1}^{n} & \left\langle \frac{\partial}{\partial x_q} \left(a_{pq} \frac{\partial(\alpha a_{nk})}{\partial x_p} \right) \frac{\partial \varphi}{\partial x_k}, \varphi \right\rangle \\ &= \sum_{p,q,k=1}^{n} \int_\Omega \frac{\partial}{\partial x_q} \left(a_{pq} \frac{\partial(\alpha a_{nk})}{\partial x_p} \right) \frac{\partial}{\partial x_k} |\varphi|^2 \, \mathrm{d}x \\ &= - \sum_{p,q,k=1}^{n} \int_\Omega \frac{\partial^2}{\partial x_k \partial x_q} \left(a_{pq} \frac{\partial(\alpha a_{nk})}{\partial x_p} \right) |\varphi|^2 \, \mathrm{d}x \, . \end{aligned}$$

From the above formula it easily follows that

$$2 \left\| \sum_{p,q,k=1}^{n} \left\langle \frac{\partial}{\partial x_q} \left(a_{pq} \frac{\partial(\alpha a_{nk})}{\partial x_p} \right) \frac{\partial \varphi}{\partial x_k}, \varphi \right\rangle \right\| \leqslant C_1 C_3 \lambda^3 \|\varphi\|^2 \tag{15.1.19}$$

for every $\lambda \geqslant 1$, where C_2 has been defined in (15.1.18) and $C_3 > 0$ depends only on (a_{kl}) and on K.

The third term on the right-hand side of (15.1.16) is non-negative. Indeed, for every $\lambda > 0$ we have

$$4\lambda \sum_{p,k=1}^{n} \left\langle \alpha a_{pn} a_{nk} \frac{\partial \varphi}{\partial x_k}, \frac{\partial \varphi}{\partial x_p} \right\rangle = 4\lambda \int_K \alpha \sum_{k=1}^{n} \left| a_{nk} \frac{\partial \varphi}{\partial x_k} \right|^2 \mathrm{d}x \geqslant 0 \, . \tag{15.1.20}$$

For the last two terms on the right-hand side of (15.1.16) it is easy to check that

$$\begin{aligned} &\left| 4 \sum_{p,q,k=1}^{n} \left\langle \alpha a_{pq} \frac{\partial a_{nk}}{\partial x_q} \frac{\partial \varphi}{\partial x_k}, \frac{\partial \varphi}{\partial x_p} \right\rangle - 2 \sum_{k,p,q=1}^{n} \left\langle \alpha a_{nk} \frac{\partial a_{pq}}{\partial x_k} \frac{\partial \varphi}{\partial x_q}, \frac{\partial \varphi}{\partial x_p} \right\rangle \right| \\ &\qquad \leqslant C_4 \int_K \alpha |\nabla \varphi|^2 \, \mathrm{d}x \end{aligned} \tag{15.1.21}$$

for every $\lambda \geqslant 1$, where $C_4 > 0$ is a constant depending only on (a_{kl}) and on K. By combining (15.1.16), (15.1.17), (15.1.19), (15.1.20) and (15.1.21), it follows that for every $\lambda \geqslant 1$ we have

$$\langle L_3\varphi + L_4\varphi - L_5\varphi, \varphi\rangle \geqslant -C_4 \int_K \alpha|\nabla\varphi|^2 \,\mathrm{d}x - C_1C_5\lambda^3\|\varphi\|^2, \tag{15.1.22}$$

where C_1 has been defined in (15.1.18) and $C_5 = \max\{C_2, C_3\}$.

On the other hand, it is easily seen that there exists $C_6 > 0$, depending only on (a_{kl}) and on K, such that

$$\left|\langle 2\lambda^3 s^3 \alpha^3 \sum_{k=1}^n a_{nk} \frac{\partial a_{nn}}{\partial x_k}\varphi, \varphi\rangle\right| \leqslant C_6\lambda^3 s^3 \int_K \alpha^3|\varphi|^2 \,\mathrm{d}x.$$

Combining the last inequality with (15.1.8), (15.1.9) and (15.1.22), we obtain that for every $\lambda \geqslant 1$ we have

$$\begin{aligned}\|L_1\varphi\|^2 + \|L_2\varphi\|^2 + 4\lambda^4 s^3 \int_K \alpha^3 a_{nn}^2 |\varphi|^2 \,\mathrm{d}x \leqslant \|P_{s,\lambda}\varphi\|^2 + C_6\lambda^3 s^3 \int_K \alpha^3|\varphi|^2 \,\mathrm{d}x \\ + C_4\lambda s \int_K \alpha|\nabla\varphi|^2 \,\mathrm{d}x + C_1C_5\lambda^4 s\|\varphi\|^2. \end{aligned} \tag{15.1.23}$$

In order to absorb the term containing $\int_K \alpha|\nabla\varphi|^2 \,\mathrm{d}x$ from the right-hand side of the above estimate we note that from (15.1.6) it follows that

$$\begin{aligned} -\langle L_1\varphi, \alpha\varphi\rangle &= \sum_{k,l=1}^n \int_K \alpha a_{kl} \frac{\partial\varphi}{\partial x_k}\frac{\partial\varphi}{\partial x_l} \,\mathrm{d}x \\ &\quad + \sum_{l=1}^n \lambda \int_K \alpha a_{nl}\frac{\partial\varphi}{\partial x_l}\varphi \,\mathrm{d}x - \lambda^2 s^2 \int_K \alpha^3 a_{nn}|\varphi|^2 \,\mathrm{d}x \\ &\geqslant \delta \int_K \alpha|\nabla\varphi|^2 \,\mathrm{d}x - C_1C_7\lambda\|\varphi\|^2 - \lambda^2 s^2 \int_K \alpha^3 a_{nn}|\varphi|^2 \,\mathrm{d}x, \end{aligned}$$

where δ is the constant from (15.1.1), C_1 has been defined in (15.1.18) and $C_7 > 0$ depends only on (a_{kl}) and on K. It follows that

$$\begin{aligned} C_4\lambda s \int_K \alpha|\nabla\varphi|^2 \,\mathrm{d}x \leqslant \delta^{-1}C_4\lambda s\|L_1\varphi\| \cdot \|\alpha\varphi\| \\ + \delta^{-1}C_4\lambda^3 s^3 \int_K \alpha^3 a_{nn}|\varphi|^2 \,\mathrm{d}x + \delta^{-1}C_1C_4C_7\lambda^2 s\|\varphi\|^2. \end{aligned} \tag{15.1.24}$$

From the above inequality it follows that for $\lambda, s \geqslant 1$ and $\varepsilon > 0$ we have

$$\begin{aligned} C_4\lambda s \int_K \alpha|\nabla\varphi|^2 \,\mathrm{d}x \leqslant \delta^{-1}C_4\varepsilon\|L_1\varphi\|^2 + \varepsilon^{-1}\delta^{-1}C_4\lambda^2 s^2 \int_K \alpha^2|\varphi|^2 \,\mathrm{d}x \\ + \delta^{-1}C_4\lambda^3 s^3 \int_K \alpha^3 a_{nn}|\varphi|^2 \,\mathrm{d}x + \delta^{-1}C_1C_4C_7\lambda^2 s\|\varphi\|^2. \end{aligned}$$

By using the above inequality, with ε chosen such that $\delta^{-1}C_4\varepsilon < \frac{1}{2}$, in (15.1.23) we obtain that

$$\frac{1}{2}\|L_1\varphi\|^2 + \|L_2\varphi\|^2 + \left\langle s^3\alpha^3(4\lambda^4 a_{nn}^2 - \delta^{-1}C_4\lambda^3 a_{nn} - C_6\lambda^3)\varphi, \varphi\right\rangle$$
$$\leqslant \|P_{s,\lambda}\varphi\|^2 + \varepsilon^{-1}\delta^{-1}C_4\lambda^2 s^2\int_K \alpha^2|\varphi|^2\,\mathrm{d}x + \delta^{-1}C_1C_4C_6\lambda^2 s\|\varphi\|^2 + C_1C_5\lambda^4 s\|\varphi\|^2.$$

Using again (15.1.24) in the above inequality we obtain that

$$\begin{aligned} &C_4\lambda s\int_K \alpha|\nabla\varphi|^2\,\mathrm{d}x + \left\langle s^3\alpha^3(4\lambda^4 a_{nn}^2 - 2\delta^{-1}C_4\lambda^3 a_{nn} - C_6\lambda^3)\varphi, \varphi\right\rangle \\ &\leqslant \|P_{s,\lambda}\varphi\|^2 + 2\varepsilon^{-1}\delta^{-1}C_4\lambda^2 s^2\int_K \alpha^2|\varphi|^2\,\mathrm{d}x + 2\delta^{-1}C_1C_4C_6\lambda^2 s\|\varphi\|^2 \\ &\qquad + C_1C_5\lambda^4 s\|\varphi\|^2. \end{aligned} \tag{15.1.25}$$

Since C_4 and C_6 depend only on (a_{kl}) and on K, there exists $\lambda_0 > 0$ such that

$$4\lambda_0^4 a_{nn}^2 - 2\delta^{-1}C_4\lambda_0^3 a_{nn} - C_6\lambda_0^3 > 2\lambda_0^4 a_{nn}^2.$$

Choosing $\lambda = \lambda_0$ in (15.1.25) we obtain that

$$\begin{aligned} &C_4\lambda_0 s\int_K \alpha|\nabla\varphi|^2\,\mathrm{d}x + 2\lambda_0^4 s^3\int_K \alpha^3 a_{nn}^2|\varphi|^2\,\mathrm{d}x \\ &\leqslant \|P_{s,\lambda_0}\varphi\|^2 + 2\varepsilon^{-1}\delta^{-1}C_4\lambda_0^2 s^2\int_K \alpha^2|\varphi|^2\,\mathrm{d}x + 2\delta^{-1}C_1C_4C_6\lambda_0^2 s\|\varphi\|^2 \\ &\qquad + C_1C_5\lambda_0^4 s\|\varphi\|^2 \end{aligned} \tag{15.1.26}$$

for every $s \geqslant 1$. Since the constants (C_k) involved in (15.1.26) are independent of s, all but the first term on the right-hand side of (15.1.26) can be absorbed by the terms on the left-hand side, provided that s is large enough. This fact clearly implies the conclusion (15.1.1). $\square$

The result in the above theorem still holds if we perturb the operator P by lower-order terms. More precisely, we have the following corollary.

Corollary 15.1.2. *With the notation in Theorem* 15.1.1, *let* $b \in L^\infty(\Omega;\mathbb{R}^n)$, $c \in L^\infty(\Omega;\mathbb{R})$ *and let* $\widetilde{P} = P + Q$ *where*

$$Q\varphi = b\cdot\nabla\varphi + c\varphi \qquad \forall\, \varphi \in H^2(\Omega).$$

Then there exist $C > 0$, $\lambda_0 > 0$ *and* $s_0 > 0$ *such that for every* $s \geqslant s_0$ *and every* $\varphi \in H^2(\Omega)$ *supported in* K,

$$s\|\nabla\varphi\|^2 + s^3\|\varphi\|^2 \leqslant C\|\widetilde{P}_{s,\lambda_0}\varphi\|^2, \tag{15.1.27}$$

where $\widetilde{P}_{s,\lambda} = e^{s\alpha}\widetilde{P}e^{-s\alpha}$ *for every* $s,\ \lambda > 0$.

Proof. We first remark that by using (15.1.4) we have

$$\sum_{l=1}^{n} b_l \frac{\partial}{\partial x_l}(e^{-s\alpha}\varphi) = e^{-s\alpha}\left(\sum_{l=1}^{n} b_l \frac{\partial \varphi}{\partial x_l} - \lambda s\alpha b_n \varphi\right),$$

so that, for every $s,\ \lambda > 0$ we have

$$\widetilde{P}_{s,\lambda}\varphi = P_{s,\lambda}\varphi + Q\varphi - \lambda s\alpha b_n\varphi.$$

The above formula together with (15.1.1) imply that

$$\begin{aligned} s\|\nabla\varphi\|^2 + s^3\|\varphi\|^2 &\leqslant C\|P_{s,\lambda_0}\varphi\|^2 \\ &\leqslant C\|\widetilde{P}_{s,\lambda_0}\varphi - Q\varphi + \lambda_0 s\alpha b_n\varphi\|^2 \leqslant 3C\left(\|\widetilde{P}_{s,\lambda_0}\varphi\|^2 + \|Q\varphi\|^2 + \lambda_0^2 s^2\|\alpha b_n\varphi\|^2\right). \end{aligned}$$

The above inequality easily implies that (15.1.27) holds for s large enough. □

15.2 The unique continuation results

In this section we apply the results from the previous one to prove unique continuation results for second-order elliptic operators.

For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball of center x and radius r. We also use the notation $B_r = B(0, r)$. The Euclidian norm of $x \in \mathbb{R}^n$ will be denoted by $|x|$, while the norm in $L^2(\Omega)$ (with $\Omega \subset \mathbb{R}^n$) will be denoted by $\|\cdot\|$.

The main result of this section is the following.

Theorem 15.2.1. *Let Ω be an open bounded and connected subset of $\mathbb{R}^n$, let*

$$(a_{kl})_{k,l\in\{1,\dots,n\}} \in C^2\left(\Omega, \mathbb{R}^{n^2}\right), \qquad b \in L^\infty(\Omega;\mathbb{R}^n), \qquad c \in L^\infty(\Omega;\mathbb{R})$$

and let $\phi \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$ be such that

$$\sum_{k,l=1}^{n} \frac{\partial}{\partial x_k}\left(a_{kl}\frac{\partial \phi}{\partial x_l}\right) + b\cdot\nabla\phi + c\phi = 0 \quad \text{in } L^2(\Omega). \tag{15.2.1}$$

Moreover, assume that there exists an open subset $\mathcal{O}$ of Ω such that

$$\phi(x) = 0 \qquad \forall\, x \in \mathcal{O}.$$

Then $\phi = 0$ in Ω.

Proof. We also denote by ϕ the extension of the original ϕ to $\mathbb{R}^n$ obtained by setting $\phi = 0$ outside Ω. According to Lemma 13.4.11, we have $\phi \in \mathcal{H}_0^1(\mathbb{R}^n)$. Recall from Section 13.2 that the support of ϕ, denoted $\operatorname{supp}\phi$, is the complement in $\mathbb{R}^n$

of the largest open set $\mathcal{G}$ such that the restriction of ϕ to $\mathcal{G}$ is the zero distribution on $\mathcal{G}$ (clearly $\mathcal{O} \subset \mathcal{G}$). Therefore, in order to prove the theorem, it suffices to show that $\operatorname{supp} \phi = \emptyset$. This will be proved by a contradiction argument. If $\operatorname{supp} \phi$ is not empty, take $x \in \Omega \setminus \operatorname{supp} \phi$ such that x is closer to $\operatorname{supp} \phi$ than to $\partial\Omega$. Let r_0 be the distance from x to $\operatorname{supp} \phi$. Then it follows that $B(x, r_0) \subset \Omega \setminus \operatorname{supp} \phi$ and $\partial B(x, r_0)$ contains at least one point $y \in \operatorname{supp} \phi$ (see Figure 15.1).

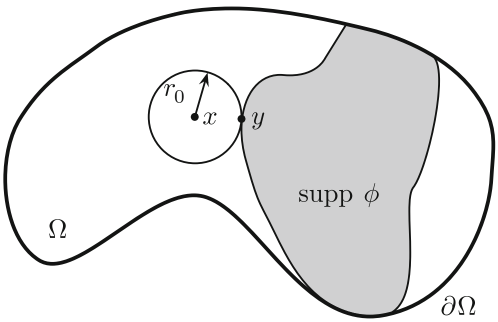

Figure 15.1: The point x is closer to $\operatorname{supp} \phi$ than to $\partial\Omega$. The ball $B(x, r_0)$ is in the complement of $\operatorname{supp} \phi$ in Ω and $y \in \operatorname{supp} \phi \cap \partial B(x, r_0)$.

It is easy to check that there exists a local system of curvilinear coordinates $(\tilde{x}_1, \dots, \tilde{x}_n)$ with the origin in y (i.e., $\tilde{x}(y) = 0$) such that, for some $r_1 > 0$,

$$B_{r_1} \subset \Omega, \qquad \operatorname{supp} \phi \cap \{\tilde{x} \in B_{r_1} \mid \tilde{x}_n \geqslant 0\} = \{0\}, \tag{15.2.2}$$

as illustrated in Figure 15.2. Using this new system of coordinates, (15.2.1) implies that

$$\widetilde{P}\phi(\tilde{x}) = 0 \qquad (\tilde{x} \in B_{r_1}),$$

where $\widetilde{P}$ is a differential operator as in Corollary 15.1.2, with appropriate coefficients (a_{kl}), b and c (expressed as functions of the new coordinates $\tilde{x}$). Let $r_2 \in (0, r_1)$ and let $\chi \in \mathcal{D}(B_{r_1})$ be such that $\chi = 1$ on B_{r_2} (see again Figure 15.2).

It is clear that

$$\operatorname{supp}(\nabla\chi) \subset \{x \in \mathbb{R}^n \mid r_2 \leqslant |x| \leqslant r_1\}. \tag{15.2.3}$$

By applying Corollary 15.1.2 with $\varphi = \chi e^{s\alpha}\phi$, where $\alpha = \alpha(x_n) = e^{\lambda x_n}$, it follows that there exist the constants $s_0, \lambda_0, C > 0$ such that, for $\lambda = \lambda_0$,

$$s\left\|\nabla(\chi e^{s\alpha}\phi)\right\|^2 + s^3\left\|\chi e^{s\alpha}\phi\right\|^2 \leqslant C\left\|\widetilde{P}_{s,\lambda_0}(\chi e^{s\alpha}\phi)\right\|^2 \qquad \forall\, s \geqslant s_0,$$

where $\widetilde{P}_{s,\lambda} = e^{s\alpha}\widetilde{P}e^{-s\alpha}$. Since $\widetilde{P}_{s,\lambda_0}(\chi e^{s\alpha}\phi) = e^{s\alpha}\widetilde{P}(\chi\phi)$, it follows that

$$s\left\|\nabla(\chi e^{s\alpha}\phi)\right\|^2 + s^3\left\|\chi e^{s\alpha}\phi\right\|^2 \leqslant C\left\|e^{s\alpha}\widetilde{P}(\chi\phi)\right\|^2 \qquad \forall\, s \geqslant s_0. \tag{15.2.4}$$

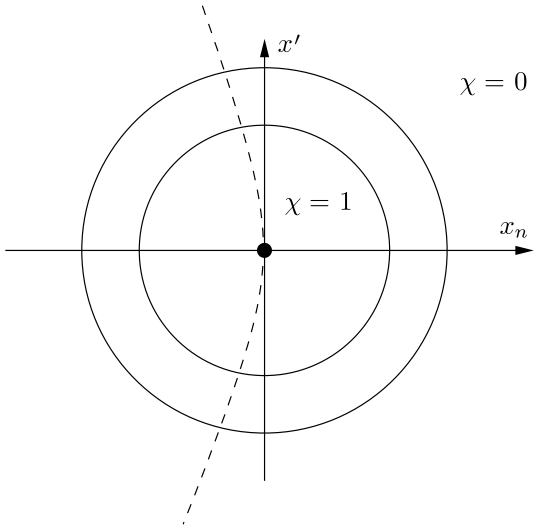

Figure 15.2: The set $\operatorname{supp}\phi$ (in the new coordinates $\tilde{x}$) is to the left of the dashed curve. Here $\tilde{x}' = (\tilde{x}_1, \dots, \tilde{x}_{n-1})$ and the point 0 corresponds to y in Figure 15.1.

On the other hand,

$$\nabla(\chi\phi) = \phi\nabla\chi + \chi\nabla\phi,$$

$$\sum_{k,l=1}^{n} \frac{\partial}{\partial x_k}\left[a_{kl}\frac{\partial}{\partial x_l}(\chi\phi)\right] = \phi \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k}\left(a_{kl}\frac{\partial \chi}{\partial x_l}\right) + 2\sum_{k,l=1}^{n} a_{kl}\frac{\partial\chi}{\partial x_k}\frac{\partial\phi}{\partial x_l} + \chi\sum_{k,l=1}^{n}\frac{\partial}{\partial x_k}\left(a_{kl}\frac{\partial\phi}{\partial x_l}\right).$$

The above two formulas and the fact that $\widetilde{P}\phi = 0$ imply that

$$\widetilde{P}(\chi\phi) = \phi\sum_{k,l=1}^{n}\frac{\partial}{\partial x_k}\left(a_{kl}\frac{\partial\chi}{\partial x_l}\right) + \phi b\cdot\nabla\chi + 2\sum_{k,l=1}^{n} a_{kl}\frac{\partial\chi}{\partial x_k}\frac{\partial\phi}{\partial x_l},$$

so that

$$\operatorname{supp}\widetilde{P}(\chi\phi) \subset \operatorname{supp}\phi\cap\operatorname{supp}\nabla\chi.$$

The above inclusion, together with (15.2.2) and (15.2.3), implies that there exists $\varepsilon > 0$ such that

$$x_n \leqslant -\varepsilon \qquad \forall\, x\in\operatorname{supp}\widetilde{P}(\chi\phi). \tag{15.2.5}$$

If we multiply both sides of (15.2.4) by $e^{-2s\alpha(-\varepsilon)}$, it follows that

$$s^3\left\|\chi e^{s(\alpha-\alpha(-\varepsilon))}\phi\right\|^2 \leqslant C\left\|e^{s(\alpha-\alpha(-\varepsilon))}\widetilde{P}(\chi\phi)\right\|^2 \qquad \forall\, s\geqslant s_0.$$

From (15.2.5) it follows that the right-hand side of the above estimate tends to zero when $s \to \infty$, so that the left-hand side has the same property. This means that

$$\operatorname{supp} \chi\phi \subset \{x \in \mathbb{R}^n \mid \alpha(x_n) - \alpha(-\varepsilon) \leqslant 0\} = \{x \in \mathbb{R}^n \mid x_n \leqslant -\varepsilon\},$$

which contradicts (15.2.2). $\square$

The above theorem implies a unique continuation result from the boundary. For the sake of simplicity, we give this result only for second-order operators having the Laplacian as principal part.

Corollary 15.2.2. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, connected and with Lipschitz boundary, let $b \in L^\infty(\Omega, \mathbb{C}^n)$, $c \in L^\infty(\Omega)$, and let $\phi \in \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)$ such that*

$$\Delta\phi + b \cdot \nabla\phi + c\phi = 0 \qquad \forall\, x \in \Omega. \tag{15.2.6}$$

Moreover, assume that there exists an open subset Γ of $\partial\Omega$ such that

$$\frac{\partial\phi(x)}{\partial\nu}(x) = 0 \qquad \forall\, x \in \Gamma.$$

Then $\phi = 0$ in Ω.

Proof. Let $x_0 \in \Gamma$ and let $\varepsilon > 0$ be such that the ball of center x_0 and radius ε, denoted by $B(x_0, \varepsilon)$, satisfies the condition

$$B(x_0, \varepsilon) \cap \partial\Omega \subset \Gamma.$$

We denote $\Omega_\varepsilon = \Omega \cup B\left(x_0, \frac{\varepsilon}{2}\right)$. By using the fact that $\partial\Omega$ is Lipschitz (see Definition 13.5.1) it follows that $\Omega_\varepsilon \setminus \operatorname{clos}\Omega$ is a non-empty open set. We extend ϕ to a function, still denoted by ϕ, defined on Ω_ε by setting $\phi(x) = 0$ for $x \in \Omega_\varepsilon \setminus \operatorname{clos}\Omega$. From Lemma 13.4.11 it follows that $\phi \in \mathcal{H}_0^1(\Omega_\varepsilon)$. This implies that

$$\langle \Delta\phi, \varphi\rangle_{\mathcal{D}'(\Omega_\varepsilon),\mathcal{D}(\Omega_e)} = \int_{\Omega_\varepsilon} \nabla\phi \cdot \nabla\varphi \,\mathrm{d}x \qquad \forall\, \varphi \in \mathcal{D}(\Omega_\varepsilon),$$

so that $\Delta\phi \in L^2(\Omega_\varepsilon)$. By applying Theorem 13.5.5 it follows that $\phi \in \mathcal{H}^2(\Omega_\varepsilon) \cap \mathcal{H}_0^1(\Omega_\varepsilon)$ and

$$\Delta\phi + b \cdot \nabla\phi + c\phi = 0 \quad \text{in} \quad \Omega_\varepsilon.$$

Since ϕ vanishes on a non-empty open subset of Ω_ε, the conclusion follows by applying Theorem 15.2.1. $\square$

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List of Notation

Index